A HOMOTOPY LIE FORMULA FOR THE P-ADIC DWORK FROBENIUS OPERATOR

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Abstract. We give a modern deformation theoretic interpretation of Dwork's theory of the zeta function of a smooth projective complete intersection variety X over a finite field. Using this interpretation, we explicitly construct a dgla (differential graded Lie algebra) for X whose cohomology gives the p-adic Dwork cohomology of X and a homotopy Lie endomorphism (so called, L_∞-endomorphism) of such dgla's, which encodes deformation data for the p-adic Dwork Frobenius operator of X. As a consequence, we will derive a formula for the p-adic Dwork Frobenius operator, whose characteristic polynomial computes the zeta function of X, in terms of $L_\infty-$morphisms and the Bell polynomials.

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1. Introduction

The goal of this paper is to recapture the deformation theory of the zeta functions of an algebraic variety, which was invented by Dwork (see [5] and [6] for example) and developed by Adolphson and Sperber (see [1] and [2] for example), using the modern deformation theoretic view point based on the dgla (differential graded Lie algebra). As an application, we derive an explicit algebraic formula for the p-adic Dwork Frobenius operator whose characteristic polynomial computes the zeta function of a smooth projective complete intersection variety, in terms of homotopy Lie morphisms (so called, L_∞-homotopy morphisms) and the Bell polynomials.

We fix a rational odd prime number p and a positive integer a. Let $\mathbb{F}_q$ be the finite field of q elements where $q = p^a$. Let n and k be positive integers such that $n \geq k \geq 1$. We use $\underline{x} = [x_0, x_1, \cdots, x_n]$ as a

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homogeneous coordinate system of the projective n-space $\mathbf{P}^n_{\mathbb{F}_q}$ over a finite field $\mathbb{F}_q$. Let $X = X_{\underline{G}}$ be a smooth complete intersection of dimension $n-k$ in the projective space $\mathbf{P}^n_{\mathbb{F}_q}$ and let $G_1(\underline{x}), \cdots, G_k(\underline{x})$ be the defining homogeneous polynomials in $\mathbb{F}_q[x_0, \cdots, x_n]$ such that $\deg(G_i) = d_i$ for $i = 1, \cdots, k$. It is well known (e.g. see [2]) that the zeta function of X may be written in the form

$$Z(X/\mathbb{F}_q, T) = \frac{P(T)^{(-1)^{n-k-1}}}{(1-T)(1-qT)\cdots(1-q^{n-k}T)} \tag{1.1}$$

where $P(T) \in 1 + T\mathbb{Z}[T]$. The reciprocal roots of $P(T)$ are units at all primes except the archimedean prime and those primes lying over p. At any archimedean prime, they have absolute value $q^{(n-k)/2}$ by Deligne [3]; at p-adic places, the p-adic Newton polygon lies on or above its Hodge polygon by Mazur [10] (called the Katz conjecture). In [2], Adolphson and Sperber gave a new proof of Mazur's result on the Katz conjecture by computing an explicit basis for the p-adic Dwork cohomology.

1.1. **The geometric idea and our motivation to use the homotopy Lie theory.** In order to explain the geometric idea behind the work of Dwork, Adolphson and Sperber and our idea how we came up with the homotopy Lie theory of the zeta function of X, we assume in this subsection that X is defined over the field of complex numbers $\mathbb{C}$ instead of $\mathbb{F}_q$. In this case, the zeta function of X has a cohomological interpretation. If we are interested in the cohomology of the smooth projective complete intersection variety X of dimension $n-k$, then the primitive middle dimensional cohomology $H^{n-k}_{\text{prim}}(X, \mathbb{C})$ is the most interesting piece because the other degree cohomologies and non-primitive pieces can be easily described in terms of the cohomology of the projective space $\mathbf{P}^n$ due to the weak Lefschetz theorem and the Poincare duality. For the computation of $H^{n-k}_{\text{prim}}(X, \mathbb{C})$, the Gysin sequence and "the Cayley trick" play important roles. There is a long exact sequence, called the Gysin sequence:

$$\cdots \to H^{n+k-1}(\mathbf{P}^n, \mathbb{C}) \to H^{n+k-1}(\mathbf{P}^n \setminus X, \mathbb{C}) \xrightarrow{\text{Res}_X} H^{n-k}(X, \mathbb{C}) \to H^{n+k}(\mathbf{P}^n, \mathbb{C}) \to \cdots$$

where Res_X is the residue map (see p. 96 of [4]). This sequence gives rise to an isomorphism

$$\text{Res}_X : H^{n+k-1}(\mathbf{P}^n \setminus X, \mathbb{C}) \xrightarrow{\sim} H^{n-k}_{\text{prim}}(X, \mathbb{C}).$$

The Cayley trick is about translating a computation of the cohomology of the complement of a complete intersection into a computation of the cohomology of the complement of a hypersurface in a bigger space. Let $\mathcal{E} = \mathcal{O}_{\mathbf{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^n}(d_k)$ be the locally free sheaf of $\mathcal{O}_{\mathbf{P}^n}$-modules with rank k. Let $\mathbf{P}(\mathcal{E})$ be the projective bundle associated to $\mathcal{E}$ with fiber $\mathbf{P}^{k-1}$ over $\mathbf{P}^n$. Then $\mathbf{P}(\mathcal{E})$ is the smooth projective toric variety with Picard group isomorphic to $\mathbb{Z}^2$ whose (toric) homogeneous coordinate ring is given by

$$A_{\mathbf{P}(\mathcal{E})} := \mathbb{C}[y_1, \cdots, y_k, x_0, \cdots, x_n] \tag{1.2}$$

where $y_1, \cdots, y_k$ are new variables corresponding to $G_1, \cdots, G_k$. There are two additive gradings ch and wt, called the charge and the weight, corresponding to the Picard group $\mathbb{Z}^2$:

$$ch(y_i) = -d_i, \quad \text{for } i = 1, \cdots, k, \quad ch(x_j) = 1, \quad \text{for } j = 0, \cdots, n,$$
$$wt(y_i) = 1, \quad \text{for } i = 1, \cdots, k, \quad wt(x_j) = 0, \quad \text{for } j = 0, \cdots, n.$$

Then

$$S(\underline{y}, \underline{x}) := \sum_{j=1}^{k} y_j G_j(\underline{x}) \in A_{\mathbf{P}(\mathcal{E})}$$

defines a hypersurface X_S in $\mathbf{P}(\mathcal{E})$. The natural projection map $\mathbf{P}(\mathcal{E}) \to \mathbf{P}^n$ induces a morphism $\mathbf{P}(\mathcal{E}) \setminus X_S \to \mathbf{P}^n \setminus X$ which can be checked to be a homotopy equivalence. Hence there exists an isomorphism

$$H^{n+k-1}(\mathbf{P}(\mathcal{E}) \setminus X_S, \mathbb{C}) \xrightarrow{\sim} H^{n+k-1}(\mathbf{P}^n \setminus X, \mathbb{C}).$$

The cohomology group $H^{n+k-1}(\mathbf{P}(\mathcal{E}) \setminus X_S, \mathbb{C})$ of a hypersurface complement in $\mathbf{P}(\mathcal{E})$ can be described explicitly in terms of the de-Rham cohomology of $\mathbf{P}(\mathcal{E})$ with poles along X_S. Based on this, one can further show that (see Theorem 1 in [4] or [8] and see [7] for the pioneering work of Griffiths in the case $k = 1$, the smooth projective hypersurface case)

$$H^{n+k-1}(\mathbf{P}(\mathcal{E}) \setminus X_S, \mathbb{C}) \xrightarrow{\sim} A_{\mathbf{P}(\mathcal{E})}/V_S \xrightarrow{\sim} \left(A_{\mathbf{P}(\mathcal{E})}/Jac(S)\right)_{ch=c_X}, \tag{1.3}$$

where V_S is the sum of the images of the endomorphisms $\frac{\partial}{\partial y_i} + \frac{\partial S}{\partial y_i}, \frac{\partial}{\partial x_j} + \frac{\partial S}{\partial x_j}$ of $A_{\mathbf{P}(\mathcal{E})}$ $(i = 1, \cdots, k, j = 0, \cdots, n)$, $Jac(S)$ is the Jacobian ideal of $S(\underline{y}, \underline{x})$, and

$$c_X := \sum_{i=1}^{k} d_i - (n+1).$$

Here the subindex $ch = c_X$ means the submodule in which the charge is c_X. Note that $Jac(S)$ is the sum of the images of the endomorphisms $\frac{\partial S}{\partial y_i}, \frac{\partial S}{\partial x_j}$ of $A_{\mathbf{P}(\mathcal{E})}$ $(i = 1, \cdots, k, j = 0, \cdots, n)$. These isomorphisms (1.3) lead us to consider the following Lie algebra representation. Let $z_1 = y_1, \cdots, z_k = y_k$ and $z_{k+1} = x_0, \cdots, z_{n+k+1} = x_n$. Let $\mathfrak{g}_{\mathbb{C}}$ be an abelian Lie algebra over $\mathbb{C}$ of dimension $n+k+1$. Let $u_1, u_2, \cdots, u_{n+k+1}$ be a $\mathbb{C}$-basis of $\mathfrak{g}_{\mathbb{C}}$. We associate a Lie algebra representation ρ on $A_{\mathbf{P}(\mathcal{E})}$ of $\mathfrak{g}_{\mathbb{C}}$ as follows:

$$\rho(u_i) := \frac{\partial}{\partial z_i} + \frac{\partial S(\underline{z})}{\partial z_i}, \text{ for } i = 1, 2, \cdots, n+k+1.$$

We extend this $\mathbb{C}$-linearly to get a Lie algebra representation $\rho : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(A_{\mathbf{P}(\mathcal{E})})$. Then the 0-th Lie algebra homology is isomorphic to $A_{\mathbf{P}(\mathcal{E})}/V_S$. Also, the $(n+k+1)$-th Lie algebra cohomology is isomorphic to $A_{\mathbf{P}(\mathcal{E})}/V_S$. In fact, the Chevalley-Eilenberg cohomology complex is the twisted de-Rham complex $(\Omega^{\bullet}_{\mathbb{A}^{n+k-1}}, d + dS)$ of the affine space $\mathbb{A}^{n+k-1}$. On the other hand, one can consider the Chevalley-Eilenberg homology complex. More precisely, we use the cochain complex $(\mathcal{A}^{\bullet}_{\rho}, K_\rho)$, which we call *the dual Chevalley-Eilenberg complex*, such that $H^i(\mathcal{A}^{\bullet}_{\rho}, K_\rho) \simeq H_{-i}(\mathfrak{g}_{\mathbb{C}}, A_{\mathbf{P}(\mathcal{E})})$ for $i \in \mathbb{Z}$:

$$\begin{aligned} \mathcal{A}^{\bullet}_{\rho} &= A_{\mathbf{P}(\mathcal{E})}[\eta_1, \eta_2, \cdots, \eta_N] = \mathbb{C}[\underline{z}][\eta_1, \eta_2 \cdots, \eta_N], \\ K_\rho &= \sum_{i=1}^{N} \left(\frac{\partial S(\underline{z})}{\partial z_i} + \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial \eta_i} : \mathcal{A}_\rho \to \mathcal{A}_\rho. \end{aligned}$$

We have

$$0 \longrightarrow \mathcal{A}_\rho^{-(n+k+1)} \xrightarrow{K_\rho} \mathcal{A}_\rho^{-(n+k)} \xrightarrow{K_\rho} \cdots \xrightarrow{K_\rho} \mathcal{A}_\rho^{-1} \xrightarrow{K_\rho} \mathcal{A}_\rho^{0} = A_{\mathbf{P}(\mathcal{E})} \longrightarrow 0$$

where

$$\mathcal{A}_\rho^{-s} = \bigoplus_{1 \le i_1 < \cdots < i_s \le n+k+1} A_{\mathbf{P}(\mathcal{E})} \cdot \eta_{i_1} \cdots \eta_{i_s}, \quad 0 \le s \le n+k+1.$$

Since one easily sees that $H^{n+k-1-s}(\Omega^{\bullet}_{\mathbb{A}^{n+k-1}}, d + dS) \xrightarrow{\sim} H^s(\mathcal{A}^{\bullet}_{\rho}, K_\rho)$ for $s \in \mathbb{Z}$ (using the fact that $\mathfrak{g}_{\mathbb{C}}$ is abelian), i.e. their differential module structures are isomorphic, either complexes can be used to study the primitive middle dimensional cohomology of X. In general, people preferred to use the twisted de-Rham complex (for example, [2] and [4]), sometimes called the algebraic Dwork complex over $\mathbb{C}$. But our key observation is that the multiplication structure (the $\mathbb{Z}$-graded commutative wedge product structure) on $(\Omega^{\bullet}_{\mathbb{A}^{n+k-1}}, d + dS)$ is quite different from the one (the $\mathbb{Z}$-graded commutative algebra structure with the rule $\eta_i \eta_j = -\eta_j \eta_i$) on $(\mathcal{A}_\rho, K_\rho)$ and, moreover, this product structure on $(\mathcal{A}_\rho, K_\rho)$ induces the homotopy Lie structure on $(\mathcal{A}_\rho, K_\rho)$ (by measuring the failure of K_ρ being a derivation of the product successively) which governs a deformation theory of cochain complexes and maps. This type of algebraic structures was studied systematically in [12] under the name "the descendant functor" by the second-named author with his collaborator. When we analyze the zeta function of X over $\mathbb{F}_q$, this descendant homotopy Lie structure will play a key role.

1.2. **The main results.** Now we come back to a smooth projective complete intersection variety X defined over $\mathbb{F}_q$. The previous subsection suggests to consider the dual Chevalley-Eilenberg complex defined over $\mathbb{F}_q$. Let $N = n + k + 1$ and

$$A := \mathbb{F}_q[z_\mu]_{\mu = 1, \cdots, N} = \mathbb{F}_q[\underline{z}]$$

where we recall $z_1 = y_1, \cdots, z_k = y_k$ and $z_{k+1} = x_0, \cdots, z_N = x_n$. We consider the Dwork potential

$$S(\underline{z}) := \sum_{\ell=1}^{k} y_\ell \cdot G_\ell(\underline{x}) \in \mathbb{F}_q[\underline{z}].$$

The same procedure as the previous subsection provides us a $\mathbb{Z}$-graded super-commutative algebra $\mathcal{A}$ with differential K_S:

$$\begin{aligned} \mathcal{A} = \mathcal{A}^\bullet &:= \mathbb{F}_q[\underline{z}][\underline{\eta}] = \mathbb{F}_q[z_1, z_2, \cdots, z_N][\eta_1, \eta_2, \cdots, \eta_N], \\ K_S &:= K_{\rho_S} = \sum_{i=1}^N \left(\frac{\partial S(\underline{z})}{\partial z_i} + \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial \eta_i} : \mathcal{A} \to \mathcal{A}. \end{aligned} \tag{1.4}$$

We also introduce the $\mathbb{F}_q$-linear endomorphism of $\mathcal{A}$:

$$Q_S := \sum_{i=1}^N \frac{\partial S(\underline{z})}{\partial z_i} \frac{\partial}{\partial \eta_i}, \quad \Delta := K_S - Q_S = \sum_{i=1}^N \frac{\partial}{\partial z_i} \frac{\partial}{\partial \eta_i}. \tag{1.5}$$

Note that Δ is a also a differential of degree 1, i.e., $\Delta^2 = 0$. Furthermore, $\Delta Q_S + Q_S \Delta = 0$ and

$$H^0(\mathcal{A}, Q_S) \xrightarrow{\sim} A/Jac(S).$$

Note that $\eta_i \cdot \eta_j = -\eta_j \cdot \eta_i$, which implies that $\eta_i^2 = 0$. Since $\frac{\partial S(\underline{y}, \underline{x})}{\partial z_i} \frac{\partial}{\partial \eta_i}$ is a differential operator of order 1, the differential Q_S is a derivation of the product of $\mathcal{A}$. Thus $(\mathcal{A}, \cdot, Q_S)$ is a cdga (commutative differential graded algebra). But K_S is *not* a derivation of the product, because the differential operator $\frac{\partial}{\partial z_i} \frac{\partial}{\partial \eta_i}$ has order 2. We define the ℓ_2-descendant of K_S with respect to the product $\cdot$ as follows:

$$\ell_2^{K_S}(a, b) := K_S(a \cdot b) - K_S(a) \cdot b - (-1)^{|a|} a \cdot K_S(b), \quad a, b \in \mathcal{A}.$$

Then $(\mathcal{A}, K, \ell_2^{K_S})$ becomes a shifted[1] dgla (differential graded Lie algebra).

Now the main question is whether one can find a p-adic lift of $(\mathcal{A}, \cdot, Q_S)$, which has a p-adic complete continuous endomorphism Φ_S as cochain map such that the characteristic polynomial of Φ_S is equal to the zeta function $P(q^k T)$ in (1.1). More precisely, can we find a $\mathbb{Z}$-graded p-adic Banach algebra $\tilde{\mathcal{A}}$ with the differential $\tilde{K}_S$ and a filtration $\{F^i \tilde{\mathcal{A}}\}_{i \in \mathbb{Z}}$ such that there is a $\mathbb{F}_q$-module isomorphism

$$\mathfrak{R} : (F^0 \tilde{\mathcal{A}} / F^1 \tilde{\mathcal{A}}, \tilde{K}_S) \xrightarrow{\sim} (\mathcal{A}, Q_S)$$

and there is a cochain endomorphism Φ_S of $(\tilde{\mathcal{A}}, \tilde{K}_S)$ which is p-adic completely continuous and whose characteristic polynomial is equal to $P(q^k T)$? This question was essentially answered by Dwork and his successors (notably, Adolphson and Sperber). There are two technical difficulties for achieving this. As is well-known, the differential operators like $\frac{\partial}{\partial z_i}$ behave badly in characteristic p. Even in the p-adic case, the differential operators behaves differently from the complex analytic case: the Poincare lemma fails if one considers p-adic analytic functions on the closed unit disc. Another difficulty is the p-adic convergence problem of the exponential function. In the complex analytic case, the radius of convergence of the exponential function is infinity, but in the p-adic case, the radius of convergence is $p^{-\frac{1}{p-1}} < 1$. All of these difficulties were resolved by Dwork by introducing an overconvergent module and the splitting function. Here we follow the version of Adolphson and Sperber, [2]. Our academic contribution is to change the product structure on the p-adic twisted de-Rham complex in [2] (by using the p-adic dual Chevalley-Eilenberg complex) in order to reveal the homotopy Lie structure, which put us in a natural framework of modern deformation theory.[2]

Let ζ_p be a primitive p-th root of unity in $\mathbb{C}_p$, where $\mathbb{C}_p$ is the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. For the p-adic overconvergent module, we fix a rational number $b \in \mathbb{Q}$ such that

$$\frac{1}{p-1} < b < \frac{p}{p-1} \tag{1.6}$$

and choose $M \in \mathbb{N}$ such that $\frac{Mb}{(p-1)p}, \frac{M}{p-1} \in \mathbb{Z}$.[3] Then we choose $\pi \in \mathbb{C}_p$ such that

$$\pi^M = p. \tag{1.7}$$

[1]We call it "shifted" because both K and $\ell_2^{K_S}$ has degree 1. The (usual) dgla has the Lie bracket degree 0 and the differential degree 1.

[2]There is a motto that every deformation problem in characteristic zero can be controlled by the Maurer-Cartan equation of some homotopy Lie algebra.

[3]These technical conditions on b and M are used to prove the statement (b) of Theorem 1.1

Denote $\mathbb{Q}_q$ be the fraction field of $\mathbb{Z}_q = W(\mathbb{F}_q)$, the ring of Witt vectors of $\mathbb{F}_q$. Let $\Bbbk = \mathbb{Q}_q(\zeta_p, \pi)$ be the smallest subfield of $\mathbb{C}_p$ containing ζ_p, π, and $\mathbb{Q}_q$. Denote by $\mathcal{O}_\Bbbk$ the ring of integers of $\Bbbk$.

Theorem 1.1. (a) There is an L_∞-algebra $(\widetilde{\mathcal{A}(b)}, \underline{\ell}^{\tilde{K}_S} = \ell_1^{\tilde{K}_S}, \ell_2^{\tilde{K}_S}, \cdots)$ and an L_∞-endomorphism $\underline{\phi}^{\Psi_S} = \phi_1^{\Psi_S}, \phi_2^{\Psi_S}, \cdots$ of $(\widetilde{\mathcal{A}(b)}, \underline{\ell}^{\tilde{K}_S})$ such that[4] there is a decomposition $\widetilde{\mathcal{A}(b)} = \bigoplus_{-N \le m \le 0} \widetilde{\mathcal{A}(b)}^m$ where $\widetilde{\mathcal{A}(b)}^0$ is a p-adic Banach commutative algebra over $\Bbbk$ and

$$\widetilde{\mathcal{A}(b)}^{-s} = \bigoplus_{1 \le i_1 < \cdots < i_s \le N} \widetilde{\mathcal{A}(b)}^0 \cdot \eta_{i_1} \cdots \eta_{i_s}, \quad 0 \le s \le N.$$

(b) There exists a separated and exhaustive decreasing filtration $\{F^i \widetilde{\mathcal{A}(b)}\}_{i \in \mathbb{Z}}$ consisting of $\mathcal{O}_\Bbbk$-submodules such that $(\widetilde{\mathcal{A}(b)}, \tilde{K}_S)$ is a filtered complex, and there is a $\mathbb{F}_q$-module cochain isomorphism

$$\mathcal{R} : (F^0\widetilde{\mathcal{A}(b)}/F^1\widetilde{\mathcal{A}(b)}, \tilde{K}_S) \xrightarrow{\sim} (\mathcal{A}, Q_S).$$

(c) The 0-th cohomology $H^0(\widetilde{\mathcal{A}(b)}, \tilde{K}_S)$ where $\tilde{K}_S = \ell_1^{\tilde{K}_S}$ is a finite dimensional $\Bbbk$-vector space whose dimension is equal to the degree of $P(T)$.

(d) There is a $\Bbbk$-linear completely continuous operator $\phi_1^{\Psi_S} = \Psi_S : (\widetilde{\mathcal{A}(b)}, \tilde{K}_S) \to (\widetilde{\mathcal{A}(b)}, \tilde{K}_S)$ which is a cochain map and

$$P(q^k T) = \det(1 - T \cdot \Psi_S \big| H^0(\widetilde{\mathcal{A}(b)}, \tilde{K}_S)). \tag{1.8}$$

We will also use the following notation

$$\tilde{K}_X = \tilde{K}_S, \quad \Psi_X = \Psi_S$$

to emphasize the dependence on the smooth projective complete intersection variety X. We fix a p-adic absolute value $|\cdot|_p$ and a p-adic valuation val_p on $\mathbb{C}_p$ such that $\mathrm{val}_p(p) = 1$ and $|x|_p = p^{-\mathrm{val}_p(x)}$. For $\underline{u} = (u_1, ..., u_m) \in \mathbb{R}^m$, we put $|\underline{u}| = u_1 + \cdots + u_m$. Let $\mathbb{Z}_{\ge 0}$ be the set of non-negative integers. For each b satisfying (1.6), the p-adic Banach algebra $\widetilde{\mathcal{A}(b)}^0 = \widetilde{A(b)}$ is given as follows:

$$\widetilde{A(b)} = \left\{ \xi(\underline{z}) = \sum_{(\underline{u},\underline{v}) \in \mathbb{Z}_{\ge 0}^N} a_{\underline{u},\underline{v}} \pi^{Mb|\underline{v}|} \underline{x}^{\underline{u}} \underline{y}^{\underline{v}} : a_{\underline{u},\underline{v}} \in \Bbbk \text{ and } a_{\underline{u},\underline{v}} \to 0 \text{ as } |(\underline{u},\underline{v})| \to \infty \right\} \tag{1.9}$$

where $\underline{y}^{\underline{v}} = (y_1^{v_1}, \cdots, y_k^{v_k})$ and $\underline{x}^{\underline{u}} = (x_0^{u_0}, \cdots, x_n^{u_n})$. Here the overconvergent factor $\pi^{Mb|\underline{v}|}$ is specifically designed (in [2]) to prove the Katz conjecture: $T_q \left(F^i \widetilde{A(b)} \right) \subseteq F^i \widetilde{A(bq)}$ where T_q is given (1.11). The p-adic Banach structure on $\widetilde{A(b)}$ is given by $|\xi(\underline{z})|_p = \sup_{(\underline{u},\underline{v})} |a_{\underline{u},\underline{v}}|_p$.

The L_∞-algebra and L_∞-morphism in Theorem 1.1 are obtained by applying the descendant functor, introduced in Definition 2.7, to the cochain endomorphism Ψ_S of $(\widetilde{\mathcal{A}(b)}, \tilde{K}_S)$. Using this homotopy Lie formalism, we now explain how to derive an L_∞-homotopy formula for Ψ_S. Let $\gamma \in \mathbb{Q}_q(\zeta_p) \subset \Bbbk$ be a solution of $\sum_{n=0}^\infty \frac{t^{p^n}}{p^n} = 0$ such that $\mathrm{val}_p(\gamma) = \frac{1}{p-1}$. If we define[5]

$$\tilde{\Delta} := \frac{\pi^{Mb}}{\gamma} \cdot \sum_{i=1}^N \frac{\partial}{\partial z_i} \frac{\partial}{\partial \eta_i}, \tag{1.10}$$

then $(\widetilde{\mathcal{A}(b)}, \cdot, \tilde{\Delta})$ is clearly a cochain complex with super-commutative product structure. In fact, $\frac{\partial}{\partial y_i}$ improves the p-adic convergence of $\widetilde{A(b)}$:

$$\frac{\partial}{\partial y_i}(\xi) \in \mathcal{A}\widetilde{(b+\epsilon)}, \ \epsilon > 0 \quad \text{for } \xi = \sum_{(\underline{u},\underline{v}) \in \mathbb{Z}_{\ge 0}^N} a_{\underline{u},\underline{v}} \pi^{Mb|\underline{v}|} \underline{x}^{\underline{u}} \underline{y}^{\underline{v}} \in \widetilde{A(b)},$$

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[4] An L_∞-algebra (homotopy Lie algebra) $(V, \underline{\ell})$ is a $\mathbb{Z}$-graded vector space V with an L_∞-structure $\underline{\ell} = \ell_1, \ell_2, \ell_3, \cdots$, where ℓ_1 is a differential such that (V, ℓ_1) is a cochain complex, ℓ_2 is a graded Lie bracket which satisfies the graded Jacobi identity up to homotopy ℓ_3 etc. An L_∞-morphism $\underline{\phi} = \phi_1, \phi_2, \cdots$ is a morphism between L_∞-algebras, say $(V, \underline{\ell})$ and $(V', \underline{\ell}')$, such that ϕ_1 is a cochain map of the underlying cochain complex, which is a Lie algebra homomorphism up to homotopy ϕ_2, etc. See subsection 2.1.

[5] The auxiliary factor $\frac{\pi^{Mb}}{\gamma}$ is a technical condition needed to prove (b) of Theorem 1.1

and $\frac{\partial}{\partial x_i}$ preserves the (radius of) p-adic convergence of $\widetilde{A(b)}$. Let us consider the Dwork operator $T_q : \widetilde{A(b)} \to \widetilde{A(b)}$ defined by

$$T_q\left(\sum_{\underline{w}\in\mathbb{Z}^N_{\geq 0}} a_{\underline{w}}\underline{z}^{\underline{w}}\right) = \sum_{\underline{w}\in\mathbb{Z}^N_{\geq 0}} a_{q\underline{w}}\underline{z}^{\underline{w}}. \tag{1.11}$$

Note that the image of T_q belongs to $\widetilde{A(bq)} \subset \widetilde{A(b)}$ (the operator T_q also improves the p-adic convergence like $\frac{\partial}{\partial y_i}$). By using the relation

$$T_q \circ z_i\frac{\partial}{\partial z_i} = qz_i\frac{\partial}{\partial z_i}\circ T_q,$$

one can easily cook up a cochain endomorphism of $(\widetilde{\mathcal{A}(b)}, \tilde{\Delta})$: If we define a $\Bbbk$-linear endomorphism $\Psi_{\mathbb{P}^n}$ on $\widetilde{\mathcal{A}(b)} = \bigoplus_{-N\leq m\leq 0}\widetilde{\mathcal{A}(b)}^m$ by additivity and the following formula

$$\Psi^{-m}_{\mathbf{P}^n}(\xi(\underline{z})\cdot\eta_{i_1}\cdots\eta_{i_m}) := q^m\frac{T_q\left(\xi(\underline{z})\cdot\prod_{\substack{j\in\{1,\cdots,N\}\\ j\neq i_1,\cdots,i_m}} z_j\right)}{\prod_{\substack{j\in\{1,\cdots,N\}\\ j\neq i_1,\cdots,i_m}} z_j}\cdot\eta_{i_1}\cdots\eta_{i_m}, \tag{1.12}$$

for each $m\geq 0$ and $\xi(\underline{z}) \in \mathcal{A}(b) = \widetilde{\mathcal{A}(b)}^0$, then $\Psi_{\mathbf{P}^n} : (\widetilde{\mathcal{A}(b)}, \tilde{\Delta}) \to (\widetilde{\mathcal{A}(b)}, \tilde{\Delta})$ is a cochain map. Then we apply the descendant functor to $\Psi_{\mathbf{P}^n}$ to obtain an L_∞-endomorphism $\underline{\phi}^{\Psi_{\mathbf{P}^n}} = \phi_1^{\Psi_{\mathbf{P}^n}}, \phi_2^{\Psi_{\mathbf{P}^n}}, \cdots$ of an L_∞-algebra $(\mathcal{A}(b), \underline{\ell}^{\tilde{\Delta}} = \ell_1^{\tilde{\Delta}}, \ell_2^{\tilde{\Delta}}, \cdots)$. Explicit formulas can be given as follows: Define $\ell_1^{\tilde{\Delta}} = \tilde{\Delta}$ and

$$\begin{aligned}&\ell_n^{\tilde{\Delta}}(x_1,\cdots,x_{n-1},x_n) = \ell_{n-1}^{\tilde{\Delta}}(x_1,\cdots,x_{n-2},x_{n-1}\cdot x_n)\\ &-\ell_{n-1}^{\tilde{\Delta}}(x_1,\cdots,x_{n-1})\cdot x_n - (-1)^{|x_{n-1}|(1+|x_1|+\cdots+|x_{n-2}|)}x_{n-1}\cdot\ell_{n-1}^{\tilde{\Delta}}(x_1,\cdots,x_{n-2},x_n),\quad n\geq 2.\end{aligned}$$

It can be easily checked that $\ell_m^{\tilde{\Delta}} = 0$ for $m \geq 3$. Define $\phi_1^{\Psi_{\mathbf{P}^n}} = \Psi_{\mathbf{P}^n}$ and

$$\phi_m^{\Psi_{\mathbf{P}^n}}(x_1,\cdots,x_m) = \phi_{m-1}^{\Psi_{\mathbf{P}^n}}(x_1,\cdots,x_{m-2},x_{m-1}\cdot x_m) - \sum_{\substack{\pi\in P(m),|\pi|=2\\ m-1\nsim_\pi m}}\phi^{\Psi_{\mathbf{P}^n}}(x_{B_1})\cdot\phi^{\Psi_{\mathbf{P}^n}}(x_{B_2}),\quad m\geq 2,$$

where $P(m)$ is the set of partitions of $\{1,\cdots,m\}$ and we refer to Definition 2.6 for details.

The deformation theory based on the Maurer-Cartan equation attached to $(\mathcal{A}(b), \cdot, \widetilde{\Delta})$ naturally leads to the following formula. Let $\tilde{F}(\underline{z})$ be the Teichmüller lifting of the polynomial $F(\underline{z}) \in \mathbb{F}_q[\underline{z}]$. In other words, if we write $F(\underline{z}) = \sum_{\underline{w}\in\mathbb{Z}^N_{\geq 0}} f_{\underline{w}}\underline{z}^{\underline{w}}$, then we have $\tilde{F}(\underline{z}) = \sum_{\underline{w}\in\mathbb{Z}^N_{\geq 0}} \tilde{f}_{\underline{w}}\underline{z}^{\underline{w}} \in \mathbb{Z}_q[\underline{z}]$ where $(\tilde{f}_{\underline{w}})^q = \tilde{f}_{\underline{w}}$ and $\tilde{f}_{\underline{w}} \equiv f_{\underline{w}} (\mathrm{mod}\, p)$.

Theorem 1.2. Suppose the Teichmüller lifting of each $G_i(\underline{x})$ is written as

$$\tilde{G}_i(\underline{x}) = \sum_{\underline{u}\in\mathbb{Z}^{n+1}_{\geq 0}} \tilde{g}_{i,\underline{u}}\underline{x}^{\underline{u}}$$

and define

$$\tilde{\Gamma} := \sum_{i,\underline{u}}\sum_{\ell\geq 0}\gamma_\ell\tilde{g}_{i,\underline{u}}^{p^\ell}y_i^{p^\ell}\underline{x}^{p^\ell\underline{u}}\quad (\gamma_\ell := \sum_{i=0}^{\ell}\frac{\gamma^{p^i}}{p^i}),\qquad \Gamma = \sum_{j=0}^{a-1}\sum_{i,\underline{u}}\sum_{\ell\geq 0}\frac{\gamma^{p^\ell}\left(\tilde{g}_{i,\underline{u}}y_i\underline{x}^{\underline{u}}\right)^{p^{j+\ell}}}{p^\ell}.$$

For any homogeneous $\lambda \in \widetilde{\mathcal{A}(b)}$, we have

$$\begin{aligned}&\tilde{K}_S(\lambda) = \tilde{\Delta}(\lambda) + \sum_{m\geq 1}\frac{1}{m!}\ell_{m+1}^{\tilde{\Delta}}(\tilde{\Gamma},\cdots,\tilde{\Gamma},\lambda) = \tilde{\Delta}(\lambda) + \ell_2^{\tilde{\Delta}}(\tilde{\Gamma},\lambda),\\ &\Psi_S(\lambda) = \Psi_{\mathbf{P}^n}(\lambda) + \sum_{m\geq 1}\sum_{\substack{j+k=m\\ j,k\geq 0}}\frac{1}{j!k!}B_j(\phi_1^{\Psi_{\mathbf{P}^n}}(\Gamma),\cdots,\phi_j^{\Psi_{\mathbf{P}^n}}(\Gamma,\cdots,\Gamma))\cdot\phi_{k+1}^{\Psi_{\mathbf{P}^n}}(\Gamma,\cdots,\Gamma,\lambda),\end{aligned}$$

where $B_n(x_1,\cdots,x_n)$ is the Bell polynomial defined in (3.8), and $\phi_1^{\Psi_X}(\Gamma,\lambda) := \phi_1^{\Psi_{\mathbf{P}^n}}(\lambda) = \Psi_{\mathbf{P}^n}(\lambda)$.

The above homotopy Lie formula for $\tilde{K}_S$ is not new but the formula for Ψ_S which naturally arises paired with $\tilde{K}_S$ from the deformation theory based on the L_∞-algebra appears to be new. A merit of this formalism is that any smooth complete intersection $X = X_{\underline{G}} \subseteq \mathbf{P}^n_{\mathbb{F}_q}$ can be regarded as one deformed from the projective space $\mathbf{P}^n_{\mathbb{F}_q}$. Hence we may prove many properties of $(\widetilde{\mathcal{A}(b)}, \tilde{K}_S)$ by checking it on the corresponding properties on the projective space and transporting it to X via the deformation. It would be a good project to see whether this formula for Ψ_S is actually helpful for an algorithmic computation of the zeta function.

Now we briefly explain the contents of each section. The section 2 is about a general theory of homotopy Lie algebras and their (formal) deformation theory and the section 3 explains the consequence of applying the general theory to a smooth projective complete intersection variety X over $\mathbb{F}_q$. The subsection 2.1 is devoted to a brief explanation of the category of homotopy Lie algebras. Then we explain the descendant functor (our key technical tool to analyze the zeta function of X) and its explicit relationship to the deformation theory of descendant homotopy Lie algebras in the subsection 2.2. In the subsection 2.3, we explain how to modify the usual deformation theory so that the deformation of the p-adic Dwork Frobenius operator makes sense later.

In the section 3, we provide detailed proofs of Theorems 1.1 and 1.2. For this, in the subsection 3.1, we construct a dgla, (more precisely, a GBV(Gerstenhaber-Batalin-Vilkovisky) algebra) for a projective space which serves as a base point of the deformation theory. Then, in the subsection 3.2, we study a filtration structure on such GBV algebra to prepare for the proof of (b) of Theorem 1.1. The subsection 3.3 is devoted to the proof of $(a), (b)$ of Theorem 1.1. The subsections 3.4 and 3.5 are devoted to the proofs of (c) and (d) of Theorem 1.1, respectively. In the subsection 3.6, we briefly review the Bell polynomials and prove Theorem 1.2 by applying all the machineries developed so far.

Finally, in the section 4, we add an appendix which explains how to compute all the cohomologies of the dgla $(\mathcal{A}, K_S, \ell_2^{K_S})$ over $\mathbb{F}_q$.

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2. Formal deformation theory of the cochain complex with multiplication

2.1. **Homotopy Lie algebra and the BV algebra.** Roughly speaking, a homotopy Lie algebra (L_∞-algebra) is a "differential Lie algebra up to homotopy". We refer to section 13.2, [9] for the precise theoretical definition (as an algebra over a particular algebraic operad Lie) and its basic properties. Here we only briefly review an explicit description following the appendix 5.2, [12].

Let k be a field of characteristic zero (we need this since we have to divide $m!$ in the definition of homotopy Lie algebras). Let $\mathbf{Art}_k^{\mathbb{Z}}$ denote the category of $\mathbb{Z}$-graded artinian local k-algebras with residue field k and $\widehat{\mathbf{Art}}_k^{\mathbb{Z}}$ be the category of complete $\mathbb{Z}$-graded noetherian local k-algebras. For $\mathfrak{a} \in \mathrm{Ob}(\mathbf{Art}_k^{\mathbb{Z}})$, $\mathfrak{m}_\mathfrak{a}$ denotes the maximal ideal of $\mathfrak{a}$ which is a nilpotent $\mathbb{Z}$-graded super-commutative and associative k-algebra without unit. Let $V = \bigoplus_{i\in\mathbb{Z}} V^i$ be a $\mathbb{Z}$-graded vector space over k. If $x \in V^i$, we say that x is a homogeneous element of degree i; let $|x|$ be the degree of a homogeneous element of V. For each $n \geq 1$ let $S(V) = \bigoplus_{n=0}^\infty S^n(V)$ be the free $\mathbb{Z}$-graded super-commutative and associative algebra over k generated by V, which is the quotient algebra of the free tensor algebra $T(V) = \bigoplus_{n=0}^\infty T^n(V)$ by the ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$. Here $T^0(V) = k$ and $T^n(V) = V^{\otimes n}$ for $n \geq 1$.

Definition 2.1 (L_∞-algebra)**.** The triple $V_L = (V, \underline{\ell}, 1_V)$ is a unital L_∞-algebra over k if $1_V \in V^0$ and $\underline{\ell} = \ell_1, \ell_2, \cdots$ be a family of k-linear maps such that

- $\ell_n \in \mathrm{Hom}(S^n(V), V)^1$ for all $n \geq 1$.
- $\ell_n(v_1, \cdots, v_{n-1}, 1_V) = 0$, for all $v_1, \cdots, v_{n-1} \in V$, $n \geq 1$.

- for any $\mathfrak{a} \in \mathrm{Ob}\left(\mathbf{Art}_k^{\mathbb{Z}}\right)$ and for all $n \geq 1$

$$\sum_{k=1}^{n} \frac{1}{(n-k)!k!} \ell_{n-k+1}\left(\ell_k\left(\gamma, \cdots, \gamma\right), \gamma, \cdots, \gamma\right) = 0,$$

whenever $\gamma \in (\mathfrak{m}_{\mathfrak{a}} \otimes V)^0$, where

$$\begin{aligned}&\ell_n\big(a_1 \otimes v_1, \cdots, a_n \otimes v_n\big)\\ &= (-1)^{|a_1|+|a_2|(1+|v_1|)+\cdots+|a_n|(1+|v_1|+\cdots+|v_{n-1}|)} a_1 \cdots a_n \otimes \ell_n\left(v_1, \cdots, v_n\right).\end{aligned}$$

Definition 2.2 (L_∞-morphism)**.** A morphism of unital L_∞-algebras from V_L into V'_L is a family $\underline{\phi} = \phi_1, \phi_2, \cdots$ such that

- $\phi_n \in \mathrm{Hom}(S^n V, V')^0$ for all $n \geq 1$.
- $\phi_1(1_V) = 1_{V'}$ and $\phi_n(v_1, \cdots, v_{n-1}, 1_V) = 0$, $v_1, \cdots, v_{n-1} \in V$, for all $n \geq 2$.
- for any $\mathfrak{a} \in \mathrm{Ob}\left(\mathbf{Art}_k^{\mathbb{Z}}\right)$ and for all $n \geq 1$

$$\begin{aligned}&\sum_{j_1+j_2=n} \frac{1}{j_1! j_2!} \phi_{j_1+1}\left(\ell_{j_2}(\gamma, \cdots, \gamma), \gamma, \cdots, \gamma\right)\\ &= \sum_{j_1+\cdots+j_r=n} \frac{1}{r!} \frac{1}{j_1! \cdots j_r!} \ell'_r\left(\phi_{j_1}(\gamma, \cdots, \gamma), \cdots, \phi_{j_r}(\gamma, \cdots, \gamma)\right),\end{aligned}$$

whenever $\gamma \in (\mathfrak{m}_{\mathfrak{a}} \otimes V)^0$, where

$$\begin{aligned}&\phi_n\big(a_1 \otimes v_1, \cdots, a_n \otimes v_n\big)\\ &= (-1)^{|a_2||v_1|+\cdots+|a_n|(|v_1|+\cdots+|v_{n-1}|)} a_1 \cdots a_n \otimes \phi_n\left(v_1, \cdots, v_n\right).\end{aligned}$$

If we forget the unity 1_V, then we call a pair $(V, \underline{\ell})$ an L_∞-algebra. We can similarly define an L_∞-morphism without the condition on 1_V. One can define the composition of L_∞-morphism and it can be checked that unital L_∞-algebras over k and L_∞-morphisms form a category.

Definition 2.3. The cohomology H of the L_∞-algebra $(V, \underline{\ell})$ is the cohomology of the underlying complex $(V, K = \ell_1)$. An L_∞-morphism $\underline{\phi}$ is an L_∞-quasi-isomorphism if ϕ_1 induces an isomorphism on cohomology.

Definition 2.4. An L_∞-algebra $(V, \underline{\ell})$ is called a (shifted) dgla(differential graded Lie algebra) if $\ell_m = 0$ for $m \geq 3$. This means that (V, ℓ_1) is a cochain complex (ℓ_1 has degree 1), i.e. $\ell_1 \circ \ell_1 = 0$ and (V, ℓ_2) is a graded Lie bracket (ℓ_2 has also degree 1), i.e.

$$\begin{aligned}\ell_2(x_1, x_2) &= (-1)^{|x_1||x_2|}\ell_2(x_2, x_1)\\ 0 &= \ell_2(\ell_2(x_1, x_2), x_3) + (-1)^{|x_1|}\ell_2(x_1, \ell_2(x_2, x_3)) + (-1)^{(|x_1|+1)|x_2|}\ell_2(x_2, \ell_2(x_1, x_3)),\end{aligned}$$

and ℓ_1 is a graded derivation of the Lie bracket

$$\ell_1\left(\ell_2(x_1, x_2)\right) = -\ell_2(\ell_1(x_1), x_2) + (-1)^{|x_1|+1}\ell_2(x_1, \ell_1(x_2)).$$

In fact, any L_∞-algebra can be strictified to give a dgla, i.e. any L_∞-algebra is L_∞-quasi-isomorphic to a dgla. Now we give definitions of G-algebra, GBV algebra, and dGBV algebra, slight variations of a dgla, which are suitable for our analysis on the p-adic Dwork complex; In Theorem 1.1, we only mentioned L_∞-algebras and L_∞-morphisms for simplicity, but we will discover more, i.e. a dGBV algebra structure on the p-adic Dwork complex in addition to the L_∞-structure.

Definition 2.5. *Let k be a field. Let $(\mathcal{C}, \cdot)$ be a unital $\mathbb{Z}$-graded super-commutative and associative k-algebra. Let $[\bullet, \bullet] : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ be a bilinear map of degree 1.*

(a) $(\mathcal{C}, \cdot, [\cdot, \cdot])$ is called a G-algebra (Gerstenhaber algebra) over k if

$$\begin{aligned}[a, b] &= (-1)^{|a||b|}[b, a],\\ [a, [b, c]] &= (-1)^{|a|+1}[[a, b], c] + (-1)^{(|a|+1)(|b|+1)}[b, [a, c]],\\ [a, b \cdot c] &= [a, b] \cdot c + (-1)^{(|a|+1)\cdot|b|} b \cdot [a, c],\end{aligned}$$

for any homogeneous elements $a, b, c \in \mathcal{C}$.

(b) $(\mathcal{C}, \cdot, K, \ell_2^K)$ *is called a GBV(Gerstenhaber-Batalin-Vilkovisky)-algebra*[6] *over* k *where*

$$\ell_2^K(a,b) := K(a\cdot b) - K(a)\cdot b - (-1)^{|a|} a \cdot K(b), \quad a, b \in \mathcal{C}, \tag{2.1}$$

if $(\mathcal{C}, K, \ell_2^K)$ *is a (shifted) dgla and* $(\mathcal{C}, \cdot, \ell_2^K)$ *is a G-algebra,*

(c) $(\mathcal{C}, \cdot, K, \ell_2^K, Q)$*, where* $Q : \mathcal{C} \to \mathcal{C}$ *is a linear map of degree 1, is called a dGBV(differential Gerstenhaber-Batalin-Vilkovisky) algebra if* $(\mathcal{C}, \cdot, K, \ell_2^K(\cdot,\cdot))$ *is a GBV algebra and* $(\mathcal{C}, \cdot, Q)$ *is a cdga(commutative differential graded algebra), i.e.*

$$Q^2 = 0, \quad Q(a\cdot b) = Q(a)\cdot b + (-1)^{|a|} a \cdot Q(b), \quad a, b \in \mathcal{C}.$$

2.2. **The descendant functor and a formal deformation theory.** In this subsection, we will study the deformations of the data $(\mathcal{C}, \cdot, K, \Psi)$ where

(1) $(\mathcal{C}, \cdot)$ is a $\mathbb{Z}$-graded super-commutative associative algebra algebra over k,
(2) $(\mathcal{C}, K)$ is a cochain complex over k, and
(3) $\Psi : \mathcal{C} \to \mathcal{C}$ is a k-linear cochain map.

The category $\mathfrak{C}_k$ is defined such that objects are triples $(\mathcal{C}, \cdot, K)$, where the pair $(\mathcal{C}, \cdot)$ is a $\mathbb{Z}$-graded super-commutative associative k-algebra while the pair $(\mathcal{C}, K)$ is a cochain complex over k, and morphisms are cochain maps. Two of the salient properties of the category $\mathfrak{C}_k$ are that

(i) morphisms are not required to be algebra homomorphisms,
(ii) the differential and multiplication in an object have no compatibility condition.

If we denote by $\mathfrak{C}_k$ the category of triples $(\mathcal{C}, \cdot, K)$ satisfying (1) and (2) above with (k-linear) cochain maps, then the given data naturally sit in the category whose objects are endomorphisms in $\mathfrak{C}_k$ and morphisms are the ones in $\mathfrak{C}_k$ commuting both endomorphisms. The deformation theory of objects in $\mathfrak{C}_k$ was studied in [12]. We briefly review it here.

Definition 2.6. A *partition* $\pi = B_1 \cup B_2 \cup \cdots$ of the set $[n] = \{1, 2, \cdots, n\}$ is a decomposition of $[n]$ into a pairwise disjoint non-empty subsets B_i, called *blocks*. Blocks are ordered by the minimum element of each block and each block is ordered by the ordering induced from the ordering of natural numbers. The notation $|\pi|$ means the number of blocks in a partition π and $|B|$ means the size of the block B. If k and k' belong to the same block in π, then we use the notation $k \sim_\pi k'$. Otherwise, we use $k \nsim_\pi k'$. Let $P(n)$ be the set of all partitions of $[n]$.

Definition 2.7. For a given object $(\mathcal{C}, \cdot, K)$ in $\mathfrak{C}_k$, we define $\mathfrak{Des}(\mathcal{C}, \cdot, K) = (\mathcal{C}, \underline{\ell}^K)$, where $\underline{\ell}^K = \ell_1^K, \ell_2^K, \cdots$ is the family of linear maps $\ell_n^K : S^n(\mathcal{A}) \to \mathcal{A}$, inductively defined by the formula: $\ell_1^K = K$ and

$$\begin{aligned}
&\ell_n^K(x_1, \cdots, x_{n-1}, x_n) = \ell_{n-1}^K(x_1, \cdots, x_{n-2}, x_{n-1}\cdot x_n) \\
&-\ell_{n-1}^K(x_1, \cdots, x_{n-1})\cdot x_n - (-1)^{|x_{n-1}|(1+|x_1|+\cdots+|x_{n-2}|)} x_{n-1}\cdot \ell_{n-1}^K(x_1, \cdots, x_{n-2}, x_n), \quad n \geq 2,
\end{aligned}$$

for any homogeneous elements $x_1, x_2, \cdots, x_n \in \mathcal{C}$.

For a given morphism $f : (\mathcal{C}, \cdot, K) \to (\mathcal{C}', \cdot, K')$ in $\mathfrak{C}_k$, we define $\mathfrak{Des}(f) = \underline{\phi}^f = \phi_1^f, \phi_2^f, \cdots$ as a family of k-linear maps $\phi_n^f : S^n(\mathcal{C}) \to \mathcal{C}'$ defined inductively by the formula: $\phi_1^f = f$ and

$$\phi_m^f(x_1, \cdots, x_m) = \phi_{m-1}^f(x_1, \cdots, x_{m-2}, x_{m-1}\cdot x_m) - \sum_{\substack{\pi \in P(m), |\pi|=2 \\ m-1 \nsim_\pi m}} \phi^f(x_{B_1})\cdot \phi^f(x_{B_2}), \quad m \geq 2,$$

for any homogeneous elements $x_1, x_2, \cdots, x_m \in \mathcal{C}$. Here we use the following notation:

$$\begin{aligned}
x_B &= x_{j_1} \otimes \cdots \otimes x_{j_r} \text{ if } B = \{j_1, \cdots, j_r\}, \\
\phi^f(x_B) &= \phi_r^f(x_{j_1}, \cdots, x_{j_r}) \text{ if } B = \{j_1, \cdots, j_r\}.
\end{aligned}$$

Proposition 2.8. Let $\mathfrak{L}_k$ be the category of L_∞-algebras. The above assignment $\mathfrak{Des}$ is a functor from $\mathfrak{C}_k$ to $\mathfrak{L}_k$.

Proof. See subsection 3.2 in [12]. □

[6] $(\mathcal{C}, \cdot, K)$ is called a BV algebra, if $(\mathcal{C}, \cdot, K, \ell_2^K)$ is a GBV algebra

We will call $\underline{\ell}^K$ and $\underline{\phi}^f$ a *descendant* L_∞-algebra and morphism respectively. In short, the functor $\mathfrak{Des} : \mathfrak{C}_k \to \mathfrak{L}_k$ takes

(i) an object $(\mathcal{C}, \cdot, K)$ in $\mathfrak{C}_k$ to a descendant L_∞-algebra $(\mathcal{C}, \underline{\ell}^K = \ell_1^K, \ell_2^K, \ell_3^K, \cdots)$, where $\ell_1^K = K$ and $\ell_2^K, \ell_3^K, \cdots$ measure the failure and higher failures of K being a derivation of the multiplication in $\mathcal{C}$,

(ii) a morphism f in $\mathfrak{C}_k$ to a descendant L_∞-morphism $\underline{\phi}^f = \phi_1^f, \phi_2^f, \phi_3^f, \cdots$, where $\phi_1^f = f$ and $\phi_2^f, \phi_3^f, \cdots$ measure the failure and higher failures of f being an algebra homomorphism.

For $\mathfrak{a} \in \mathbf{Art}_k^{\mathbb{Z}}$ denote $\mathfrak{m}_{\mathfrak{a}}$ its maximal ideal. In what follows we endow $\mathfrak{a} \otimes \mathcal{C}$ the natural $\mathbb{Z}$-grading. We need the following lemmas to see a precise relationship between a formal deformation theory for $\mathfrak{C}_k$ (based on the Maurer-Cartan equation) and the descendant functor $\mathfrak{Des}$.

Lemma 2.9. For $\Gamma \in (\mathfrak{m}_{\mathfrak{a}} \otimes \mathcal{C})^0$ and homogeneous $\lambda \in \mathfrak{a} \otimes \mathcal{C}$, denote

$$L^K(\Gamma) := \sum_{n \geq 1} \frac{1}{n!} \ell_n^K(\Gamma, \cdots, \Gamma), \quad L_\Gamma^K(\lambda) := K\lambda + \sum_{n \geq 2} \frac{1}{(n-1)!} \ell_n^K(\Gamma, \cdots, \Gamma, \lambda).$$

Then we have the following identities:

$$K(e^\Gamma - 1) = L^K(\Gamma) \cdot e^\Gamma, \quad K(\lambda \cdot e^\Gamma) = L_\Gamma^K(\lambda) \cdot e^\Gamma + (-1)^{|\lambda|} \lambda \cdot K(e^\Gamma - 1).$$

Proof. See Lemma 3.1, [12]. □

Lemma 2.10. For $\Gamma \in (\mathfrak{m}_{\mathfrak{a}} \otimes \mathcal{C})^0$ and homogeneous $\lambda \in \mathfrak{a} \otimes \mathcal{C}$, denote

$$\Phi^f(\Gamma) := \sum_{n \geq 1} \frac{1}{n!} \phi_n^f(\Gamma, \cdots, \Gamma), \quad \Phi_\Gamma^f(\lambda) := \phi_1^f(\lambda) + \sum_{n \geq 2} \frac{1}{(n-1)!} \phi_n^f(\Gamma, \cdots, \Gamma, \lambda).$$

Then we have the following identities:

$$f(e^\Gamma - 1) = e^{\Phi^f(\Gamma)} - 1, \quad f(\lambda \cdot e^\Gamma) = \Phi_\Gamma^f(\lambda) \cdot e^{\Phi^f(\Gamma)}.$$

Proof. See Lemma 3.3, [12]. □

The identities in lemma 2.9 say that if $\Gamma \in (\mathfrak{m}_{\mathfrak{a}} \otimes \mathcal{C})^0$ satisfies the Maurer-Cartan equation:

$$K(e^\Gamma - 1) = 0 \iff L^K(\Gamma) = \sum_{n \geq 1} \frac{1}{n!} \ell_n^K(\Gamma, \cdots, \Gamma) = 0$$

then we can equate

$$L_\Gamma^K = e^{-\Gamma} \circ K \circ e^\Gamma$$

so $K_\Gamma := L_\Gamma^K$ becomes a differential on $\mathfrak{a} \otimes \mathcal{C}$, i.e., $(\mathfrak{a} \otimes \mathcal{C}, \cdot, K_\Gamma)$ is again an object of $\mathfrak{C}_k$, which is a (formal) deformation of $(\mathcal{C}, \cdot, K)$ by a Maurer-Cartan solution Γ.

Now we deform a cochain map $f : (\mathcal{C}, \cdot, K) \to (\mathcal{C}', \cdot, K')$. If we assume that $K(e^\Gamma - 1)$ for some $\Gamma \in (\mathfrak{m}_{\mathfrak{a}} \otimes \mathcal{C})^0$, and $K'(e^{\Gamma'} - 1) = 0$ for some $\Gamma' \in (\mathfrak{m}_{\mathfrak{a}} \otimes \mathcal{C}')^0$, then

$$f_{\Gamma', \Gamma} := e^{-\Gamma'} \circ f \circ e^\Gamma : \mathfrak{a} \otimes \mathcal{C} \longrightarrow \mathfrak{a} \otimes \mathcal{C}' \ ,$$

then f_Γ is clearly a cochain map from $(\mathfrak{a} \otimes \mathcal{C}, \cdot, K_\Gamma)$ to $(\mathfrak{a} \otimes \mathcal{C}', \cdot, K'_{\Gamma'})$. In particular, we can deform a cochain endomorphism $\Psi : (\mathcal{C}, K) \to (\mathcal{C}, K)$ using a Maurer-Cartan solution. Then $\Psi_\Gamma := \Psi_{\Gamma, \Gamma}$ is a cochain endomorphism of $(\mathfrak{a} \otimes \mathcal{C}, \cdot, K_\Gamma)$. Unfortunately, this formal deformation is not suitable for a p-adic deformation of the p-adic Banach algebra $(\widetilde{A(b)}, \cdot, \tilde{\Delta}) \in \mathrm{Ob}(\mathfrak{C}_k)$ which was given in (1.9) and (1.10).[7] Thus, in the next subsection, we study a slightly enhanced deformation theory for $\mathfrak{C}_k$.

[7]We will explain this further later in subsection 3.3.

2.3. **Deformation theory for the q-power map.** For a p-adic deformation of the p-adic Banach algebra $(\widetilde{A(b)}, \cdot, \tilde{\Delta}) \in \mathrm{Ob}(\mathfrak{C}_k)$, we consider a graded k-algebra endomorphism[8] $\sigma : \mathfrak{a} \to \mathfrak{a}$. Since σ is an algebra map, we have $\sigma(\mathfrak{m}_{\mathfrak{a}}) \subseteq \mathfrak{m}_{\mathfrak{a}}$ by the nilpotency of $\mathfrak{m}_{\mathfrak{a}}$, and

$$Ke^{\sigma\Gamma} = K(\sigma e^{\Gamma}) = \sigma(Ke^{\Gamma})$$

so the triple $(\mathfrak{a} \otimes \mathcal{C}, \cdot, K_{\sigma\Gamma})$, where $K_{\sigma\Gamma} = e^{-\sigma\Gamma} \circ K \circ e^{\sigma\Gamma}$, is also an object of $\mathfrak{C}_k$ whenever $Ke^{\Gamma} = 0$. Let Ψ be a cochain endomorphism of $(\mathcal{C}, K)$. Now define

$$\Psi_{\sigma,\Gamma} := e^{-\sigma\Gamma} \circ \Psi \circ e^{\Gamma} : \mathfrak{a} \otimes \mathcal{C} \longrightarrow \mathfrak{a} \otimes \mathcal{C}$$

Then

$$\begin{aligned}
\Psi_{\sigma,\Gamma} \circ K_{\Gamma} &= e^{-\sigma\Gamma} \circ \Psi \circ e^{\Gamma} \circ e^{-\Gamma} \circ K \circ e^{\Gamma} \\
&= e^{-\sigma\Gamma} \circ \Psi \circ K \circ e^{\Gamma} \\
&= e^{-\sigma\Gamma} \circ K \circ \Psi \circ e^{\Gamma} \\
&= e^{-\sigma\Gamma} \circ K \circ e^{\sigma\Gamma} \circ e^{-\sigma\Gamma} \circ \Psi \circ e^{\Gamma} \\
&= K_{\sigma\Gamma} \circ \Psi_{\sigma,\Gamma}
\end{aligned}$$

shows that

$$\Psi_{\sigma,\Gamma} : (\mathfrak{a} \otimes \mathcal{C}, \cdot, K_{\Gamma}) \longrightarrow (\mathfrak{a} \otimes \mathcal{C}, \cdot, K_{\sigma\Gamma})$$

defines a morphism in $\mathfrak{C}_k$. Since $K_{\Gamma} \neq K_{\sigma\Gamma}$ in general, $\Psi_{\sigma,\Gamma}$ is not an endomorphism in general. However, multiplication by e^{x} has inverse e^{-x} so via the commutative diagram

$$\begin{array}{ccc}
H^{\bullet}(\mathfrak{a} \otimes \mathcal{C}, K_{\Gamma}) & \xrightarrow{\Psi_{\sigma,\Gamma}} & H^{\bullet}(\mathfrak{a} \otimes \mathcal{C}, K_{\sigma\Gamma}) \\
{\scriptstyle e^{\Gamma}}\downarrow\wr & & {\scriptstyle e^{\sigma\Gamma}}\downarrow\wr \\
\mathfrak{a} \otimes H^{\bullet}(\mathcal{C}, K) & \xrightarrow[\Psi]{} & \mathfrak{a} \otimes H^{\bullet}(\mathcal{C}, K)
\end{array}$$

We may regard $\Psi_{\sigma,\Gamma}$ as an endomorphism on the cohomology space. Moreover, this diagram also shows that $\Psi_{\sigma,\Gamma}$ on the cohomology space depends only on the Ψ on the cohomology space.

3. Homotopy Lie theory for the zeta function

3.1. **A GBV algebra associated to $\mathbf{P}^n$.** We are interested in a p-adic homotopy Lie algebra and the zeta function (Theorem 1.1) of a smooth projective complete intersection variety in $\mathbf{P}^n$ over $\mathbb{F}_q$ defined by $G_1(\underline{x}), \cdots, G_k(\underline{x})$. In order to study them, we start from $\mathbf{P}^n$.

Recall that $\Bbbk = \mathbb{Q}_q(\zeta_p, \pi)$ is a finite extension of $\mathbb{Q}_p$ with the uniformizer π and the p-adic Banach algebra over $\Bbbk$ given in (1.9). Let $\mathfrak{g}_{\Bbbk}$ be an abelian Lie algebra over $\Bbbk$ of dimension $N = n + k + 1$. Let $\beta_1, \beta_2, \cdots, \beta_N$ be a $\Bbbk$-basis of $\mathfrak{g}_{\Bbbk}$. We associate a Lie algebra representation $\tilde{\rho}$ on $\widetilde{A(b)}$ of $\mathfrak{g}_{\Bbbk}$ as follows:

$$\tilde{\rho}(\beta_i) := \frac{\pi^{Mb}}{\gamma}\left(\frac{\partial}{\partial z_i}\right), \text{ for } i = 1, 2, \cdots, N.$$

We extend this $\Bbbk$-linearly to get a Lie algebra representation $\rho_X : \mathfrak{g}_{\Bbbk} \to \mathrm{End}_{\Bbbk}(A(b))$. Then we consider *the dual Chevalley-Eilenberg complex* $(\widetilde{\mathcal{A}(b)}, \tilde{\Delta})$ associated to $\tilde{\rho}$ as in the introduction. The $\mathbb{Z}$-graded super-commutative algebra $\mathcal{A}_{\tilde{\rho}}$ and the differential $\tilde{\Delta}$ is given explicitly as follows:

$$\begin{aligned}
\widetilde{\mathcal{A}(b)} &= \mathcal{A}(b)[\underline{\eta}] = \mathcal{A}(b)[\eta_1, \eta_2, \cdots, \eta_N], \\
\tilde{\Delta} &= \frac{\pi^{Mb}}{\gamma} \cdot \frac{\partial}{\partial z_i}\frac{\partial}{\partial \eta_i} : \widetilde{\mathcal{A}(b)} \to \widetilde{\mathcal{A}(b)}.
\end{aligned}$$

We have

$$0 \longrightarrow \widetilde{\mathcal{A}(b)}^{-N} \xrightarrow{\tilde{\Delta}} \widetilde{\mathcal{A}(b)}^{-N+1} \xrightarrow{\tilde{\Delta}} \cdots \xrightarrow{\tilde{\Delta}} \widetilde{\mathcal{A}(b)}^{-1} \xrightarrow{\tilde{\Delta}} \widetilde{\mathcal{A}(b)}^{0} = \widetilde{\mathcal{A}(b)} \longrightarrow 0$$

[8] In our application to the zeta function of X, $\mathfrak{a}$ will be a formal power series ring with k number of variables $t_1, \cdots, t_k$ and σ will be a q-power map of those variables.

where

$$\widetilde{\mathcal{A}(b)}^{-s} = \bigoplus_{1\le i_1<\cdots<i_s\le N} \mathcal{A}(b)\cdot\eta_{i_1}\cdots\eta_{i_s}, \quad 0\le s\le N.$$

Then $(\widetilde{\mathcal{A}(b)},\cdot,\tilde{\Delta})$ is an object of $\mathfrak{C}_{\Bbbk}$. If we apply the descendant functor to it, we obtain the following proposition.

Proposition 3.1. $(\widetilde{\mathcal{A}(b)},\cdot,\tilde{\Delta},\ell_2^{\tilde{\Delta}})$ is a GBV-algebra over $\Bbbk$.

Proof. We need to show that $(\widetilde{\mathcal{A}(b)},\tilde{\Delta},\ell_2^{\tilde{\Delta}})$ is a dgla and $(\widetilde{\mathcal{A}(b)},\cdot,\ell_2^{\tilde{\Delta}})$ is a G-algebra. The fact that $\tilde{\Delta}$ is a homogeneous differential operator of order 2 implies that $\ell_m^{\tilde{\Delta}}=0$ for $m\ge 3$, which implies that $(\widetilde{\mathcal{A}(b)},\tilde{\Delta},\ell_2^{\tilde{\Delta}})$ is a dgla. It follows that $(\widetilde{\mathcal{A}(b)},\cdot,\ell_2^{\tilde{\Delta}})$ is a G-algebra by a straightforward computation. □

We can also prove a finite field version of the above proposition. Note that $(\mathcal{A},\cdot,K_S)$ given in (1.4) is an object of $\mathfrak{C}_{\mathbb{F}_q}$[9]

Proposition 3.2. $(\mathcal{A},\cdot,\Delta,\ell_2^{\Delta})$ is a GBV algebra.

3.2. **A π-adic filtered complex.** Here we like to understand a precise relationship between $(\widetilde{\mathcal{A}(b)},\cdot,\tilde{\Delta})$ and $(\mathcal{A},\cdot,\Delta)$. For this, we define a filtration on $(\widetilde{\mathcal{A}(b)},\cdot,\tilde{\Delta})$ which is compatible with the differential $\tilde{\Delta}$. The technical conditions on b and M and the factor $\frac{\pi^{Mb}}{\gamma}$ is designed to accomplish this by Adolphson and Sperber in [2]. Recall that π is a uniformizer for $\mathcal{O}_{\Bbbk}$. Following section 3, [2], define a decreasing filtration $\{F^s\widetilde{\mathcal{A}(b)}\}_{s\in\mathbb{Z}}$ on $\widetilde{\mathcal{A}(b)}$:

$$F^s\widetilde{\mathcal{A}(b)}^{-m} = \bigoplus_{r+t=m}\ \bigoplus_{\substack{1\le i_1<\cdots<i_r\le k\\ k+1\le j_1<\cdots<j_t\le N}} \pi^{Mb(k-r)}\tilde{F}^s\widetilde{\mathcal{A}(b)}\cdot\eta_{i_1}\cdots\eta_{i_r}\cdot\eta_{j_1}\cdots\eta_{j_t}, \quad 0\le m\le N, \tag{3.1}$$

where

$$\tilde{F}^s\widetilde{\mathcal{A}(b)} = \left\{ \sum_{(\underline{u},\underline{v})\in\mathbb{Z}^N_{\ge 0}} a_{\underline{u},\underline{v}}\pi^{Mb|\underline{v}|}\underline{x}^{\underline{u}}\underline{y}^{\underline{v}} : a_{\underline{u},\underline{v}}\in\pi^s\mathcal{O}_{\Bbbk} \text{ for all } (\underline{u},\underline{v})\in\mathbb{Z}^N_{\ge 0}\right\}.$$

Note that

$$F^s\widetilde{\mathcal{A}(b)} = F^s\widetilde{\mathcal{A}(b)}^0 = \left\{ \sum_{(\underline{u},\underline{v})\in\mathbb{Z}^N_{\ge 0}} a_{\underline{u},\underline{v}}\pi^{Mb(|\underline{v}|+k)}\underline{x}^{\underline{u}}\underline{y}^{\underline{v}} : a_{\underline{u},\underline{v}}\in\pi^s\mathcal{O}_{\Bbbk} \text{ for all } (\underline{u},\underline{v})\in\mathbb{Z}^N_{\ge 0}\right\}.$$

Then a simple calculation confirms that $\tilde{\Delta}(F^s\widetilde{\mathcal{A}(b)}^{-m})\subset F^s\widetilde{\mathcal{A}(b)}^{-m+1}$, using the fact $\frac{1}{p-1}<b<\frac{p}{p-1}$. (here the factor $\frac{\pi^{Mb}}{\gamma}$ in $\tilde{\Delta}=\frac{\pi^{Mb}}{\gamma}\cdot\sum_{i=1}^N\frac{\partial}{\partial z_i}\frac{\partial}{\partial\eta_i}$ plays a role). Therefore the filtration (3.1) makes $(\widetilde{\mathcal{A}(b)},\tilde{\Delta})$ into a π-adic filtered complex.

For each $0\le m\le N$, define a $\Bbbk$-linear map $\mathcal{R}: F^0\widetilde{\mathcal{A}(b)}^{-m}\to\mathcal{A}^{-m}$, where $\mathcal{A}$ was given in (1.4), by additivity and the formula:

$$\sum_{(\underline{u},\underline{v})\in\mathbb{Z}^N_{\ge 0}} a_{\underline{u},\underline{v}}\pi^{Mb(|\underline{v}|+k-r)}\underline{x}^{\underline{u}}\underline{y}^{\underline{v}}\cdot\eta_{i_1}\cdots\eta_{i_r}\cdot\eta_{j_1}\cdots\eta_{j_t}\mapsto \sum_{(\underline{u},\underline{v})\in\mathbb{Z}^N_{\ge 0}}\overline{a}_{\underline{u},\underline{v}}\underline{x}^{\underline{u}}\underline{y}^{\underline{v}}\cdot\eta_{i_1}\cdots\eta_{i_r}\cdot\eta_{j_1}\cdots\eta_{j_t},$$

where $\overline{a}_{\underline{u},\underline{v}}$ is the reduction of $a_{\underline{u},\underline{v}}$ modulo the maximal ideal of $\mathcal{O}_{\Bbbk}$. Since $a_{\underline{u},\underline{v}}\to 0$ as $|(\underline{u},\underline{v})|\to\infty$, the image $\mathcal{R}(\xi)$ for $\xi\in\widetilde{\mathcal{A}(b)}^{-m}$ is a finite sum. It is not difficult to see that this $\Bbbk$-linear map is surjective with kernel $F^1\widetilde{\mathcal{A}(b)}^{-m}$ (using the conditions $\frac{1}{p-1}<b$ and $\frac{Mb}{(p-1)p},\frac{M}{p-1}\in\mathbb{Z}$), hence $\mathcal{R}$ induces a linear isomorphism

$$\mathcal{R}: F^0\widetilde{\mathcal{A}(b)}^{-m}/F^1\widetilde{\mathcal{A}(b)}^{-m}\simeq\mathcal{A}^{-m},$$

for each $0\le m\le N$. Note that this map $\mathcal{R}$ is not a ring homomorphism. We can choose a $\Bbbk$-linear section $s_{\mathcal{R}}:\mathcal{A}\to F^0\widetilde{\mathcal{A}(b)}$ such that $\mathcal{R}\circ s_{\mathcal{R}}=\mathrm{id}$.

[9]Even though $\mathbb{F}_q$ is of characteristic p, a notion of dgla still makes sense.

Proposition 3.3. The $\Bbbk$-linear map $\mathcal{R}$ induces an isomorphism of cochain complexes over $\mathbb{F}_q$

$$\mathcal{R} : (F^0\widetilde{\mathcal{A}(b)}/F^1\widetilde{\mathcal{A}(b)}, \tilde{\Delta}) \simeq (\mathcal{A}, 0),$$

where $(\mathcal{A}, 0)$ is the cochain complex $\mathcal{A}$ with the zero differential.

3.3. **A dGBV algebra associated to X.** Here we will prove (a) and (b) of Theorem 1.1. The main tool is to apply the deformation formalism of the subsection 2.3 to $(\widetilde{\mathcal{A}(b)}, \cdot, \tilde{\Delta})$. Let $X_{\underline{t}} \subseteq \mathbf{P}^n$ be a family of smooth projective complete intersections parametrized by variables $\underline{t} = t_1, \cdots, t_k$. For $i = 1, \cdots, k$, let $G_i(\underline{t}, \underline{x}) \in \mathbb{F}[\underline{t}, \underline{x}]$ be homogeneous polynomials of $\underline{x}$-degree d_i, i.e.

$$G_i(\underline{t}, \underline{x}) = \sum_{\underline{w} \in \mathbb{Z}^k_{\geq 0}} G_{i,\underline{w}}(\underline{x})\underline{t}^{\underline{w}} \in \mathbb{F}_q[\underline{t}, \underline{x}], \quad G_{i,\underline{w}}(\underline{x}) = \sum_{\underline{u} \in \mathbb{Z}^{n+1}_{\geq 0}} g_{i,\underline{w},\underline{u}}\underline{x}^{\underline{u}} \in \mathbb{F}_q[\underline{x}],$$

where $\deg(G_{i,\underline{w}}(\underline{x})) = d_i$, such that $G_i(\underline{x}) = G_i(\underline{1}, \underline{x})$ and $G_i(\underline{0}, \underline{x}) = 0$. Let $\tilde{G}_i(\underline{t}, \underline{x})$ be the Teichmüller lifting:

$$\tilde{G}_i(\underline{t}, \underline{x}) = \sum_{\underline{u} \in \mathbb{Z}^{n+1}_{\geq 0}, \underline{w} \in \mathbb{Z}^k_{\geq 0}} \tilde{g}_{i,\underline{u},\underline{w}}\underline{x}^{\underline{u}}\underline{t}^{\underline{w}}$$

where $\tilde{g}_{i,\underline{u},\underline{w}} \in \mathbb{Z}_q$ is the Teichmüller lifting of $g_{i,\underline{u},\underline{w}} \in \mathbb{F}_q$. Let $S(\underline{t}, \underline{z}) = \sum_{i=1}^k y_i G_i(\underline{t}, \underline{x})$ and let $\tilde{S}(\underline{t}, \underline{z})$ be the Teichmüller lifting of $S(\underline{t}, \underline{z})$:

$$\tilde{S}(\underline{t}, \underline{z}) = \sum_{i=1}^k y_i \tilde{G}_i(\underline{t}, \underline{x}) = \sum_{i,\underline{u},\underline{w}} y_i \tilde{g}_{i,\underline{u},\underline{w}}\underline{x}^{\underline{u}}\underline{t}^{\underline{w}} \in \mathbb{Z}_q[\underline{t}, \underline{z}] = \mathbb{Z}_q[\underline{t}, \underline{x}, \underline{y}],$$

where $\mathbb{Z}_q$ is the ring of integers of $\mathbb{Q}_q$.

A proof of (a) of Theorem 1.1. We start to give a simple proof that the $\Bbbk$-linear operator $\Psi_{\mathbf{P}^n}$ given in (1.12) is a cochain endomorphism of $(\widetilde{\mathcal{A}(b)}, \tilde{\Delta})$:

$$\begin{aligned}
&\left(\Psi_{\mathbf{P}^n}^{-m+1} \circ \tilde{\Delta}\right)(\xi(\underline{z})\eta_{i_1}\cdots\eta_{i_m}) \\
&= \frac{\pi^{Mb}}{\gamma} \cdot \Psi_{\mathbf{P}^n}^{-m+1}\left(\sum_{k=1}^m \frac{\partial \xi(\underline{z})}{\partial z_{i_k}}\eta_{i_1}\cdots\hat{\eta_{i_k}}\cdots\eta_{i_m}\right) \\
&= \frac{\pi^{Mb}}{\gamma} \cdot \sum_{k=1}^m \frac{q^{m-1}}{z_{i_k}\prod_{\substack{j\in\{1,\cdots,N\}\\ j\neq i_1,\cdots,i_m}} z_j} T_q\left(z_{i_k}\frac{\partial}{\partial z_{i_k}}\left(\xi(\underline{z})\prod_{\substack{j\in\{1,\cdots,N\}\\ j\neq i_1,\cdots,i_m}} z_j\right)\eta_{i_1}\cdots\hat{\eta_{i_k}}\cdots\eta_{i_m}\right) \\
&= \frac{\pi^{Mb}}{\gamma} \cdot \sum_{k=1}^m \frac{q^m}{\prod_{\substack{j\in\{1,\cdots,N\}\\ j\neq i_1,\cdots,i_m}} z_j}\frac{\partial}{\partial z_{i_k}}\left(T_q\left(\xi(\underline{z})\prod_{\substack{j\in\{1,\cdots,N\}\\ j\neq i_1,\cdots,i_m}} z_j\right)\frac{\partial}{\partial \eta_{i_k}}(\eta_{i_1}\cdots\eta_{i_m})\right) \\
&= \frac{\pi^{Mb}}{\gamma} \cdot \sum_{i=1}^N \frac{\partial}{\partial z_i}\frac{\partial}{\partial \eta_i}\left(\frac{q^m}{\prod_{\substack{j\in\{1,\cdots,N\}\\ j\neq i_1,\cdots,i_m}} z_j} T_q\left(\xi(\underline{z})\prod_{\substack{j\in\{1,\cdots,N\}\\ j\neq i_1,\cdots,i_m}} z_j\right)\eta_{i_1}\cdots\eta_{i_m}\right) \\
&= \left(\tilde{\Delta} \circ \Psi_{\mathbf{P}^n}^{-m}\right)(\xi(\underline{z})\eta_{i_1}\cdots\eta_{i_m}).
\end{aligned}$$

Hence $\Psi_{\mathbf{P}^n} : (\widetilde{\mathcal{A}(b)}, \cdot, \tilde{\Delta}) \to (\widetilde{\mathcal{A}(b)}, \cdot, \tilde{\Delta})$ is indeed a morphism of the category $\mathfrak{C}_\Bbbk$. Then we apply the deformation formalism in the subsection 2.3. Let $\mathfrak{a} = \Bbbk[[t_1, \cdots, t_k]] \in \mathrm{Ob}(\widehat{\mathbf{Art}^{\mathbb{Z}}_\Bbbk})$ with degree $|t_i| = 0$. Since $\widetilde{\mathcal{A}(b)}^1 = 0$, the Maurer-Cartan equation $\tilde{\Delta}(e^\Gamma) = 0$ is vacuous, i.e., we can deform $(\widetilde{\mathcal{A}(b)}, \cdot, \tilde{\Gamma})$ formally by any element $\tilde{\Gamma}$ in $(\mathfrak{m}_\mathfrak{a} \otimes \Bbbk[[\underline{z}]])^0$. We take

$$\sigma : \Bbbk[[t_1, \cdots, t_k]] \longrightarrow \Bbbk[[t_1, \cdots, t_k]] \qquad t_i \longmapsto t_i^q$$

and

$$\tilde{\Gamma}_{\underline{t}} := \sum_{i,\underline{u},\underline{w}} \sum_{\ell \geq 0} \gamma_\ell y_i^{p^\ell} \tilde{g}_{i,\underline{u},\underline{w}}^{p^\ell} \underline{x}^{p^\ell \underline{u}} \underline{t}^{p^\ell \underline{w}} \in ((\underline{t}) \otimes \Bbbk[[\underline{z}]])^0, \tag{3.2}$$

and consider

$$e^{\tilde{\Gamma}_{\underline{t}}} = \exp\left(\sum_{i,\underline{u},\underline{w}} \sum_{\ell \geq 0} \gamma_\ell y_i^{p^\ell} \tilde{g}_{i,\underline{u},\underline{w}}^{p^\ell} \underline{x}^{p^\ell \underline{u}} \underline{t}^{p^\ell \underline{w}} \right) = \prod_{i,\underline{u},\underline{w}} \exp\left(\sum_{\ell \geq 0} \gamma_\ell y_i^{p^\ell} \tilde{g}_{i,\underline{u},\underline{w}}^{p^\ell} \underline{x}^{p^\ell \underline{u}} \underline{t}^{p^\ell \underline{w}} \right).$$

Note that neither $\tilde{\Gamma}_{\underline{t}}$ nor $e^{\tilde{\Gamma}_{\underline{t}}}$ belong to $\mathfrak{m}_{\mathfrak{a}} \otimes \widetilde{A(b)} = (\mathfrak{m}_{\mathfrak{a}} \otimes \widetilde{\mathcal{A}(b)})^0$. But the formal deformation of $\tilde{\Delta}$ by $\tilde{\Gamma}_{\underline{t}}$ (note that $\tilde{\Delta}$ is $\mathfrak{a}$-linear) can be computed as follows:

$$\tilde{\Delta}_{\tilde{\Gamma}_{\underline{t}}} := e^{-\tilde{\Gamma}_{\underline{t}}} \circ \tilde{\Delta} \circ e^{\tilde{\Gamma}_{\underline{t}}} = \frac{\pi^{Mb}}{\gamma} \cdot \sum_{i=1}^{N} \left(\frac{\partial \hat{S}(\underline{t}, \underline{z})}{\partial z_i} + \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial \eta_i},$$

(the equalities here are formal ones in $\mathfrak{a} \otimes \Bbbk[[\underline{z}]]$ without considering p-adic covergence) where

$$\frac{\partial \hat{S}(\underline{t}, \underline{z})}{\partial z_i} = \frac{\partial}{\partial z_i} \left(\tilde{\Gamma}|_{\underline{t}=\underline{1}} \right), \quad i = 1, \cdots, N,$$

by using Lemma 2.9 and the fact $\ell_m^{\tilde{\Delta}} = 0,\ m \geq 3$. Then the expression $\frac{\pi^{Mb}}{\gamma} \cdot \sum_{i=1}^N \left(\frac{\partial \hat{S}(\underline{t},\underline{z})}{\partial z_i} + \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial \eta_i}$ actually makes sense as a $\Bbbk$-linear operator of $\mathfrak{a} \otimes \widetilde{\mathcal{A}(b)}$, since

$$\frac{\partial \hat{S}(\underline{t}, \underline{z})}{\partial z_i} \in (\mathfrak{a} \otimes \widetilde{\mathcal{A}(b)})^0, \quad i = 1, \cdots, N,$$

by using the estimate (3.6).

To compute $\Psi_{\mathbf{P}^n, \sigma, \tilde{\Gamma}_{\underline{t}}}$ (the deformation of $\Psi_{\mathbf{P}^n}$ by σ and $\tilde{\Gamma}_{\underline{t}} = \tilde{\Gamma}_{\underline{t}}(\underline{z})$), observe that

$$E(\underline{z}, \underline{t}^q) \circ T_q = T_q \circ E(\underline{z}^q, \underline{t}) \quad \text{in } \mathfrak{a} \otimes \Bbbk[[\underline{z}]]$$

for any power series $E(\underline{z}, \underline{t})$ so formally (note that $\Psi_{\mathbf{P}^n}$ is $\mathfrak{a}$-linear)

$$\Psi_{\mathbf{P}^n, \sigma, \tilde{\Gamma}_{\underline{t}}} = e^{-\sigma(\tilde{\Gamma}_{\underline{t}})} \circ \Psi_{\mathbf{P}^n} \circ e^{\tilde{\Gamma}_{\underline{t}}} = \Psi_{\mathbf{P}^n} \circ \frac{e^{\tilde{\Gamma}_{\underline{t}}(\underline{z})}}{e^{\tilde{\Gamma}_{\underline{t}}(\underline{z}^q)}} = \Psi_{\mathbf{P}^n} \circ e^{\Gamma_{\underline{t}}},$$

where

$$\Gamma_{\underline{t}} = \Gamma_{\underline{t}}(\underline{z}) := \sum_{j=0}^{a-1} \sum_{i,\underline{u},\underline{w}} \sum_{\ell \geq 0} \frac{\gamma^{p^\ell} \left(\tilde{g}_{i,\underline{u}} y_i \underline{x}^{\underline{u}} \underline{t}^{\underline{w}} \right)^{p^{j+\ell}}}{p^\ell}. \tag{3.3}$$

Note that the last (formal) equality $\frac{e^{\tilde{\Gamma}_{\underline{t}}(\underline{z})}}{e^{\tilde{\Gamma}_{\underline{t}}(\underline{z}^q)}} = e^{\Gamma_{\underline{t}}}$ crucially uses the fact $\tilde{G}_i(\underline{t}, \underline{x})$ is the Teichmüller lifting of $G_i(\underline{t}, \underline{x})$: $\tilde{g}_{i,\underline{u},\underline{w}}^q = \tilde{g}_{i,\underline{u},\underline{w}}$. The key point here is that the expression $\Psi_{\mathbf{P}^n} \circ e^{\Gamma_{\underline{t}}}$ makes sense[10] as a $\Bbbk$-linear operator of $\mathfrak{a} \otimes \widetilde{\mathcal{A}(b)}$, since $e^{\Gamma_{\underline{t}}}$ belongs to $\mathfrak{m}_{\mathfrak{a}} \otimes \widetilde{A(\frac{b}{q})}$, $\Psi_{\mathbf{P}^n} : \widetilde{A(\frac{b}{q})} \to \widetilde{A(b)}$, and $\widetilde{A(b)} \subset \widetilde{A(\frac{b}{q})}$. By evaluating at $\underline{t} = \underline{1}$, we define

$$\begin{aligned} \tilde{K}_X = \tilde{K}_S &:= \tilde{\Delta}_{\tilde{\Gamma}_{\underline{t}}}\Big|_{\underline{t}=\underline{1}} \\ \Psi_X = \Psi_S &:= \Psi_{\mathbf{P}^n, \sigma, \tilde{\Gamma}_{\underline{t}}}\Big|_{\underline{t}=\underline{1}}. \end{aligned} \tag{3.4}$$

By the formal identity in the deformation theory, we conclude that $\Psi_X : (\widetilde{\mathcal{A}(b)}, \cdot, \tilde{K}_X) \to (\widetilde{\mathcal{A}(b)}, \cdot, \tilde{K}_X)$ is a morphism of $\mathfrak{C}_\Bbbk$. Now the part (a) follows directly by applying the descendant functor to Ψ_X. □

The differential $\tilde{K}_X$ is given as follows:

$$\tilde{K}_X = \tilde{K}_S = \frac{\pi^{Mb}}{\gamma} \cdot \sum_{i=1}^{N} \left(\frac{\partial \hat{S}(\underline{z})}{\partial z_i} + \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial \eta_i} : \widetilde{\mathcal{A}(b)} \to \widetilde{\mathcal{A}(b)}.$$

[10] If we use $\Psi_{\mathbf{P}^n, \tilde{\Gamma}_{\underline{t}}}$ instead of $\Psi_{\mathbf{P}^n, \sigma, \tilde{\Gamma}_{\underline{t}}}$, then we can not achieve this.

We also define another differential operator

$$\tilde{Q}_X = \tilde{Q}_S = \frac{\pi^{Mb}}{\gamma} \cdot \sum_{i=1}^{N} \frac{\partial \hat{S}(\underline{z})}{\partial z_i} \frac{\partial}{\partial \eta_i} : \widetilde{\mathcal{A}(b)} \to \widetilde{\mathcal{A}(b)}.$$

The following proposition holds which can be view as a slight strengthening of (a) of Theorem 1.1.

Proposition 3.4. $(\widetilde{\mathcal{A}(b)}, \cdot, \tilde{K}_X, \ell_2^{\tilde{K}_X}, \tilde{Q}_X)$ is a dGBV algebra over $\Bbbk$.

A proof of (b) of Theorem 1.1. The part (b) follows by using the decreasing filtration and the $\Bbbk$-linear map $\mathcal{R}$ in the subsection 3.2. A same computation confirms that

$$(F^0\widetilde{\mathcal{A}(b)}/F^1\widetilde{\mathcal{A}(b)}, \tilde{K}_S) \xrightarrow{\mathcal{R}} (\mathcal{A}, Q_S), \tag{3.5}$$

where $(\mathcal{A}, Q_S)$ is given in (1.5), is an isomorphism of cochain complexes over $\mathbb{F}_q$. ☐

3.4. **The computation of cohomology.** Here we prove (c) of Theorem 1.1. For this we will compare our BV algebra $(\widetilde{\mathcal{A}(b)}, \cdot, K_X)$ with a twisted de Rham complex, so called the p-adic Dwork complex, $(\Omega_b^\bullet, \wedge, D)$ which appeared in section 2, [2]. We briefly review the Dwork construction closely following section 2, [2]. The degree m-th module of $\Omega_b^\bullet$ is given by

$$\Omega_b^m := \bigoplus_{1\le i_1<\cdots<i_m\le N} \widetilde{\mathcal{A}(b)} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_m}$$

for each $m \ge 0$. Here $\wedge$ is the wedge product on the twisted de Rham complex $\Omega_b^\bullet$ of $\widetilde{\mathcal{A}(b)}$. Note that $\Omega_b^0 = \widetilde{\mathcal{A}(b)}$ and $\Omega_b^\bullet = \bigoplus_{0\le m\le N} \Omega_b^m$. The differential D is defined by

$$D(w) = \frac{\pi^{Mb}}{\gamma}\left(dw + d\hat{S}(\underline{z}) \wedge w\right), \quad w \in \Omega_b^m$$

for any $m \ge 0$.

Proposition 3.5. We have the following relationship between $(\Omega_b^\bullet, D)$ and $(\widetilde{\mathcal{A}(b)}, \tilde{K}_X)$;

(a) For each $s \in \mathbb{Z}$, if we define a $\Bbbk$-linear map $J : (\Omega_b^s, D) \to (\widetilde{\mathcal{A}(b)}^{s-N}, \tilde{K}_X)$ by

$$dz_{i_1} \cdots dz_{i_s} \mapsto (-1)^{i_1+\cdots+i_s-s}(\cdots \hat{\eta_{i_1}} \cdots \hat{\eta_{i_s}} \cdots)$$

for $1 \le i_1 < \cdots < i_s \le N = n + k + 1$ and extending it $\Bbbk$-linearly, then $J \circ D = \tilde{K}_X \circ J$ and J induces an isomorphism

$$H^s(\Omega_b^\bullet, D) \simeq H^{s-N}(\widetilde{\mathcal{A}(b)}, \tilde{K}_X).$$

for every $s \in \mathbb{Z}$.

(b) The map J satisfies that $J \circ \partial_X = \tilde{Q}_X \circ J$, where ∂_X is the wedge product with $\frac{\pi^{Mb}}{\gamma} d\hat{S}(\underline{z})$.

Proof. These follow from direct computations. ☐

A proof of (c) of Theorem 1.1. By Proposition 3.5,

$$H^N(\Omega_b^\bullet, D) \simeq H^0(\widetilde{\mathcal{A}(b)}, \tilde{K}_X).$$

Because the $\Bbbk$-dimension of $H^N(\Omega_b^\bullet, D)$ is shown to be the degree of $P(T)$ in [2], we conclude that the degree of $P(T)$ is equal to the $\Bbbk$-dimension of $H^0(\widetilde{\mathcal{A}(b)}, \tilde{K}_X)$. ☐

Remark 3.6. Proposition 3.5 implies that two cochain complexes $(\Omega_b^\bullet, D)$ and $(\widetilde{\mathcal{A}(b)}, \tilde{K}_X)$ are degree-twisted isomorphic to each other. But we emphasize that the natural product structure, the wedge product, on $\Omega_b^\bullet$ and the super-commutative product $\cdot$ on $\widetilde{\mathcal{A}(b)}$ are quite different and J is *not* a ring isomorphism. It is crucial for us to use the super-commutative product $\cdot$ on $\widetilde{\mathcal{A}(b)}$ to get all the main theorems of this article.

$(\Omega_b^\bullet, \wedge, D)$ is a p-adic twisted de-Rham algebra.	$(\widetilde{\mathcal{A}(b)}, \cdot, \tilde{K}_X, \ell_2^{\tilde{K}_X})$ is a p-adic GBV algebra.
$(\Omega_b^\bullet, \wedge, \partial_X)$ is not a cdga.	$(\widetilde{\mathcal{A}(b)}, \cdot, \tilde{Q}_X)$ is a cdga.
$(\Omega_b^\bullet, \wedge, \frac{\pi^{Mb}}{\gamma} d)$ is a cdga.	$(\widetilde{\mathcal{A}(b)}, \cdot, \tilde{\Delta})$ is not a cdga.

TABLE 3.1. The Dwork twisted de-Rham algebra versus the GBV algebra

3.5. **A character sum and the Dwork splitting function.** We will prove (d) of Theorem 1.1 this subsection. Following [2], we consider

$$\hat{\theta}(t) := \prod_{i=0}^{\infty} \theta(t^{p^i}) \quad \text{where} \quad \theta(t) := E(\gamma t) = \sum_{i=0}^{\infty} \lambda_i t^i \in \Bbbk[[t]],$$

where $E(t)$ is the Artin-Hasse exponential series

$$E(t) = \exp\left(\sum_{n=0}^{\infty} \frac{t^{p^n}}{p^n}\right) \in \Bbbk[[t]].$$

One can easily check that

$$\hat{\theta}(t) = \exp\left(\sum_{\ell=0}^{\infty} \gamma_\ell t^{p^\ell}\right) \quad \text{where} \quad \gamma_\ell := \sum_{i=0}^{\ell} \frac{\gamma^{p^i}}{p^i}.$$

Note that

$$\text{val}(\gamma_\ell) \geq \frac{p^{\ell+1}}{p-1} - (\ell+1). \tag{3.6}$$

If, for $a \in \mathbb{R}_{\geq 0}$, we use the following notations

$$D(a) = \{x \in \mathbb{C}_p : |x|_p \leq a\}, \quad D(a^-) = \{x \in \mathbb{C}_p : |x|_p < a\},$$

then $\theta(t)$ converges on $D(p^{\frac{1}{p-1}-})$ (since $\text{val}_p(\lambda_i) \geq \frac{i}{p-1}$) and $E(t)$ converges only on $D(1^-)$. Here $\theta(t)$ is called the Dwork splitting function because of the following lemma of Dwork.

Lemma 3.7. If we define a function $\psi_q : \mathbb{F}_q \to \mathbb{C}_p^*$ by the formula

$$\psi_q(x) := \theta(\tilde{x}) \cdot \theta(\tilde{x}^p) \cdots \theta(\tilde{x}^{p^{a-1}})$$

for $x \in \mathbb{F}_q$ and the Teichmüller representative $\tilde{x} \in \mathbb{Z}_q$ of x, then ψ_q is an additive character of $\mathbb{F}_q$.

Let us write $G_\ell(\underline{x}) = \sum_{\underline{v} \in \mathbb{Z}_{\geq 0}^{n+1}} g_{v,\ell} \underline{x}^{\underline{v}} \in \mathbb{F}_q[\underline{x}]$ for each $\ell = 1, \cdots, k$. Then its Teichmüller lift $\tilde{G}_\ell(\underline{x})$ can be written as $\sum_{\underline{v} \in \mathbb{Z}_{\geq 0}^{n+1}} \tilde{g}_{v,\ell} \underline{x}^{\underline{v}} \in \mathbb{Z}_q[\underline{x}]$ and

$$\tilde{S}(\underline{z}) = \sum_{\ell=1}^{k} y_\ell \cdot \tilde{G}_\ell(\underline{x}) = \sum_{\underline{w} \in \mathbb{Z}_{\geq 0}^N} \tilde{s}_{\underline{w}} \underline{z}^{\underline{w}} = \sum_{\ell=1}^{k} \sum_{\underline{u} \in \mathbb{Z}_{\geq 0}^{n+1}} \tilde{g}_{u,\ell} \cdot y_\ell \underline{x}^{\underline{u}}.$$

Let $\tau \in \text{Gal}(\Bbbk/\mathbb{Q}_p)$ be an automorphism of $\Bbbk$ which lifts the Frobenius automorphism $x \mapsto x^p$ of $\mathbb{F}_q$ such that $\tau(\zeta_p) = \zeta_p$ and $\tau(\pi) = \pi$. Then $\tau(\tilde{a}) = \tilde{a}^p$ for $\tilde{a}$ which is the Teichmüller lifting of a. Moreover, we define

$$\hat{S}(\underline{z}) = \sum_{m=0}^{\infty} \gamma_m \tilde{S}^{\tau^m}(\underline{z}^{p^m}) := \sum_{m=0}^{\infty} \sum_{\ell=1}^{k} \gamma_m \cdot y_\ell^{p^m} \sum_{\underline{u} \in \mathbb{Z}_{\geq 0}^{n+1}} \tau^m(\tilde{g}_{\underline{u},\ell}) \cdot \underline{x}^{\underline{u} \cdot p^m} = \sum_{m,\ell,\underline{u}} \gamma_m \cdot y_\ell^{p^m} (\tilde{g}_{\underline{u},\ell} \underline{x}^{\underline{u}})^{p^m}.$$

Following [2] again, we use the following notation:

$$E_{\tilde{S}}(\underline{z}) := \prod_{j=0}^{a-1} \prod_{\underline{w}} \theta((\tilde{s}_{\underline{w}} \underline{z}^{\underline{w}})^{p^j}), \quad \hat{E}_{\tilde{S}}(\underline{z}) := \prod_{\underline{w}} \hat{\theta}(\tilde{s}_{\underline{w}} \underline{z}^{\underline{w}})$$

so that $E_{\tilde{S}}(\underline{z}) = \hat{E}_{\tilde{S}}(\underline{z}) / \hat{E}_{\tilde{S}}(\underline{z}^q)$ where $\underline{z}^q = (z_1^q, \cdots, z_N^q)$. Then it is a straightforward computation that

$$\frac{\partial \hat{S}(\underline{z})}{\partial z_i} = \frac{1}{\hat{E}_{\tilde{S}}(\underline{z})} \frac{\partial E_{\tilde{S}}(\underline{z})}{\partial z_i}, \quad i = 1, \cdots, N = n+k+1.$$

As an operator, we have the following identity:

$$\left(\frac{\partial}{\partial z_i}+\frac{\partial \hat{S}(\underline{z})}{\partial z_i}\right)(f)=\hat{E}_{\tilde{S}}(\underline{z})^{-1}\cdot\frac{\partial}{\partial z_i}\left(\hat{E}_{\tilde{S}}(\underline{z})\cdot f\right),\quad i=1,\cdots,N=n+k+1,$$

for $f\in\widetilde{A(b)}$. Now we define a $\Bbbk$-linear operator $\hat{\Psi}_X:\widetilde{A(b)}\to\widetilde{A(b)}$ by

$$\hat{\Psi}_X:=T_q\circ E_{\tilde{S}}(\underline{z})$$

where $\hat{\Psi}_X$ means the multiplication by $E_{\tilde{S}}(\underline{z})$ followed by T_q. Since $b<\frac{p}{p-1}$, $E_{\tilde{S}}(\underline{z})\in\mathcal{A}\left(\frac{b}{q}\right)$ and $\hat{\Psi}_X$ is a well-defined endomorphism of $\widetilde{A(b)}$:

$$\widetilde{A(b)}\hookrightarrow\mathcal{A}\left(\frac{b}{q}\right)\xrightarrow{E_{\tilde{S}}(\underline{z})}\mathcal{A}\left(\frac{b}{q}\right)\xrightarrow{T_q}\widetilde{A(b)}.$$

Then $\hat{\Psi}_X$ is a completely continuous $\Bbbk$-linear operator on $\widetilde{A(b)}$; see [2].

Definition 3.8. We define a completely continuous $\Bbbk$-linear endomorphism $\tilde{\Psi}_X$ on $\widetilde{\mathcal{A}(b)}=\bigoplus_{m\le 0}\widetilde{\mathcal{A}(b)}^m$ by additivity and the following formula

$$\tilde{\Psi}_X^{-m}(\xi(\underline{z})\cdot\eta_{i_1}\cdots\eta_{i_m}):=q^m\frac{\hat{\Psi}_X\left(\xi(\underline{z})\cdot\prod_{\substack{j\in\{1,\cdots,N\}\\ j\neq i_1,\cdots,i_m}}z_j\right)}{\prod_{\substack{j\in\{1,\cdots,N\}\\ j\neq i_1,\cdots,i_m}}z_j}\cdot\eta_{i_1}\cdots\eta_{i_m},$$

for each $m\ge 0$ and $\xi(\underline{z})\in\widetilde{A(b)}=\widetilde{\mathcal{A}(b)}^0$.

Lemma 3.9. The map Ψ_X is equal to $\tilde{\Psi}_X$ and so the $\Bbbk$-linear map $\tilde{\Psi}_X:(\widetilde{\mathcal{A}(b)},\tilde{K}_X)\to(\widetilde{\mathcal{A}(b)},\tilde{K}_X)$ is a cochain map.

Proof. This follows from (3.9) and a straightforward computation. □

A proof of (d) of Theorem 1.1. First note that

$$e^{\tilde{\Gamma}_{\underline{t}}}=\prod_{i,\underline{u},\underline{w}}\hat{\theta}\left(y_i\tilde{g}_{i,\underline{u},\underline{w}}\underline{x}^{\underline{u}}\underline{t}^{\underline{w}}\right)=\hat{E}_{\tilde{S}(\underline{t},\underline{z})}(\underline{z}),\quad e^{\sigma(\tilde{\Gamma}_{\underline{t}})}=\hat{E}_{\tilde{S}(\underline{t}^q,\underline{z})}(\underline{z}),$$

where $\tilde{\Gamma}_{\underline{t}}$ is given in (3.2). Similarly, we have that

$$e^{\Gamma_{\underline{t}}}=E_{\tilde{S}(\underline{t},\underline{z})}(\underline{z}),$$

where $\Gamma_{\underline{t}}$ is given in (3.3). Because

$$\tilde{\Delta}_{\tilde{\Gamma}_{\underline{t}}}:=e^{-\tilde{\Gamma}_{\underline{t}}}\circ\tilde{\Delta}\circ e^{\tilde{\Gamma}_{\underline{t}}}=\hat{E}_{\tilde{S}(\underline{t},\underline{z})}(\underline{z})^{-1}\circ\tilde{\Delta}\circ\hat{E}_{\tilde{S}(\underline{t},\underline{z})}(\underline{z})=\frac{\pi^{Mb}}{\gamma}\cdot\sum_{i=1}^{N}\left(\frac{\partial\hat{S}(\underline{t},\underline{z})}{\partial z_i}+\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial\eta_i},$$

$$\Psi_{\mathbf{P}^n,\sigma,\tilde{\Gamma}_{\underline{t}}}=e^{-\sigma(\tilde{\Gamma}_{\underline{t}})}\circ\Psi_{\mathbf{P}^n}\circ e^{\tilde{\Gamma}_{\underline{t}}}=\hat{E}_{\tilde{S}(\underline{t}^q,\underline{z})}(\underline{z})^{-1}\circ\Psi_{\mathbf{P}^n}\circ\hat{E}_{\tilde{S}(\underline{t},\underline{z})}(\underline{z})=\Psi_{\mathbf{P}^n}\circ E_{\tilde{S}(\underline{t},\underline{z})}(\underline{z}),$$

we see that the operator α_{n+k+1} on $H^{n+k+1}(\Omega_b^\bullet,D)$, in Corollary 6.5, [2], corresponds exactly (by (3.9) and Lemma 3.9) to the operator Ψ_X on $H^0(\widetilde{\mathcal{A}(b)},\tilde{K}_X)$ under the isomorphism $H^{n+k+1}(\Omega_b^\bullet,D)\simeq H^0(\widetilde{\mathcal{A}(b)},\tilde{K}_X)$ in Proposition 3.5. Then the part (d) of Theorem 1.1

$$P(q^kT)=\det(1-T\cdot\Psi_X\big|H^0(\widetilde{\mathcal{A}(b)},\tilde{K}_S))\tag{3.7}$$

follows from Corollary 6.5, [2]. □

3.6. **The Bell polynomials and a deformation formula for the Dwork operator Ψ_X.** The goal here is to prove Theorem 1.2. The *partial Bell polynomials* $B_{n,k}(x_1,\cdots,x_{n-k+1})$ are defined by the power series expansion

$$\exp\left(u\sum_{i=1}^{\infty}x_i\frac{t^i}{i!}\right)=\sum_{n,k\geq 0}B_{n,k}(x_1,\cdots,x_{n-k+1})\frac{t^n}{n!}u^k=1+\sum_{n\geq 1}\frac{t^n}{n!}\left(\sum_{k=1}^{n}u^kB_{n,k}(x_1,\cdots,x_{n-k+1})\right)$$

which gives the following formula:

$$B_{n,k}(x_1,\cdots,x_{n-k+1})=\sum\frac{n!}{i_1!\cdots i_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{i_1}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{i_{n-k+1}}$$

where the sum runs over all sequences $i_1,\cdots,i_{n-k+1}$ of non-negative integers such that

$$i_1+\cdots+i_{n-k+1}=k,\quad i_1+2i_2+\cdots+(n-k+1)i_{n-k+1}=n$$

and the *(complete) Bell polynomials* $B_n(x_1,\cdots,x_n)$ are defined by the power series expansion

$$\exp\left(\sum_{i\geq 1}x_i\frac{t^i}{i!}\right)=1+\sum_{n\geq 1}B_n(x_1,\cdots,x_n)\frac{t^n}{n!} \tag{3.8}$$

in other words,

$$B_n(x_1,\cdots,x_n)=\sum_{k=1}^{n}B_{n,k}(x_1,\cdots,x_{n-k+1}),\qquad B_0=1.$$

There are recurrence relations

$$B_{n+1}(x_1,\cdots,x_{n+1})=\sum_{i=0}^{n}\binom{n}{i}B_{n-i}(x_1,\cdots,x_{n-i})x_{i+1}$$

where $B_0=1$, and

$$B_{n,k}(x_1,\cdots,x_{n-k+1})=\sum_{i=1}^{n-k+1}\binom{n-1}{i-1}x_iB_{n-i,k-1}(x_1,\cdots,x_{n-i+k})$$

where $B_{0,0}=1$, $B_{n,0}=0$ for $n\geq 1$, and $B_{0,k}=0$ for $k\geq 1$. For example, $B_1(x_1)=x_1, B_2(x_1,x_2)=x_1^2+x_2$, $B_3(x_1,x_2,x_3)=x_1^3+3x_1x_2+x_3,\cdots$.

A proof of Theorem 1.2. Recall that

$$\begin{aligned}\tilde{K}_X=\tilde{K}_S:=\tilde{\Delta}_{\tilde{\Gamma}_{\underline{t}}}\Big|_{\underline{t}=\underline{1}}&=\frac{\pi^{Mb}}{\gamma}\cdot\sum_{i=1}^{N}\left(\frac{\partial\hat{S}(\underline{z})}{\partial z_i}+\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial\eta_i}\\ \Psi_X=\Psi_S:=\Psi_{\mathbf{P}^n,\sigma,\tilde{G}_{\underline{t}}}\Big|_{\underline{t}=\underline{1}}&=\Psi_{\mathbf{P}^n}\circ E_{\tilde{S}(\underline{z})}(\underline{z}).\end{aligned} \tag{3.9}$$

By Lemma 2.9, we have

$$\tilde{K}_X(\lambda)=e^{-\tilde{\Gamma}_{\underline{t}}}\circ\tilde{\Delta}\circ e^{\tilde{\Gamma}_{\underline{t}}}\Big|_{\underline{t}=\underline{1}}(\lambda)=\tilde{\Delta}(\lambda)+\sum_{n\geq 2}\frac{1}{(n-1)!}\ell_n^{\tilde{\Delta}}(\tilde{\Gamma},\cdots,\tilde{\Gamma},\lambda),\qquad \lambda\in\widetilde{\mathcal{A}(b)}.$$

where we recall that

$$\tilde{\Gamma}|_{\underline{t}=1}=\tilde{\Gamma}:=\sum_{i,\underline{u}}\sum_{\ell\geq 0}\gamma_\ell\tilde{g}_{i,\underline{u}}^{p^\ell}y_i^{p^\ell}\underline{x}^{p^\ell\underline{u}}\quad(\gamma_\ell:=\sum_{i=0}^{\ell}\frac{\gamma^{p^i}}{p^i}).$$

Since $\ell_m^{\tilde{\Delta}}=0$ for $m\geq 3$, the first part of Theorem 1.2 follows. Moreover, one can check that

$$\ell_2^{\tilde{\Delta}}(\tilde{\Gamma},\lambda)=\left(\frac{\pi^{Mb}}{\gamma}\cdot\sum_{i=1}^{N}\frac{\partial\hat{S}(\underline{z})}{\partial z_i}\frac{\partial}{\partial\eta_i}\right)(\lambda),\quad \lambda\in\widetilde{\mathcal{A}(b)}.$$

By Lemma 2.10, we have

$$\begin{aligned}\Psi_X(\lambda) &= e^{-\sigma(\tilde{\Gamma}_{\underline{t}})} \circ \Psi_{\mathbf{P}^n} \circ e^{\tilde{\Gamma}_{\underline{t}}}\Big|_{\underline{t}=\underline{1}}(\lambda)\\ &= \Psi_{\mathbf{P}^n}\left(E_{\tilde{S}(\underline{z})}(\underline{z})\cdot\lambda\right)\\ &= \left(\phi_1^{\Psi_{\mathbf{P}^n}}(\lambda) + \sum_{n\geq 2}\frac{1}{(n-1)!}\phi_n^{\Psi_{\mathbf{P}^n}}(\Gamma,\cdots,\Gamma,\lambda)\right)\cdot e^{\Phi^{\Psi_{\mathbf{P}^n}}(\Gamma)}, \quad \lambda\in\widetilde{\mathcal{A}(b)},\end{aligned}$$

where we recall that

$$\Gamma = \sum_{j=0}^{a-1}\sum_{i,\underline{u}}\sum_{\ell\geq 0}\frac{\gamma^{p^\ell}\tau^\ell(\tilde{g}_{i,\underline{u}})^{p^j}\,(y_i\underline{x}^{\underline{u}})^{p^{j+\ell}}}{p^\ell}.$$

We want to expand this as

$$\Psi_X(\lambda) = \Psi_0(\Gamma)+\Psi_1(\Gamma)+\Psi_2(\Gamma)+\Psi_3(\Gamma)+\cdots,$$

where $\Psi_n(c\Gamma) = c^n\Psi_n(\Gamma)$ for $c\in\Bbbk$. By the definition of the Bell polynomials, we have

$$e^{\Phi^{\Psi_{\mathbf{P}^n}}(\Gamma)} = \exp\left(\sum_{n\geq 1}\frac{1}{n!}\phi_n^{\Psi_{\mathbf{P}^n}}(\Gamma,\cdots,\Gamma)\right) = 1+\sum_{n\geq 1}\frac{1}{n!}B_n(\phi_1^{\Psi_{\mathbf{P}^n}}(\Gamma),\cdots,\phi_n^{\Psi_{\mathbf{P}^n}}(\Gamma,\cdots,\Gamma)).$$

This expansion has an advantage that

$$B_n\left(\phi_1^{\Psi_{\mathbf{P}^n}}(c\cdot\Gamma),\cdots,\phi_n^{\Psi_{\mathbf{P}^n}}(c\cdot\Gamma,\cdots,c\cdot\Gamma)\right) = c^n\cdot B_n\left(\phi_1^{\Psi_{\mathbf{P}^n}}(\Gamma),\cdots,\phi_n^{\Psi_{\mathbf{P}^n}}(\Gamma,\cdots,\Gamma)\right).$$

for $c\in\Bbbk$. Therefore we get the expansion of desired form:

$$\Psi_X(\lambda) = \Psi_{\mathbf{P}^n}(\lambda) + \sum_{m\geq 1}\sum_{\substack{j+k=m\\ j,k\geq 0}}\frac{1}{j!k!}B_j(\phi_1^{\Psi_{\mathbf{P}^n}}(\Gamma),\cdots,\phi_j^{\Psi_{\mathbf{P}^n}}(\Gamma,\cdots,\Gamma))\cdot\phi_{k+1}^{\Psi_{\mathbf{P}^n}}(\Gamma,\cdots,\Gamma,\lambda).$$

□

4. Appendix

In this section, we give a computation of the cohomology of $H^m(\mathcal{A},K_S)$ for all $m\in\mathbb{Z}$, where $(\mathcal{A},\cdot,K_S)$ is a BV algebra over $\mathbb{F}_q$. This will be done by comparing $(\mathcal{A},\cdot,K_S)$ with the algebraic Dwork complex (algebraic twisted de-Rham complex) over $\mathbb{F}_q$ in [1].[11] For a given cochain complex $(\mathcal{C},K)$, we will use two notations interchangeably:

$$H^i(\mathcal{C},K) = H^i_K(\mathcal{C}), \quad i\in\mathbb{Z}.$$

By additive quantum number we mean a rule of assigning an integer to each monomial in the BV algebra $\mathcal{A}=\mathcal{A}_S$ over $\mathbb{F}_q$ such that the number is additive when two monomials are multiplied. We are going to introduce three additive quantum number; ghost number, charge and weight.

We assign $gh(z_\mu)=0$ and $gh(\eta_\mu)=-1$. We have the ghost number decomposition of $\mathcal{A}$;

$$\mathcal{A} = \bigoplus_{i\in\mathbb{Z}}\mathcal{A}^i.$$

In fact, the ghost number is the cohomology grading and $\mathcal{A}$ is super-commutative with respect to the ghost numbers; $xy = (-1)^{gh(x)gh(y)}yx$ where x,y are homogeneous polynomials in $\mathcal{A}$. Sometimes we abbreviate $|x| = gh(x)$. For each z_μ assign a non-zero integer $ch(z_\mu)$ called charge of z_μ as follows:

$$ch(z_i) = ch(y_i) = -d_i, \quad \text{for } i=1,\cdots,k, \quad ch(z_i) = ch(x_{N-k-1}) = 1, \quad \text{for } i=k+1,\cdots,N.$$

Also assign $ch(\eta_\mu) := -ch(z_\mu)$. Define the background charge c_X by

$$c_X := -\sum_\mu ch(z_\mu) = \sum_{i=1}^k d_i - (n+1)$$

[11] The cohomology $H^m(\widetilde{\mathcal{A}(b)},\tilde{K}_S)$ for all $m\in\mathbb{Z}$ can be also computed by using the π-adic filtration structure in the subsection 3.2 and the appendix of [2], which summarizes basics results on lifting cohomology from characteristic p to characteristic zero.

Note that $c_X = 0$ means that X is Calabi-Yau. Associated with $ch(z_\mu)_{\mu=1,\cdots,N}$ we define the Euler vector

$$E_{ch} = \sum_{\mu=1}^{N} ch(z_\mu) z_\mu \frac{\partial}{\partial z_\mu} + \sum_{\mu=1}^{N} ch(\eta_\mu)\eta_\mu \frac{\partial}{\partial \eta_\mu}.$$

For any monomial $M \in \mathcal{A}$ we have $E_{ch}(M) = ch(M)M$. Hence we have the charge decomposition of $\mathcal{A}$;

$$\mathcal{A} = \bigoplus_{\lambda\in\mathbb{Z}} \mathcal{A}_\lambda,$$

where $\mathcal{A}_\lambda$ is the $\mathbb{C}$-subspace of $\mathcal{A}$ generated by the monomials of charge λ. We also introduce the notion of weight such that $wt(y_\ell) = 1$ for all $\ell = 1, \cdots, k$ and $wt(x_i) = 0$ for $i = 0, 1, \cdots, n$. Define $wt(\eta_\mu) = 1 - wt(z_\mu)$ for all $\mu = 1, \cdots, N$. Associated with the weight we define the Euler vector

$$E_{wt} = \sum_{\mu=1}^{N} wt(z_\mu) z_\mu \frac{\partial}{\partial z_\mu} + \sum_{\mu=1}^{N} wt(\eta_\mu)\eta_\mu \frac{\partial}{\partial \eta_\mu} = \sum_{\ell=1}^{k} y_\ell \frac{\partial}{\partial y_\ell} + \sum_{\ell=k+1}^{N} \eta_\ell \frac{\partial}{\partial \eta_\ell}.$$

For any monomial $M \in \mathcal{A}$ we have $E_{wt}(M) = wt(M)M$. Hence we have the weight decomposition of $\mathcal{A}$;

$$\mathcal{A} = \bigoplus_{w\geq 0} \mathcal{A}_{(w)},$$

where $\mathcal{A}_{(w)}$ is the $\mathbb{C}$-subspace of $\mathcal{A}$ generated by the monomials of weight w. Note that $\mathcal{A} = A[\underline{\eta}]$, where $\eta_\mu\eta_\nu = -\eta_\nu\eta_\mu$, and

$$gh(z_\mu) + gh(\eta_\mu) = -1, \quad ch(z_\mu) + ch(\eta_\mu) = 0, \quad wt(z_\mu) + wt(\eta_\mu) = 1.$$

We then have decomposition

$$\mathcal{A} = \bigoplus_{-N\leq j\leq 0} \bigoplus_{\lambda\in\mathbb{Z}} \bigoplus_{w\geq 0} \mathcal{A}^j_{\lambda,(w)},$$

where $u \in \mathcal{A}^j_{\lambda,(w)}$ if and only if $E_{ch}(u) = \lambda \cdot u$, $E_{wt}(u) = w \cdot u$, and $gh(u) = j$. Note that $S(\underline{z}) \in \mathcal{C}^0_{0,(1)}$, and Q_S preserves ch and wt though Δ decreases wt by 1. Also note that K preserves the charge grading. Hence we have the following proposition.

Proposition 4.1. For each charge $\lambda \in \mathbb{Z}$ and weight $w \in \mathbb{Z}$ we have a cochain subcomplex $(\mathcal{A}_\lambda, K)$ and $(\mathcal{A}_{\lambda,(w)}, Q)$.

It follows that the cohomology $H = H_K(\mathcal{A})$ of the cochain complex $(\mathcal{A}, K) = (\mathcal{A}, K_S)$ has a charge decomposition

$$H = \oplus_{\lambda\in\mathbb{Z}} H_\lambda, \qquad H_\lambda := H_K(\mathcal{A}_\lambda).$$

Associated with E_{ch} we define

$$R := \sum_\mu ch(z_\mu) z_\mu \eta_\mu \in \mathcal{C}^{-1}_{0,(1)}.$$

Note that, for any $f \in \mathcal{C}$,

$$\delta_R f := \ell_2^K(R, f) = \sum_\mu ch(z_\mu) \left(z_\mu \frac{\partial}{\partial z_\mu} - \eta_\mu \frac{\partial}{\partial \eta_\mu} \right) f$$

Hence for any $f \in \mathcal{C}_\lambda$ we have $\delta_R f = \lambda f$. Note also that $QR = 0$, while

$$KR = -c_X, \qquad c_X = \text{the background charge} = \sum_{i=1}^{k} d_i - (n+1).$$

Proposition 4.2. The cohomology $H = H_{K_S}(\mathcal{A})$ is concentrated on charge c_X, i.e. $H = H_{c_X}$.

Proof. From

$$\begin{aligned} K(f \cdot R) &= \ell_2^K(R, f) + KR \cdot f + Kf \cdot R \\ &= \delta_R f - c_X f + Kf \cdot R \\ &= (\lambda - c_X) f + Kf \cdot R \text{ (if } f \in \mathcal{A}_\lambda) \end{aligned}$$

we see that any $f \in \mathcal{C}_\lambda \cap \operatorname{Ker} K$ belongs to $\operatorname{Im} K$ unless $\lambda = c_X$, since $(\lambda - c_X) f = K(f \cdot R)$. □

There also exists the algebraic Dwork complex over $\mathbb{F}_q$. Consider the de Rham complex $\Omega^\bullet_{\mathbb{F}_q[\underline{z}]/\mathbb{F}_q}$ and define the boundary map ∂_S on $\Omega^\bullet_{\mathbb{F}_q[\underline{z}]/\mathbb{F}_q}$ by $\partial_S(\omega) = dS \wedge \omega$ where $S(\underline{z}) = \sum_{\ell=1}^k y_\ell \cdot G_\ell(\underline{x})$. Let $D_S = d + \partial_S$. In [1], Adolphson and Sperber introduced the bigrading $(\deg_1, \deg_2)$ on $\Omega^\bullet_{\mathbb{F}_q[\underline{z}]/\mathbb{F}_q}$ as follows;

$$\deg_1(x_i) = \deg_1(dx_i) = 1, \quad i = 0, \cdots, n, \qquad \deg_2(x_i) = \deg_2(dx_i) = 0, \quad i = 0, \cdots, n,,$$
$$\deg_1(y_i) = \deg_1(dy_i) = -d_i, \quad i = 1, \cdots, k, \qquad \deg_2(y_i) = \deg_2(dy_i) = 1, \quad i = 1, \cdots, k.$$

Then $(\Omega^\bullet_{\mathbb{F}_q[\underline{y},\underline{x}]/\mathbb{F}_q}, \partial_S)$ is a bigraded cochain complex of bidegree $(0,1)$. For $(u,v) \in \mathbb{Z}^2$, let us denote by $\Omega^{s,(u,v)}_{\mathbb{F}_q[\underline{y},\underline{x}]/\mathbb{F}_q}$ the submodule of homogeneous elements of bidegree (u,v) in $\Omega^s_{\mathbb{F}_q[\underline{y},\underline{x}]/\mathbb{F}_q}$.

Lemma 4.3. We have the following relationship between $(\Omega^\bullet_{\mathbb{F}_q[\underline{z}]/\mathbb{F}_q}, D_S)$ and $(\mathcal{A}, K_S)$;

(a) For each $m \in \mathbb{Z}$, if we define a $\mathbb{F}_q$-linear map $J : (\Omega^m_{\mathbb{F}_q[\underline{z}]/\mathbb{F}_q}, D_S) \to (\mathcal{A}^{m-N}, K_S)$ by sending $dz_{i_1} \cdots dz_{i_m}$ to $(-1)^{i_1+\cdots+i_m-m}(\cdots \hat{\eta_{i_1}} \cdots \hat{\eta_{i_m}} \cdots)$ for $1 \le i_1 < \cdots < i_m \le N = n+k+1$ and extending it $\mathbb{F}_q$-linearly, then $J \circ D_S = K_S \circ J$ and J induces an isomorphism

$$H^m_{D_S}(\Omega^\bullet_{\mathbb{F}_q[\underline{y},\underline{x}]/\mathbb{F}_q}) \simeq H^{m-(n+k+1)}_{K_S}(\mathcal{A}^\bullet).$$

for every $m \in \mathbb{Z}$.

(b) The map J induces a $\mathbb{F}_q$-linear map from $\Omega^{m,(u,v)}_{\mathbb{F}_q[\underline{z}]/\mathbb{F}_q}$ to $\mathcal{A}^{m-N}_{S,c,(w)}$ where $c = u + c_X$ and $w + m - N = v - k$.

(c) The map J satisfies that $J \circ \partial_S = Q_S \circ J$ and $J \circ d = \Delta \circ J$.

Proposition 4.4. Assume that $n > k \ge 1$.[12] Then we have the following description of the total cohomology of $(\mathcal{A}, K_S)$;

$$H^m_{K_S}(\mathcal{A}) = 0, \quad \text{for} \quad m \ne k-n-1, -1, 0,$$
$$H^{-1}_{K_S}(\mathcal{A}) \simeq H^0_{K_S}(\mathcal{A}), \quad \text{and} \quad H^{k-n-1}_{K_S}(\mathcal{A}) \simeq \mathbb{F}_q.$$

Proof. We consider the increasing weight filtration $F^\bullet_{wt}\mathcal{A}$ defined by

$$F^j_{wt}\mathcal{A} = \bigoplus_{0 \le w \le j} \mathcal{A}_{S,(w)}.$$

This follows from a spectral sequence associated to the filtered cochain complex $(F^\bullet_{wt}\mathcal{A}, K_S)$ and Theorem 1.6 in [1]. We shift the degree of $(\mathcal{A}, K) = (\mathcal{A}, K_S)$ to consider $\mathcal{C} = \mathcal{A}[-N]$ with $N = n+k+1$ so that $\mathcal{C}^i = \mathcal{A}^{i-N}$ for each $i \in \mathbb{Z}$. [13] Then we have a cochain complex $(\mathcal{C}, K)$;

$$0 \longrightarrow \mathcal{C}^0 \xrightarrow{K} \mathcal{C}^1 \xrightarrow{K} \cdots \xrightarrow{K} \mathcal{C}^{n+2} \longrightarrow 0$$

We define a filtered complex as follows;

$$\mathcal{C} =: F^0\mathcal{C} \supset F^1\mathcal{C} \supset \cdots \supset F^{n-1}\mathcal{C} \supset F^n\mathcal{C} = \{0\}$$

where the decreasing filtration is given by the weights

$$F^i\mathcal{C} = \bigoplus_{k \le n-1-i} \mathcal{C}_{(k)}, \quad i \ge 1.$$

The associated graded complex to a filtered complex $(F^p\mathcal{C}, K = K_S)$ is the complex

$$Gr\mathcal{C} = \bigoplus_{p \ge 0} Gr^p\mathcal{C}, \quad Gr^p\mathcal{C} = \frac{F^p\mathcal{C}}{F^{p+1}\mathcal{C}},$$

where the differential is the obvious one induced from K. We observe that this differential d_0 is, in fact, induced from $Q = Q_S$, because we have that $K = \Delta + Q$, $\Delta : F^p\mathcal{C} \to F^{p+1}\mathcal{C}$ and $Q : F^p\mathcal{C} \to F^p\mathcal{C}$. This will imply that $E_1^{p,q} = H_Q^{p+q}(Gr^p\mathcal{C})$.

The filtration $F^p\mathcal{C}$ on $\mathcal{C}$ induces a filtration $F^pH_K(\mathcal{C})$ on the cohomology $H_K(\mathcal{C})$ by

$$F^pH^q_K(\mathcal{C}) = \frac{F^pZ^q}{F^pB^q},$$

[12] If $n = k$, then the same results holds except $\dim_{\mathbb{F}_q} H^{-1}_{K_S}(\mathcal{A}) = \dim_{\mathbb{F}_q} H^0_{K_S}(\mathcal{A}) + 1$; see (1.12), [1].

[13] The reason for ghost number shifting is to get a spectral sequence in the first quadrant

where $Z^q = \ker(K : \mathcal{C}^q \to \mathcal{C}^{q+1})$ and $B^q = K(\mathcal{C}^{q-1})$. The associated graded cohomology is

$$GrH_K(\mathcal{C}) = \bigoplus_{p,q} Gr^p H^q_K(\mathcal{C}), \quad Gr^p H^q_K(\mathcal{C}) = \frac{F^p H^q_K(\mathcal{C})}{F^{p+1} H^q_K(\mathcal{C})}.$$

Then the general theory of filtered complexes implies that there is a spectral sequence $\{E_r, d_r\}$ $(r \geq 0)$ with

$$E_0^{p,q} = \frac{F^p\mathcal{C}^{p+q}}{F^{p+1}\mathcal{C}^{p+q}}, \quad E_1^{p,q} = H^{p+q}(Gr^p\mathcal{C}), \quad E_\infty^{p,q} = Gr^p H_K^{p+q}(\mathcal{C}).$$

Note that

$$E_r = \bigoplus_{p,q\geq 0} E_r^{p,q}, \quad d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, d_r^2 = 0.$$

In [12], it was shown that the classical to quantum spectral sequence $\{E_r\}$ satisfies

$$E_1^{p,q} \simeq Gr^p H_Q^{p+q}(\mathcal{C}), \tag{4.1}$$
$$E_2^{p,q} \overset{\simeq}{\Rightarrow} E_\infty^{p,q} = Gr^p H_K^{p+q}(\mathcal{C}).$$

In particular, $\{E_r\}$ degenerates at E_2. Then we have

$$H_K^p(\mathcal{C}) = H_{K_S}^{p-N}(\mathcal{A}), \quad Gr^p H_K^{p+q}(\mathcal{C}) = Gr^p H_{K_S}^{p+q-N}(\mathcal{A}), \quad p, q \geq 0.$$

Note that Proposition 3.5 implies that $H_Q^p(\mathcal{C}) \simeq H_{Q_S}^p(\Omega^\bullet_{\mathbb{F}_q[\underline{z}]/\mathbb{F}_q})$ for $p \geq 0$ and Proposition 4.2 says that $H_{K_S}^q(\mathcal{A}) = H_{K_S}^q(\mathcal{A}_{S,c_X})$ for $q \leq 0$. Now it is easy to see that Theorem 1.6 in [1] implies the desired result combined with Proposition 4.2, (4.1), and Proposition 3.5. The (1.10) and (1.11) of Theorem 1.6, [1] can be interpreted as follows; unless p divides $d_1 \cdots d_k$, $n+k$ is odd, and $v = \frac{n+k+1}{2}$ or $v = \frac{n+k+1}{2} - 1$, then

$$\dim_{\mathbb{F}_q} H_{Q_S}^{-1}(\mathcal{A}_{c_X,(v)}) = \dim_{\mathbb{F}_q} H_{Q_S}^0(\mathcal{A}_{c_X,(v)});$$

p divides $d_1 \cdots d_k$, $n+k$ is odd, and $v = \frac{n+k+1}{2}$ or $v = \frac{n+k+1}{2} - 1$, then

$$\dim_{\mathbb{F}_q} H_{Q_S}^{-1}(\mathcal{A}_{c_X,(v)}) = \dim_{\mathbb{F}_q} H_{Q_S}^0(\mathcal{A}_{c_X,(v)}) + \begin{cases} 1 & \text{if } v = \frac{n+k+1}{2} - 1, \\ -1 & \text{if } v = \frac{n+k+1}{2}. \end{cases}.$$

Since for each $m \in \mathbb{Z}$

$$\dim_{\mathbb{F}_q} H_{K_S}^m(\mathcal{A}) = \sum_{v=0}^\infty \dim_{\mathbb{F}_q} H_{Q_S}^m(\mathcal{A}_{c_X,(v)}),$$

(which is actually a finite sum) the desired result $\dim_{\mathbb{F}_q} H_{K_S}^{-1}(\mathcal{A}) = \dim_{\mathbb{F}_q} H_{K_S}^0(\mathcal{A})$ follows. In fact, the map from $H_{Q_S}^0(\mathcal{A}_{c_X,(v)})$ to $H_{Q_S}^{-1}(\mathcal{A}_{c_X,(v+1)})$ is given by $[f] \mapsto [R \cdot f]$ where $f \in \mathcal{A}_{K_S}^0$ and $R = \sum_\mu ch(q_\mu) q_\mu \eta_\mu \in \mathcal{C}_0^{-1}$ (compare this map with the map θ of Theorem 4.4 of [1]). The cohomology $H_{K_S}^{k-n-1}(\mathcal{A})$ is generated by

$$[J\,(dy_1 \wedge \cdots \wedge dy_k \wedge dG_1 \wedge \cdots \wedge dG_k)]$$

where J is the map in Proposition 3.5. □

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