CONTINUED FRACTION EXPANSIONS IN THE REAL QUADRATIC FIELD OF DISCRIMINANT FIVE AND AN INVARIANT MEASURE OF GAUSS–KUZMIN TYPE IN DIMENSION TWO

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ABSTRACT. We investigate a two-dimensional dynamical system which models continued fraction expansions with coefficients in the real quadratic field of discriminant five. A Markov partition is exhibited, which we use to analyze spectral properties of the transfer operator. As a consequence, we show that it has a unique invariant probability measure whose density function is continuous with respect to the Lebesgue measure.

1. INTRODUCTION

Gauss, in his letter to Laplace, introduced the probability measure

$$\mu_G := \frac{dx}{(\log 2)\,(1+x)} \tag{1.1}$$

on the unit interval $[0,1]$. He postulated that when a typical real number $x \in [0,1]$ is expanded as a regular continued fraction, namely an iterated fraction of the form

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}, \qquad a_n \in \mathbb{Z}_{>0},\ n = 1, 2, \cdots, \tag{1.2}$$

the integral $\int_0^t \mu_G$ would be the likelihood for a truncation of (1.2) belonging to the interval $[0,t]$. Kuzmin [8] made a decisive progress by confirming the prediction of Gauss and we call μ_G the Gauss–Kuzmin distribution. See Knuth [6, p.362-366] for more details and Baladi–Vallée [1] for a modern refinement.

In this article, we take[1] θ to be an algebraic number satisfying $\theta^2 = \theta + 1$ and investigate continued fractions of the form

$$z = \frac{\epsilon_1}{a_1 + \frac{\epsilon_2}{a_2 + \cdots}} \qquad a_n, \epsilon_n \in \mathbb{Z}[\theta],\ n = 1, 2, \cdots. \tag{1.3}$$

Our main result, which we will be able to state precisely only after defining necessary terms, establishes that there is an associated probability measure playing the role of μ_G. We note that Hensley [5, § 5.7] treated the continued

Date: Jan 2025.

[1] $\mathbb{Q}(\theta)$ is the unique real quadratic field of discriminant five.

fractions of the form (1.2) where a_n's are Gaussian integers, and proved the existence of an invariant probability measure. We also note that Nakada and his collaborators [3] gave a different proof.

The continued fraction (1.3) shall be regarded as an expansion of an element $z \in \mathbb{R}^2$ in the following sense. Recall that θ satisfies $\theta^2 = \theta + 1$. The polynomial equation $t^2 = t + 1$ has two real zeros α and β, which we order so that $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Represent an element in $\mathbb{Q}(\theta)$ in the form $m + n\theta$ with $m, n \in \mathbb{Q}$, and define the embedding $\mathbb{Q}(\theta) \hookrightarrow \mathbb{R}^2$ by sending $m + n\theta$ to $(m + n\alpha, m + n\beta)$. The embedding extends to an isomorphism $\mathbb{Q}(\theta) \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^2$ between rings. Now we interpret (1.3) as a sequences of elements in $\mathbb{R}^2$ converging to z.

Now we turn to arithmetic. As we will show later in Lem. 4.2, the expansion we will define in this paper will terminate for all $z \in \mathbb{Q}(\theta)$. The reason for the termination lies in that the ring of integers of $\mathbb{Q}(\theta)$ is Euclidean. Our work suggests that the complexity of the associated the Euclidean algorithm may be analyzed from the perspective of [1], although we do not claim any results in this direction.

To motivate the statement of our main result, we recall that the dynamical system underlying μ_G is the so-called Gauss map; $x \mapsto x^{-1} - \lfloor x^{-1} \rfloor$, where $\lfloor r \rfloor$ for $r \in \mathbb{R}$ denotes the unique integer such that $r - \lfloor r \rfloor \in [0, 1)$. In this paper, we will introduce a domain $I \subset \mathbb{R}^2$ which plays the role of $[0, 1]$ and a family of maps

$$T_d \colon I \to I$$

indexed by integers $d \geq 1$. For each d, the pair (I, T_d) is regarded as a two-dimensional analogue of the interval $[0, 1]$ equipped with the Gauss map. In particular, for each d, the map T_d determines an expansion of an element $z \in \mathbb{R}^2$ as a continued fraction of the form (1.3). See § 2 for the definitions of I and T_d. Here is our main result.

Theorem 1.4. *For $d \geq 3$, the system (I, T_d) has a unique invariant probability measure whose density function is continuous with respect to the Lebesgue measure.*

Remark 1.5. When $d \leq 2$, we are unable to produce such a measure, due to the failure of Proposition 5.6.

We outline the main body of the paper. In § 2-3 we define (I, T_d) and introduce notations for local inverse branches of T_d. The definition of (I, T_d) is combinatorially involved, whose complexity will be justified in the subsequent sections. In § 4-5 we establish basic properties of local inverse branches. The properties rely on the combinatorial structure of the system (I, T_d) and will play a crucial role in the proof of our main theorem. In § 6-7, we prove our main theorem, where key arguments are similar to those of [5, § 5.7].

For a general introduction to Euclidean quadratic fields, we refer the reader to [4]. The authors of [9] revealed a characteristic property of $\mathbb{Q}(\theta)$ among norm Euclidean real quadratic fields.

2. Construction of I and T_d

Recall the embedding $\mathbb{Q}(\theta) \hookrightarrow \mathbb{R}^2$ from the introduction. The image of $\mathbb{Z}[\theta]$ is a lattice in $\mathbb{R}^2$. An important ingredient of our construction is the choice of a particular fundamental domain I for $\mathbb{Z}[\theta]$. The interior of I has three connected components. Its boundary is shown in Figure 1.

Remark 2.1. For an intuitive understanding of I, we relate it to the parallelogram R spanned by $\theta = (\alpha, \beta)$ and $1-\theta = (\beta, \alpha)$. Since $\mathbb{Z}[\theta]$ as a lattice is spanned by θ and $1-\theta$, R is a fundamental domain for $\mathbb{R}^2/\mathbb{Z}[\theta]$. In order to obtain R from I, translate the part of I lying on the second quadrant by θ and translate the part on the fourth quadrant by $1-\theta$. By our reconstruction of R from I, it is clear that I is another fundamental domain. The authors do not know how to generalize this construction to other real quadratic fields.

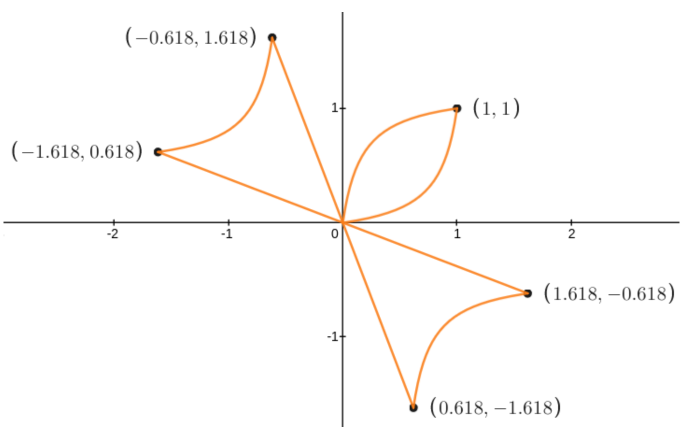


Figure 1. The curve surrounding the region $I \subset \mathbb{R}^2$

We write down explicit inequalities for I. The boundary of I is covered by six algebraic curves. To list their equations, it is convenient to introduce

$$f_1(x,y) := \alpha^2 x + y \tag{2.2}$$

$$f_2(x,y) := \sqrt{5}xy - x + y \tag{2.3}$$

and put $f_3(x,y) := f_2(x-\alpha, y-\beta)$. Then, the equations are $f_i(x,y) = 0$ and $f_i(y,x) = 0$ for $i = 1,2,3$.

Let I^o be the interior of I. We claim that the family

$$I^o + a, \qquad a \in \mathbb{Z}[\theta] \tag{2.4}$$

consists of disjoint open sets whose union is dense in $\mathbb{R}^2$. To see this, consider four subsets

$$A_1 = \{(x,y) \in I^o \colon x > y > 0\} \tag{2.5}$$

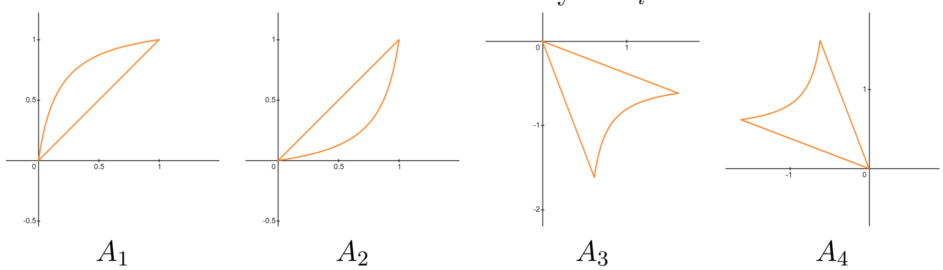
$$A_2 = \{(x,y) \in I^o \colon y > x > 0\} \tag{2.6}$$

$$A_3 = \{(x,y) \in I^o \colon x > 0 > y\} \tag{2.7}$$

$$A_4 = \{(x,y) \in I^o \colon y > 0 > x\}. \tag{2.8}$$

They are shown in Figure 2.

Figure 2. Boundary of A_i



Then, as we mentioned in Remark 2.1, A_1, A_2, $A_3 - \theta + 1$, $A_4 + \theta$ are disjoint and their union is dense in the parallelogram spanned by θ and $1-\theta$. Since θ and $1-\theta$ generate $\mathbb{Z}[\theta]$ as an abelian group, the claim follows.

Throughout, we will fix a subset

$$\mathcal{F} \subset I \tag{2.9}$$

such that $I^o \subset \mathcal{F} \subset I$ and that the family

$$\mathcal{F} + a, \qquad a \in \mathbb{Z}[\theta] \tag{2.10}$$

is a set-theoretic partition of $\mathbb{R}^2$. If $z \in \mathbb{R}^2$, we define $\lfloor z \rfloor_{\mathcal{F}}$ to be the unique element $a \in \mathbb{Z}[\theta]$ with $z - a \in \mathcal{F}$.

3. The dynamical system (I, T_d) and its inverse branches

Let $u \colon I \to \mathbb{Z}[\theta]$ be a function satisfying

$$u(z) = \begin{cases} 1 & \text{if } z \in A_3 \cup A_4, \\ \theta & \text{if } z \in A_2, \text{ and} \\ 1-\theta & \text{if } z \in A_1. \end{cases}$$

The above conditions do not determine $u(z)$ when $z \notin A_i$ for all i. To pin down $u(z)$, we extend $u(z)$ so that it is constant on locally closed subsets. Such an extension is not unique, but we may ignore this indeterminacy in the sequel for two reasons. First, the ambiguity lies on a set of measure zero, in which case it gives rise to a well-defined action on a measure that is absolutely continuous with respect to Lebesgue measure. Second, the local inverse branches we construct in § 3 will be independent from the choice.

For each positive integer d, define u_d by $u_d(z) := d \cdot u(z)$. Finally, we define

$$T_d\colon z \longmapsto u_d(z)z^{-1} - \lfloor u_d(z)z^{-1} \rfloor_{\mathcal{F}}$$

for $z \in I - \{0\}$ and put $T(0) := 0$.

We analyze local inverse branches of T_d. For each $i \in \{1, 2, 3, 4\}$, put $R_i := T_d^{-1}(A_i)$.

Remark 3.1. We give an intuitive interpretation of R_i. For each $i \in \{1, 2, 3, 4\}$, regard A_i as an *ideal tile*. Then, define an open subset $B \subset I$ as a *tile of shape* A_i if T maps B bi-analytically onto A_i.

The next proposition will be important for us.

Proposition 3.2. *Fix $i, j \in \{1, 2, 3, 4\}$. If $X \subset R_i \cap A_j$ is a connected component, then*

$$T_d\colon X \to A_i \tag{3.3}$$

is bi-analytic.

Proof. We factor $T_d|_{R_i \cap A_j}\colon R_i \cap A_j \to A_i$ through

$$B_j := \left\{u_d(z)z^{-1}\colon z \in A_j\right\} \tag{3.4}$$

via the maps

$$\begin{aligned} \iota\colon R_i \cap A_j &\longrightarrow B_j \\ z &\longmapsto u_d(z)z^{-1} \end{aligned}$$

and

$$\begin{aligned} \tau\colon B_j &\longrightarrow A_i \\ w &\longmapsto w - \lfloor w \rfloor. \end{aligned}$$

Since ι is a bi-analytic map from A_j onto B_j, the assertion of the proposition would follow from the following lemma.

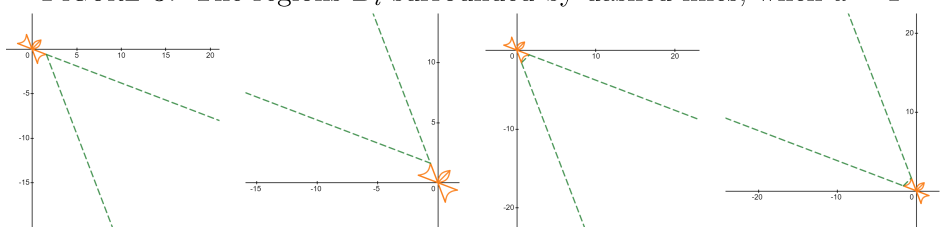
Lemma 3.5. *Let $b \in \mathbb{Z}[\theta]$. If $(b + A_i) \cap B_j$ is non-empty, then $(b + A_i) \subset B_j$.*

Indeed, assuming the lemma, if $X \subset R_i \cap A_j$ is a connected component, then $\iota(X)$ will satisfy $\iota(X) = b + A_i$ for some b. That is, $\iota(X) \to A_i$ is the bi-analytic map induced by adding b. It follows that $X \to A_i$ is a bi-analytic map, because both $\iota|_X\colon X \to \iota(X)$ and $\tau|_{\iota(X)}\colon \iota(X) \to A_i$ are bi-analytic.

proof of lemma. Each i and j can take four values, and there are sixteen cases in total. All cases are similar and proved by inspection. We outline the necessary computation. When $d = 1$, the regions B_i for $i = 1, 2, 3, 4$ is shown in Figure 3. There are two assertions. First, the boundary of B_i is either a line segment or a half-line, all of whose endpoints lie in $\mathbb{Z}[\theta]$. Second, the tangential direction of each line segment or half-line is 1, θ, or θ^{-1}. In order to verify the assertions for a positive integer d, it suffices to verify them for $d = 1$ because the general case is obtained by multiplying d to them, which preserve the two assertions. That is, by the multiplication-by-d map, $\mathbb{Z}[\theta]$ is preserved and the tangential direction of a line remain equal. For $d = 1$, it is straightforward to verify the two assertions by direct calculation. $\square$

The proof of the proposition is complete. $\square$

FIGURE 3. The regions B_i surrounded by dashed lines, when $d = 1$



B_1 B_2 B_3 B_4

In view of Proposition 3.2, define

$$\mathcal{X}_i^j := \{X \subset R_i \cap A_j \colon X \text{ is a connected component.}\} \tag{3.6}$$

for $i, j \in \{1, 2, 3, 4\}$.

If $X \in \mathcal{X}_i^j$, then there exist $\epsilon(X), a(X) \in \mathbb{Z}[\theta]$ such that

$$T_d(z) = \frac{\epsilon(X)}{z} - a(X)$$

for all $z \in X$. The inverse of $T_d|_X$ is given by

$$z \mapsto \frac{\epsilon(X)}{z + a(X)}. \tag{3.7}$$

Since $X \subset A_j$, the above map takes values in A_j for all $X \in \mathcal{X}_i^j$. It results in the family of maps

$$h_X \colon A_i \longrightarrow A_j \tag{3.8}$$

$$z \longmapsto \frac{\epsilon(X)}{z + a(X)} \tag{3.9}$$

indexed by $X \in \mathcal{X}_i^j$.

Definition 3.10. We call h_X in (3.8) a local inverse branch.

We finish this section by recording a lemma for later purposes. For $A \subset \mathbb{R}^2$, let $\bar{A}$ be the closure of A.

Lemma 3.11. *Any local inverse branch h_X extends to a continuous map from $\bar{A}_i$ to $\bar{A}_j$.*

Proof. Each X is the image of the map (3.7), so the closure of X does not meet axes. It follows that the Jacobian of $T_d|_X\colon X \to A_i$ has absolute value greater than a fixed positive constant. Thus, the map $T_d|_X\colon X \to A_i$ extend to a invertible continuous map from $\bar{X}$ to $\bar{A}_i$. Taking the inverse, the assertion of the lemma follows. □

4. The backward orbit of the origin

In this section we analyze the backward orbit of 0 under the map T_d. The aim is to prove the next proposition, which will be used later.

Proposition 4.1. *For any integer $d \geq 1$, the set $\bigcup_{n\geq 1} T_d^{-n}(0)$ is dense in I.*

Proof. An immediate consequence of the Lemma 4.2. □

Lemma 4.2. *Let $d \geq 1$ be any integer. For any $a \in \mathbb{Q}(\theta) \cap I$, there is some $N \geq 0$ such that $T_d^N(a) = 0$.*

Proof. Recall that we chose a fundamental domain $\mathcal{F}$ in (2.9). Note that $T_d^n(z) \in \mathcal{F}$ for any $z \in I$ and any $n \geq 1$. Define the function $\mathbb{Z}[\theta] \xrightarrow{||\cdot||} \mathbb{Z}_{\geq 0}$ by mapping $z = (x, y)$ to $||z|| := |xy|$. Note that $z \in \mathcal{F}$ implies $||z|| < 1$.

Recall from the definition of T_d that if $a = \frac{u}{v}$ with $u, v \in \mathbb{Z}[\theta]$, then we have

$$T_d(a) = \frac{edv - uq}{u}$$

for some $e \in \{1, \theta, 1-\theta\}$ and $q \in \mathbb{Z}[\theta]$. We claim that

$$\max\{||u||, ||v||\} > \max\{||edv - uq||, ||u||\}. \tag{4.3}$$

Indeed, $a \in I$ implies that $\max\{||u||, ||v||\} \geq ||u||$ while $T_d(a) \in \mathcal{F}$ implies that $\max\{||edv - uq||, ||u||\} = ||edv - uq||$ and that $||u|| > ||edv - uq||$. Putting two inequalities together, we obtain (4.3).

Now consider a sequence $(u_n, v_n) \in \mathbb{Z}[\theta]^2$, beginning with $(u_0, v_0) = (u, v)$ such that $v_{n+1} = u_n$ and that $T_d^n(a) = \frac{u_n}{v_n}$ for $n \geq 0$. Then $c_n := \max\{||u_n||, ||v_n||\}$ is a decreasing sequence of nonnegative integers, which implies that $c_N = 0$ for some N. We conclude that $T_d^N(a) = 0$. □

5. Complex analytic extension of local inverse branches

In this section, we assume $h_X\colon A_i \to A_j$ is a local inverse branch. We will simply write $h_X = h$. Then, h is given by

$$h(z) = \frac{\epsilon}{z + a} \tag{5.1}$$

for some $\epsilon, a \in \mathbb{Z}[\theta]$.

By Lemma 3.11, h extends to the closure of A_i. Here we are interested in extending h to some domain in $\mathbb{C}^2$. For $\delta > 0$, let $D_\delta \subset \mathbb{C}$ be the open disc of radius δ centered at the origin. For a subset $A \subset \mathbb{R}^2$, define

$$A^\delta := \left\{(x + x', y + y') \in \mathbb{C}^2 \colon (x, y) \in A \text{ and } x', y' \in D_\delta\right\}. \tag{5.2}$$

Recall that h is of the form $h(x,y) = (h_1(x), h_2(y))$ for single variable functions h_1 and h_2. Let $f^{(r)}$ be the r-th derivative of such a function. Put

$$M_r = \sup_{1\le i\le 4} \sup_{(x,y)\in A_i} \max\left(\left|h_1^{(r)}\right|, \left|h_2^{(r)}\right|\right) \tag{5.3}$$

for $r = 1, 2$.

Proposition 5.4. *Suppose that $M_1 < 1$. Then, for any $\delta > 0$ satisfying $\delta < (1 - M_1)M_2^{-1}$, the map h uniquely extends to a holomorphic function*

$$\tilde{h} \colon A_i^\delta \to A_j^\delta \tag{5.5}$$

whose image is relatively compact in A_j^δ.

Proof. Since h is a rational function, it extends to a function $\tilde{h}\colon A_i^\delta \to \mathbb{C}^2$ if δ is not too large. So we need to verify that its image is contained in A_j^δ when δ is small. By the intermediate value theorem, we have an estimate for any $x' \in D_\delta$

$$\left|\tilde{h}_1(x + x') - \left(h_1(x) + x' h_1^{(1)}(x)\right)\right| < \delta^2 M_2,$$

which yields

$$\left|\tilde{h}_1(x + x') - h_1(x)\right| < \delta^2 M_2 + \delta M_1.$$

Our assumption, $\delta < (1 - M_1)M_2^{-1}$, implies that the right-hand-side of the above inequality is less than δ. The argument for h_2 is identical. This finishes the proof of the proposition. □

The rest of the section regards the hypothesis of Proposition 5.4.

Proposition 5.6. *For any local inverse branch h, we have $M_1 < \frac{\alpha^2}{d}$. In particular, if $d \ge 3$, then for any h we have $M_1 \le \frac{\alpha^2}{3} \approx 0.79$.*

Proof. Recall that h is the inverse of $T_d|_X \to A_i$. Thus, it suffices to find a lower bound for its partial derivatives with respect to x and y. Also recall that $T_d|_X$ is given by $z \mapsto \frac{\epsilon}{z} - a$ for some ϵ and a which deped on X. Since the arguments for the variables x and y are the same, we only treat the derivative with respect to x. Write $\epsilon = (\epsilon_1, \epsilon_2)$ and $a = (a_1, a_2)$. Then, the derivative of $x \mapsto \frac{\epsilon_1}{x} - a_1$ is $\epsilon_1 x^{-2}$. We will bound this by considering two cases; $j = 1, 2$ and $j = 3, 4$. If $j = 1, 2$, then $|\epsilon_1| \ge d\alpha^{-1}$ and $|x| \le 1$. If $j = 3, 4$, then $|\epsilon_1| = d$ and $|x| \le \alpha$. Summing up, we have a uniform bound

$$\left|\epsilon_1 x^{-2}\right| \ge d\alpha^{-2}.$$

The above implies the claim of the proposition. □

6. Transfer operator

Throughout this section, we fix $d \geq 3$. We also choose $\delta > 0$ given by Proposition 5.4.

For each A_i let $C(A_i)$ be the space of continuous real-valued functions on A_i which extend to bounded complex-analytic functions on A_i^δ. We regard it as a Banach space with respect to the supremum norm. For $1 \leq i, j \leq 4$, we define the partial transfer operator $L_i^j \colon C(A_j) \to C(A_i)$ by

$$\left(L_i^j f_j\right)(z) = \sum_{X \in \mathcal{X}_i^j} |J_X(z)| \cdot f_j \circ h_X(z).$$

A key finiteness property regards the convergence of the series

$$\sum_{X \in \mathcal{X}_i^j} |J_X(z)| \tag{6.1}$$

for $z \in \overline{A}_i$.

Proposition 6.2. *For each $\mathcal{X}_i^j$, the series* (6.1) *converges uniformly on $\overline{A}_i$.*

Proof. Put $R_X = \sup_{z \in \overline{A}_i} |J_X(z)|$. It suffices to show the convergence of $\sum_{X \in \mathcal{X}_i^j} R_X$. In view of (3.8), we write h_X in the form

$$h_X(z) = \frac{\epsilon(X)}{z + a(X)}$$

for some $\epsilon(X), a(X) \in \mathbb{Z}[\theta]$. Observe that $\epsilon(X)$ is determined by i and j but does not depend on X. Define

$$\mathbb{Z}[\theta]_i^j := \left\{a(X) \colon X \in \mathcal{X}_i^j\right\}.$$

We enumerate elements of $\mathbb{Z}[\theta]_i^j$ in the following way. If $j = 3$, putting $a = m\theta - n$ for $m, n \in \mathbb{Z}$, the elements of $\mathbb{Z}[\theta]_i^j$ are given by

$$\begin{aligned}
a \in \mathbb{Z}[\theta]_1^3 \quad &\Leftrightarrow \quad d+1 \leq m, 1 \leq n \leq m \\
a \in \mathbb{Z}[\theta]_2^3 \quad &\Leftrightarrow \quad d \leq m, 1 \leq n \leq m \\
a \in \mathbb{Z}[\theta]_3^3 \quad &\Leftrightarrow \quad d \leq m, 0 \leq n \leq m \\
a \in \mathbb{Z}[\theta]_4^3 \quad &\Leftrightarrow \quad d+1 \leq m, 1 \leq n \leq m-1.
\end{aligned}$$

If $j = 1$, we also put $a = m\theta - n$ and obtain a similar description;

$$\begin{aligned}
a \in \mathbb{Z}[\theta]_1^1 \quad &\Leftrightarrow \quad d+1 \leq m, 1 \leq n \leq m-1 \\
a \in \mathbb{Z}[\theta]_2^1 \quad &\Leftrightarrow \quad d \leq m, 1 \leq n \leq m \\
a \in \mathbb{Z}[\theta]_3^1 \quad &\Leftrightarrow \quad d \leq m, 0 \leq n \leq m-1 \\
a \in \mathbb{Z}[\theta]_4^1 \quad &\Leftrightarrow \quad d+1 \leq m, 1 \leq n \leq m-2.
\end{aligned}$$

If $j = 2$ or 4, then we put $a = -m\theta + n$ and obtain similar descriptions;

$$\begin{aligned}
a \in \mathbb{Z}[\theta]^4_1 \quad &\Leftrightarrow \quad d \le m, 1 \le n \le m \\
a \in \mathbb{Z}[\theta]^4_2 \quad &\Leftrightarrow \quad d+1 \le m, 1 \le n \le m \\
a \in \mathbb{Z}[\theta]^4_3 \quad &\Leftrightarrow \quad d+1 \le m, 1 \le n \le m-1 \\
a \in \mathbb{Z}[\theta]^4_4 \quad &\Leftrightarrow \quad d \le m, 0 \le n \le m
\end{aligned}$$

and

$$\begin{aligned}
a \in \mathbb{Z}[\theta]^2_1 \quad &\Leftrightarrow \quad d \le m, 1 \le n \le m-1 \\
a \in \mathbb{Z}[\theta]^2_2 \quad &\Leftrightarrow \quad d+1 \le m, 1 \le n \le m-1 \\
a \in \mathbb{Z}[\theta]^2_3 \quad &\Leftrightarrow \quad d+1 \le m, 0 \le n \le m-2 \\
a \in \mathbb{Z}[\theta]^2_4 \quad &\Leftrightarrow \quad d \le m, 0 \le n \le m-1.
\end{aligned}$$

Write $R_X = R_{m,n}$ when $a(X) = m\theta - n$ or $a(X) = -m\theta + n$. The convergence of $\sum_{X \in \mathcal{X}^j_i} R_X$ is reduced to estimating

$$\sum_{n=l}^{u} R_{m,n} \tag{6.3}$$

where m is sufficiently large and (l, u) is one of boundary conditions listed above, namely $(1, m)$, $(0, m)$, $(1, m-1)$, $(0, m-1)$, $(1, m-2)$, or $(0, m-2)$. In all cases, (6.3) is bounded by cm^{-3} for some constant c. The desired convergence follows. ☐

Proposition 6.4. *The operator L^j_i is compact.*

Proof. Proposition 5.4 allows one to invoke Montel's theorem to obtain the compactness of the operator $f_j \mapsto j_j \circ h_X$. Also, Proposition 5.4 implies that $|J_X(z)|$ is bounded. It follows that each summand of L^j_i is a compact operator. By Proposition 6.2, L^j_i converges. Since the set of compact operators is closed, we conclude that L^j_i is compact. ☐

Summing over all partial operators, we define the transfer operator L as

$$\begin{aligned}
L\colon \prod_{i=1}^{4} C(A_i) &\longrightarrow \prod_{i=1}^{4} C(A_i) \\
(f_j)_j &\longmapsto \left(\sum_{j=1}^{4} L^j_i f_j \right)_i
\end{aligned}$$

and Proposition 6.4 implies that L is compact.

The involution $\iota\colon (x, y) \mapsto (y, x)$ induces isomorphisms $C(A_1) \simeq C(A_2)$ and $C(A_3) \simeq C(A_4)$ by sending f to $f \circ \iota$. If we put

$$\mathfrak{B} = \left\{ (f_i) \in \prod_{i=1}^{4} C(A_i)\colon f_1 = f_2 \circ \iota, f_3 = f_4 \circ \iota \right\},$$

then $\mathfrak{B}$ is a Banach space on which L acts compactly. Let r be the spectral radius of $L\colon \mathfrak{B} \to \mathfrak{B}$.

Theorem 6.5. *The operator L on $\mathfrak{B}$ has r as a simple eigenvalue and the spectral radius of $L - r$ is strictly smaller than r.*

Proof. We will use the criterion by Krasnoselskii [7]. Let $P \subset \mathfrak{B}$ be the subspace consisting of tuples $(f_i)_i$ such that each f_i takes non-negative values. Write $f \geq g$ if $f - g \in P$. Let $v = (v_i)_i \in P$ be the tuple with $v_i \equiv 1$ for all i. To apply the criterion, it suffices to show that the quadruple $(\mathfrak{B}, P, L, v)$ satisfies the properties

1. P is closed under addition and scaling by positive numbers,
2. P is a closed subset of $\mathfrak{B}$ whose interior is non-empty,
3. every $f \in \mathfrak{B}$ can be written as $f = p_1 - p_2$ with $p_1, p_2 \in P$,
4. $L(P) \subset P$, and
5. if $f \in P$ is not zero, then there exist some positive integer n and positive real numbers c_1 and c_2, such that $c_1 v \leq L^n f \leq c_2 v$.

Except for the last one, they are easy to verify. The first property follows directly from the definition. The second follows from observing that the interior of P contains tuples of functions with positive infima. To verify the third property, note that any continuous function g on a compact set can be rewritten as $g = (g + 2M) - 2M$ where M is the maximum of g. The fourth property follows from the positivity of $|J_X|$.

We verify the last property, using a pair of lemmas. For $f = (f_i)_i \in \mathfrak{B}$, put $f(0) = f_1(0) + f_2(0) + f_3(0) + f_4(0)$. Here is our first lemma.

Lemma 6.6. *If $f \in P$ is nonzero, then $(L^n f)(0) > 0$ for some $n \geq 0$.*

Proof. Assume, on the contrary, $(L^n f)(0) = 0$ for all n. It implies that, by the positivity of $|J_X(z)|$, each f_i vanishes on the set $T_d^{-n}(0) \cap A_i$ for all n. By Proposition 4.1, $\bigcup_{n \geq 1} T_d^{-n}(0)$ is dense in I. Since f_i is continuous, this forces f_i to be the zero function. This is a contradiction. □

Here is our second lemma.

Lemma 6.7. *If $f \in P$ and $f(0) > 0$, then there exists some positive c_1 with $c_1 v \leq Lf$.*

Proof. If $f(0) > 0$, then $f_j(0) > 0$ for some j. It suffices to show, for each $i = 1, 2, 3, 4$, there is some positive δ_i such that $(L_i^j f_j)(z) \geq \delta_i$ for all $z \in \overline{A}_i$. Note that, for any $Y \in \mathcal{X}_i^j$, we have

$$(L_i^j f_j)(z) = \sum_{X \in \mathcal{X}_i^j} |J_X(z)| \cdot f_j \circ h_X(z) \geq |J_Y(z)| \cdot f_j \circ h_Y(z)$$

and the infimum of $|J_Y(z)|$ over A_i is strictly positive. Thus it suffices to find some $Y \in \mathcal{X}_i^j$ and some $\epsilon > 0$ such that $f_j \circ h_Y(z) > \epsilon$ for all $z \in A_i$.

To find such Y, choose a small positive ϵ so that $f(z) > \epsilon$ for all $z \in V_s$ for some $s > 0$, where

$$V_s := \{z = (x, y) \in \overline{A}_j : x^2 + y^2 < s\}.$$

Then, $Y \subset V_s$ for all but finitely many $Y \in \mathcal{X}_i^j$. For any such Y, the desired conclusion follows. □

Combining these two lemmas, any nonzero vector $f \in P$ satisfies $c_1 v \leq L^n f$ for some n and positive c_1. On the other hand, by the boundedness of $(L^n f)_i$ for each i, the inequality $L^n f \leq c_2 v$ is met for some positive c_2. □

7. Uniqueness of the invariant measure

In this section we prove Theorem 1.4. So we assume throughout $d \geq 3$. Theorem 1.4 claims that (I, T) has a unique invariant probability measure absolutely continuous with respect to the Lebesgue measure.

For the existence, we use Theorem 6.5. Indeed, let r be the spectral radius of L. Then by Theorem 6.5 there is a unique $\psi \in \mathfrak{B}$ satisfying both $L\psi = r\psi$ and $\int_I \psi d\nu = 1$, where ν denotes the Lebesgue measure. The desired invariant probability measure μ is constructed as $d\mu := \psi d\nu$, in view of the following lemma:

Lemma 7.1. *We have $r = 1$.*

Proof. First we show $r \geq 1$. Letting L^* be the adjoint of L, it suffices to show that the spectrum of L^* contains 1. Indeed, L^* preserves the Lebesgue measure.

To show $r \leq 1$, it suffices to show $||L|| \leq 1$, for some operator norm $||-||$. Take $||-||$ to be the sum of L^1-norms;

$$||f|| := \sum_{i=1}^{4} \int_{A_i} |f_i| d\nu.$$

Then $||Lf|| \leq ||f||$ for any $f \in \mathfrak{B}$ follows from a change-of-variable argument, using $J_X(z)d\nu(z) = d\nu(w)$ for $h_X(z) = w$. □

To prove the uniqueness, we use a standard argument which can be found, for example, in [2, Thm 7.5]. Suppose that $\tilde{\mu}$ is another invariant probability measure whose Radon-Nikodym derivative with respect to ν is f. For any Borel set B we have $\int_{T^{-1}B} f d\nu = \int_B f d\nu$. Then we have $\int_B Lf d\nu = r \int_{T^{-1}B} f d\nu = r \int_B f d\nu$. It implies that $Lf = rf$ holds ν-almost everywhere. Since our space $\mathfrak{B}$ is dense in the L^1-space, r is a simple eigenvalue of L acting on the L^1-space.[2] It implies $f = \psi$ which yields $\mu = \tilde{\mu}$.

[2]This argument appeared in Prop. 7.1 of [2].

Acknowledgement. Junyeong Park was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (RS-2024-00449679) and the National Research Foundation of Korea(NRF) grant funded by the Korea government (MSIT) (RS-2024-00415601). Dohyeong Kim is supported by the NRF grant funded by the Korean government (MSIT) (No. 2020R1C1C1A01006819).

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