TRIPLE SYMBOLS IN ARITHMETIC

DOHYEONG KIM AND MASANORI MORISHITA

Abstract. Triple symbols are arithmetic analogues of the mod n triple linking number in topology, where $n > 1$ is an integer. In this paper, we introduce a cohomological formulation of a mod n triple symbol for characters over a number field containing a primitive n-th root of unity. Our definition is motivated by the arithmetic Chern–Simons theory and in this respect it differs from earlier approaches to triple symbols. We show that our symbol agrees with that of Rédei when $n = 2$ and of Amano–Mizusawa–Morishita when $n = 3$.

1. Introduction

The purpose of this paper is to introduce triple symbols $[\chi_1, \chi_2, \chi_3]_n$ cohomologically, which takes the values in the group μ_n of n-th roots of unity ($n > 1$), for certain mod n Kummer characters χ_i's over a number field F containing μ_n and to show that our construction provides a new description of known cases of triple symbols for $F = \mathbb{Q}$ and $n = 2$ [11] and for $F = \mathbb{Q}(\sqrt{-3})$ and $n = 3$ [2].

The study of triple symbols in arithmetic goes back to the work of Rédei in 1939 [11], which intended to generalize the Legendre symbol and Gauss' genus theory for quadratic fields. For certain prime numbers p_1, p_2 and p_3, Rédei's triple symbol $[p_1, p_2, p_3] \in \{\pm 1\}$ describes the decomposition law in the dihedral extension K_{p_1,p_2} of degree 8, determined by p_1 and p_2. Some variants of the Rédei symbol were also studied in [4, 5, 13] among others. In the late 1990s, the second author interpreted the Rédei symbol as an arithmetic analogue of the mod 2 triple linking number (Milnor invariant) of a link [7, 14] from the viewpoint of arithmetic topology, and also gave a description in terms of the triple Massey product of the étale cohomology of $\mathrm{Spec}(\mathbb{Z}) \setminus \{p_1, p_2, p_3\}$ (cf. [8] and [9, Ch. 9]). However, it remains a problem to extend these constructions over a number field F containing μ_n, because there are cohomological obstructions related to the unit group of F. So far only triple cubic residue symbols in $F = \mathbb{Q}(\sqrt{-3})$ could be well defined [2].

In this paper, we take a different approach to the problem to construct triple symbols over a number field F containing μ_n. We start with three mod n Kummer characters $\chi_i : G \to \mathbb{Z}/n\mathbb{Z}$ ($i = 1, 2, 3$) of the absolute Galois group $G := \mathrm{Gal}(\overline{F}/F)$ rather than three primes of F. Fixing a primitive n-th root of unity ζ, we identify $\mathbb{Z}/n\mathbb{Z}$ with μ_n by $a \bmod n \mapsto \zeta^a$. Let S_i be

---

Date: May 2024.

primes where χ_i is ramified. We assume that S_1, S_2, S_3 are disjoint, that all χ_i's are tame characters, and that the local "linking conditions"

$$[\mathrm{loc}_v^2(\chi_i \cup \chi_j)] = 0$$

holds for $i \neq j$ and all non-archimedean places v of F. Here, for a place v of F, let loc_v^i denote the localization map of continuous group cochains

$$\mathrm{loc}_v^i : C^i(G, \mathbb{Z}/n\mathbb{Z}) \to C^i(G_v, \mathbb{Z}/n\mathbb{Z}),$$

where G_v denotes the local absolute Galois group $\mathrm{Gal}(\overline{F}_v/F_v)$. Now let us take a finite set S of places of F so that S contains $S_1 \cup S_2 \cup S_3$, any place dividing n and all archimedean places of F, and let G_S be the Galois group of the maximal extension of F unramified outside S. Then, using the assumptions and the vanishing of $H^3(G_S, \mathbb{Z}/n\mathbb{Z})$, we find $\eta \in C^2(G, \mathbb{Z}/n\mathbb{Z})$ and $\eta_v \in C^2(G_v, \mathbb{Z}/n\mathbb{Z})$ such that

$$d\eta = \chi_1 \cup \chi_2 \cup \chi_3, \ \ d\eta_v = \mathrm{loc}_v^2(\chi_1 \cup \chi_2 \cup \chi_3),$$

where d denotes the coboundary map. Here η_v is required to be unramified, a condition to be introduced in the section 2. The key idea to detect the triple symbol is to look at the difference of global cochain η and local unramified cochain η_v over $v \in S$. Thus, we set

$$\mathfrak{d}(\chi_1, \chi_2, \chi_3) := \sum_{v \in S} \mathrm{inv}_v(\mathrm{loc}_v^2(\eta) - \eta_v) \in \mathbb{Z}/n\mathbb{Z},$$

where inv_v is the invariant map $H^2(G_v, \mathbb{Z}/n\mathbb{Z}) \overset{\sim}{\to} \mathbb{Z}/n\mathbb{Z}$ of local class field theory for a non-archimedean place v or a real place v with $n = 2$, and $\mathrm{loc}_v = 0$ for other cases. Here, as we will prove in Prop. 3, among other things, that the sum $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ is independent of a choice of S. We then define the triple symbol by

$$[\chi_1, \chi_2, \chi_3]_n := \zeta^{\mathfrak{d}(\chi_1, \chi_2, \chi_3)}.$$

The idea to consider $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ is motivated by the first author's computation of arithmetic Chern-Simons invariants [3].

We briefly discuss the practical aspects about computation of $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ by means of evaluating the sum above. Since its number of terms is the cardinality of S, it is desirable to work with a small S. On the other hand, S cannot be too small otherwise the necessary cochain η will not exist. Theoretically, it is convenient to require that S contains all S_i's, all places dividing n, and all archimedean places, because the combination of them will gaurantee the existence of η for any triple of χ_i's. Practically, it is computationally effecient to choose a smaller S, if possible, for the particular triple of χ_i's in consideration. For this reason, we sometimes relax the condition on S, notably in Prop. 3 and § 3.

The Rédei's symbol is then recovered from ours as follows. Choose χ_i to be the quadratic Kummer character associated to p_i. The conditions imposed on p_1, p_2, p_3 to define Rédei's symbol implies that our conditions on χ_1, χ_2, χ_3 holds true with $n = 2$ and $S = \{p_1, p_2, p_3, 2, \infty\}$. Then we show

$[p_1, p_2, p_3] = [\chi_1, \chi_2, \chi_3]_2$. Similarly, we recover the triple cubic symbol in [2] as a special case of our symbol. These are shown in § 3, 4. Moreover, for these cases, we can show that $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ gives another description of the Massey product $\langle \chi_1, \chi_2, \chi_3 \rangle$ in § 5.

Acknowledgement. This work was started during the second author's stay at Seoul National University in October of 2023. The second author is thankful to the first author and SNU for the support and hospitality. The first author is supported by the NRF[1].

2. TRIPLE SYMBOLS

Let $n \geq 2$ be an integer. Put

$$D_n := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}/n\mathbb{Z} \right\}. \tag{1}$$

When n is unambiguous we simply write $D = D_n$. This group sits in the short exact sequence of groups

$$1 \to Z \to D \to \overline{D} \to 1 \tag{2}$$

where $Z \subset D$ denotes the center and $\overline{D} := D/Z$. Concretely speaking, Z is the subgroup given by $a = c = 0$. We will identify $Z \simeq \mathbb{Z}/n\mathbb{Z}$ via the projection onto the b-component.

A standard argument [12, § I.5.7 Proposition 43] shows that for any profinite group G acting trivially on D, taking continuous cohomology sets associated to (2) yields an exact sequence of pointed sets

$$H^1(G, D) \to H^1(G, \overline{D}) \xrightarrow{\delta} H^2(G, \mathbb{Z}/n\mathbb{Z}). \tag{3}$$

Lemma 1. *A homomorphism $\varphi\colon G \to \overline{D}$ can be lifted to a continuous homomorphism $G \to D$ if and only if $[\delta\varphi] = 0$.*

Proof. Immediate from (3). □

We describe the class $\delta\varphi$ in terms of the cup product. We identify $\overline{D} \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ by sending a coset containing $\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)$ to (a, c).

Lemma 2. *Suppose that $\varphi\colon G \to \overline{D}$ is given by $\varphi(g) = (\chi_1(g), \chi_2(g))$ for some homomorphisms $\chi_1, \chi_2\colon G \to \mathbb{Z}/n\mathbb{Z}$. Then, $\delta\varphi = \chi_1 \cup \chi_2$.*

Proof. For $g \in G$, let $\tilde{\varphi}(g)$ be a lift of $\varphi(g)$ to D. Namely $\tilde{\varphi}(g) = s \circ \varphi(g)$ for some section $s\colon \overline{D} \to D$ of the projection map in (2). Then, there is a

---

[1] National Research Foundation of Korea, No. 2020R1C1C1A01006819

1-cochain $\alpha \in C^1(G, \mathbb{Z}/n\mathbb{Z})$ such that

$$\tilde{\varphi}(g) = \begin{pmatrix} 1 & \chi_1(g) & \alpha(g) \\ 0 & 1 & \chi_2(g) \\ 0 & 0 & 1 \end{pmatrix}.$$

The class $\delta[\varphi]$ is represented by the 2-cocycle $(g_1, g_2) \mapsto \tilde{\varphi}(g_1)\tilde{\varphi}(g_2)\tilde{\varphi}(g_1g_2)^{-1}$. By straightforward computation, we obtain $\tilde{\varphi}(g_1)\tilde{\varphi}(g_2)\tilde{\varphi}(g_1g_2)^{-1} = \chi_1 \cup \chi_2(g_1, g_2) + d\alpha(g_1, g_2)$. This proves the assertion of the lemma. □

Let F be a number field and n be an integer greater than one such that F contains a primitive n-th root of unity. Denote by $\mu_n \subset F$ the subset of all n-th roots of unity. We fix a generator $\zeta \in \mu_n$ and identify $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$ by sending ζ to 1. Now we take $G = \mathrm{Gal}(\overline{F}/F)$ for an algebraic closure $\overline{F}$ of F. For any set S of places of F, let G_S be the quotient of G corresponding to the maximal extension of F unramified outside S. Consider characters

$$\chi_i \colon G \to \mathbb{Z}/n\mathbb{Z}$$

for $i = 1, 2, 3$. Since F contains μ_n, χ_i is a Kummer character associated to some $\alpha_i \in F^\times$, namely, $\zeta^{\chi_i(g)} = g(\sqrt[n]{\alpha_i})/\sqrt[n]{\alpha_i}$ for $g \in G$. Denote by S_i the set of places where χ_i is ramified.

Proposition 1. *Let S be a finite set of places of F such that $S_i \subset S$ for $i = 1, 2, 3$ and that S contains both all places v dividing n and all archimedean places. There is a 2-cochain $\eta \in C^2(G, \mathbb{Z}/n\mathbb{Z})$ such that*

$$d\eta = \chi_1 \cup \chi_2 \cup \chi_3$$

and that η is in the image of the inflation map $C^2(G_S, \mathbb{Z}/n\mathbb{Z}) \to C^2(G, \mathbb{Z}/n\mathbb{Z})$.

Proof. Observe that $\chi_1 \cup \chi_2 \cup \chi_3$ represents a cocycle in $C^3(G_S, \mathbb{Z}/n\mathbb{Z})$. The assertion of the proposition would follow from $H^3(G_S, \mathbb{Z}/n\mathbb{Z}) = 0$. Indeed, this is a standard result [10] about cohomological dimension of G_S. □

For a place v of F, choose an algebraic closure $\overline{F}_v/F_v$ and put $G_v = \mathrm{Gal}(\overline{F}_v/F_v)$. Fix an embeeding $\overline{F} \to \overline{F}_v$ and regard G_v as a subgroup of G. In particular, restriction induces a map between cochains which we denote as

$$\mathrm{loc}_v^i \colon C^i(G, \mathbb{Z}/n\mathbb{Z}) \to C^i(G_v, \mathbb{Z}/n\mathbb{Z})$$

for each $i \geq 0$. By a local trivialization of a cocycle $\varphi \in C^i(G, \mathbb{Z}/n\mathbb{Z})$, we mean a cochain

$$\eta_v \in C^{i-1}(G_v, \mathbb{Z}/n\mathbb{Z})$$

such that

$$d\eta_v = \mathrm{loc}_v^i(\varphi).$$

When v is non-archimedean, we put $G_v^{\mathrm{ur}} := \mathrm{Gal}(F_v^{\mathrm{ur}}/F_v)$ where $F_v^{\mathrm{ur}} \subset \overline{F}_v$ is the maximal unramified subextension of F_v. We say a local trivialization η_v is unramified if η_v is in the image of the map

$$C^{i-1}(G_v^{\mathrm{ur}}, \mathbb{Z}/n\mathbb{Z}) \to C^{i-1}(G_v, \mathbb{Z}/n\mathbb{Z})$$

induced by inflation. If v is archimedean, by convention, a local trivialization is unramified if it is a constant function.

On the other hand, we say a cochain $\eta \in C^{i-1}(G, \mathbb{Z}/n\mathbb{Z})$ is a global trivialization unramified outside S if $d\eta = \varphi$ and η is in the image of the map

$$C^{i-1}(G_S, \mathbb{Z}/n\mathbb{Z}) \to C^{i-1}(G, \mathbb{Z}/n\mathbb{Z})$$

induced by the inflation along the natural surjection $G \to G_S$.

Proposition 2. *Let v be a non-archimdean place of F. Assume that S_1, S_2, S_3 are disjoint and that each χ_i is tame; the wild inertia group inside G_v is contained in the kernel of $\chi_i \circ \mathrm{loc}_v$. Further assume that*

$$[\mathrm{loc}_v(\chi_i \cup \chi_j)] = 0$$

for all $i \neq j$. Then, $\mathrm{loc}_v^3(\chi_1 \cup \chi_2 \cup \chi_3)$ is trivialized by an unramified cochain.

Proof. Put $\psi_i = \mathrm{loc}_v \circ \chi_i$. We first prove the assertion when v is non-archimedean. We do not lose generality by assuming that n is a power of a prime ℓ, because $C^i(-, A \oplus B) \simeq C^i(-, A) \oplus C^i(-, B)$. We claim that $\psi_1 \cup \psi_2 \cup \psi_3 = 0$ as a cochain, or all of ψ_i are unramified. It would imply the assertion of the proposition, since $H^3(G_v^{\mathrm{ur}}, \mathbb{Z}/n\mathbb{Z}) = 0$. To prove the claim, let us assume that one of ψ_i is ramified, say ψ_1. Then, by the assumptions, for $i \in \{2, 3\}$, ψ_i is unramified. We claim that $\psi_1 \cup \psi_2 = 0$ as a cochain. For $i \in \{1, 2\}$, let d_i be the order of ψ_i. Then, ψ_1 is tamely ramified so it is of the form $\psi_1 = \psi_{\mathrm{tm}}^{n/d_1}$ for a tamely ramified character ψ_{tm} of order n. On the other hand, $\psi_2 = \psi_{\mathrm{ur}}^{n/d_2}$ for an unramified character ψ_{ur} of order n. Then, the condition $\psi_1 \cup \psi_2 = 0$ is equivalent to $d_1 d_2 \mid n$, by local class field theory. Since the cup product is bilinear at the level of cochains we have a cochain-level equality $\psi_1 \cup \psi_2 = \frac{n^2}{d_1 d_2}(\psi_{\mathrm{tm}} \cup \psi_{\mathrm{ur}})$. Now $d_1 d_2 \mid n$ implies the aformentioned claim. From this, we conclude that $\psi_1 \cup \psi_2 \cup \psi_3 = 0$ as a cochain.

Now consider the case when v is archimedean. The disjointness of S_i's implies that at most one of ψ_v's is non-trivial. At least two of them are trivial, so we again conclude $\psi_1 \cup \psi_2 \cup \psi_3 = 0$ as a cochain. □

Recall that we assumed $\mu_n \subset F$ and fixed an isomorphism $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$. The local class field theory provides an isomorphism

$$\mathrm{inv}_v \colon H^2(G_v, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$$

for a non-archimedean place v. If v is archimedean we have $H^2(G_v, \mathbb{Z}/n\mathbb{Z}) = 0$ unless v is real and n is even, in which case $\mathrm{inv}_v \colon H^2(G_v, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

In the next proposition, we temporarily relax the condition on S while in Prop.1 the set S was required, among other things, to contain all places dividing n. The advantage will be justified in Rem 2.

Proposition 3. *Let S be any set of places containing all archimedean places and S_i for all i. Suppose that $\chi_1 \cup \chi_2 \cup \chi_3$ admits a global trivialization unramified outside S, say η, and that it admits a local unramified trivialization η_v for every place v of F. The sum*

$$\mathfrak{d}(\chi_1, \chi_2, \chi_3) = \sum_{v \in S} \mathrm{inv}_v \left(\mathrm{loc}_v^2 \, \eta - \eta_v \right) \in \mathbb{Z}/n\mathbb{Z}$$

is independent of the choices of trivializations. Also, $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ is independent of S.

Proof. For two global trivializations η and η', $\eta - \eta'$ is a class in $H^2(G, \mathbb{Z}/n\mathbb{Z})$. By the global reciprocity, the sum $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ remains the same. On the other hand, for a place v and two unramified trivializations η_v and η'_v, the difference $\eta_v - \eta'_v$ is unramified. In particular, the difference is the image of a class of $H^2(G_v^{\mathrm{ur}}, \mathbb{Z}/n\mathbb{Z})$. The group $H^2(G_v^{\mathrm{ur}}, \mathbb{Z}/n\mathbb{Z})$ is trivial because $G_v^{\mathrm{ur}} \simeq \hat{\mathbb{Z}}$ has cohomological dimension one. So $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ is independent of the choices of η and η_v's.

Now we prove the independence of S. It suffices to show that $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ remains the same if we replace S with a larger set S'. Suppose $\chi_1 \cup \chi_2 \cup \chi_3$ admits a global trivialization unramified outside of S, say η. If $S' \supset S$, then η is a global trivialization unramified outside S' because the surjection $G \twoheadrightarrow G_S$ factors through $G \twoheadrightarrow G_{S'}(F)$. We claim that for $v \in S' - S$, we have $\mathrm{inv}_v \left(\mathrm{loc}_v^2 \, \eta - \eta_v \right) = 0$. Indeed, $\mathrm{loc}_v^2 \, \eta$ is in the image of $C^2(G_v^{\mathrm{ur}}, \mathbb{Z}/n\mathbb{Z}) \to G^2(G_v, \mathbb{Z}/n\mathbb{Z})$, because η is in the image of the inflation map $C^2(G_S, \mathbb{Z}/n\mathbb{Z}) \to C^2(G_{S'}(F), \mathbb{Z}/n\mathbb{Z})$. It follows that $\mathrm{loc}_v^2 \, \eta - \eta_v$ is unramified. From $H^2(G_v^{\mathrm{ur}}, \mathbb{Z}/n\mathbb{Z}) = 0$, we conclude $\mathrm{inv}_v \left(\mathrm{loc}_v^2 \, \eta - \eta_v \right) = 0$. We have shown that $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ is independent of S. ☐

Remark 1. Its proof is similar to that for decomposition formula [3, § 4] in the arithmetic Dijkgraaf-Witten theory.

Remark 2. In Prop. 1 we require that S contains all places dividing n in order to ensure the existence of a global trivialization η. However, there might exist a global trivialization η' unramified outside a proper subset S' of S. On the other hand, in Prop. 3 the condition on S is partially relaxed in order to allow such S'. When η' is available, we use it instead of η to obtain a shorter sum.

Recall that we have fixed a primitive n-th root of unity ζ which maps to 1 under the isomorphism $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$.

Definition 1. Suppose that χ_1, χ_2 and χ_3 satisfy the assumptions in Prop. 3. Define

$$[\chi_1, \chi_2, \chi_3]_n := \zeta^{\mathfrak{d}(\chi_1, \chi_2, \chi_3)}. \tag{4}$$

Theorem 1. *Suppose that S_i's are disjoint and that $\mathrm{loc}_v^1 \, \chi_i$ is tame for all i and all v. If $[\mathrm{loc}_v^2 \, (\chi_i \cup \chi_j)] = 0$ for all non-arhimedean v and all $1 \le i < j \le 3$, then (4) is defined.*

Proof. By Proposition 1, global trivialization η exists if S contains both all places dividing n and all archimedean places. On the other hand, by Proposition 2, an unramified trivialization exists at every non-arhimedean v. For an archimdean place v, the disjointness of S_i's imply that at least one of χ_i's is trivial at v. It follows that $\mathrm{loc}_v^3(\chi_1 \cup \chi_2 \cup \chi_3) = 0$ so it is trivialized by an unramified cochain. The remaining independence follows from Proposition 3. ☐

Corollary 1. *For any permuation σ on the set $\{1,2,3\}$ with sign $|\sigma|$, we have*

$$[\chi_1, \chi_2, \chi_3]_n = [\chi_{\sigma(1)}, \chi_{\sigma(2)}, \chi_{\sigma(3)}]_n^{|\sigma|} \tag{5}$$

whenever the triple symbol is defined.

Proof. Since the cup product is alternating, we have

$$\chi_1 \cup \chi_2 \cup \chi_3 = |\sigma| \left(\chi_{\sigma(1)} \cup \chi_{\sigma(2)} \cup \chi_{\sigma(3)}\right).$$

The conditions in Thm. 1 are indifferent to the ordering of the characters, so the left hand side of (5) is defined if and only if the right hand side is defined. If η is a global trivialization for $\chi_1 \cup \chi_2 \cup \chi_3$, then $|\sigma|\eta$ is one for $\chi_{\sigma(1)} \cup \chi_{\sigma(2)} \cup \chi_{\sigma(3)}$. Similarly, if η_v is a local unramified trivialization for $\chi_1 \cup \chi_2 \cup \chi_3$, then $|\sigma|\eta_v$ is one for $\chi_{\sigma(1)} \cup \chi_{\sigma(2)} \cup \chi_{\sigma(3)}$. From this, one concludes (5). ☐

3. Rédei symbols

In [11], Rédei introduced a triple symbol $[d_1, d_2, d_3]$ for non-zero integers $d_i \in \mathbb{Z}$. They are defined only when certain conditions are satisfied among d_i's. Later, variants are studied in [4] and [5].

Our aim in this section is to recall the case when $d_i = p_i$ is a prime, and show that Rédei's symbol agrees with ours. Let p_1, p_2 and p_3 be distinct prime numbers such that $p_i \equiv 1 \bmod 4$ $(i = 1, 2, 3)$. When

$$\left(\frac{p_i}{p_j}\right) = 1$$

for all $1 \le i < j \le 3$, Rédei defined his triple symbol

$$[p_1, p_2, p_3] \in \{\pm 1\}$$

which we reproduce here.

Definition 2. Let p_1, p_2 be distinct prime numbers such that $p_1 \equiv p_2 \equiv 1 \pmod 4$. A Rédei extension associated to p_1 and p_2 is a Galois extension $K/\mathbb{Q}$ such that

1. $K/\mathbb{Q}$ is unramified outside p_1, p_2 and ∞,
2. the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is isomophic to D_2 in (1), the dihedral group of order 8, and
3. the ramification indices of p_1 and p_2 are exactly 2.

Proposition 4 (Rédei and Amano). *Suppose that* $\left(\frac{p_1}{p_2}\right) = 1$. *Then, a Rédei extension exists and is unique. It is given by*

$$K_{p_1,p_2} := \mathbb{Q}(\sqrt{p}_1, \sqrt{p}_2, \sqrt{\beta}) \tag{6}$$

where $\beta = x + y\sqrt{p}$ *satisfies the following conditions:*

(1) $2 \mid y$ *and* $x - y \equiv 1 \bmod 4$,
(2) $x^2 - p_1 y^2 - p_2 z^2 - 0$ *for some* $z \in \mathbb{Z}$, *and*
(3) x, y, z *are relatively prime.*

Proof. See Theorem 1.2 and Theorem 2.1 of [1]. ☐

If p_3 is a prime not dividing $p_1 p_2$ and satisfies $\left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_3}\right) = 1$, then p_3 is totally decomposed in $\mathbb{Q}(\sqrt{p}_1, \sqrt{p}_2)$. Thus, the degree of r in K_{p_1,p_2}, say $f(K_{p_1,p_2}, p_3)$, is either one or two. Rédei's triple symbol is defined as

$$[p_1, p_2, p_3] := \begin{cases} 1 & \text{if } f(K_{p_1,p_2}, p_3) = 1, \\ -1 & \text{if } f(K_{p_1,p_2}, p_3) = 2. \end{cases} \tag{7}$$

We want to recover the Rédei symbol $[p_1, p_2, p_3]$ from Definition 1.

Theorem 2. *Let* p_1, p_2 *and* p_3 *be prime numbers such that* $p_i \equiv 1 \bmod 4$ *(*$i = 1, 2, 3$*) and that* $\left(\frac{p_i}{p_j}\right) = 1$ *(*$1 \leq i < j \leq 3$*). Denote by* $\chi_i : G \to \mathbb{Z}/2\mathbb{Z}$ *the character defined by* $(-1)^{\chi_i(g)} = g(\sqrt{p_i})/\sqrt{p_i}$ *for* $g \in G$. *Then,* $[\chi_1, \chi_2, \chi_3]_2$ *is defined and satisfies*

$$[\chi_1, \chi_2, \chi_3]_2 = [p_1, p_2, p_3].$$

Proof. We first note that $[\chi_1, \chi_2, \chi_3]_2$ is defined by Theorem 1. So it remains to compare it to Rédei's symbol. Let K_{p_1,p_2} be the unique Rédei extension given by (6) in Proposition 4. It gives a homomorphism $G \to D_2$. This lifts the homomorphism $G \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ given by $\chi_1 \times \chi_2$. By Lemma 1, this gives a trivialzation $\epsilon \in C^1(G, \mathbb{Z}/2\mathbb{Z})$ for $\chi_1 \cup \chi_2$ unramified outside $\{p_1, p_2, \infty\}$. Put

$$\eta = \epsilon \cup \chi_3$$

then η is a trivialization of $\chi_1 \cup \chi_2 \cup \chi_3$ unramified outside $S = \{p_1, p_2, p_3, \infty\}$.

Table 1 shows the upshot of local computations that are needed to determine $[\chi_1, \chi_2, \chi_3]_2$. In view of Remark. 2, we note that there are no contributions from the place 2, becauce η is unramified at 2. The first row indicates that K_{p_1,p_2} is ramified at p_1 and p_2 and possibly at ∞. The second row indicates that χ_3 is ramified exactly at p_3 and trivial at either p_1 or p_2. The third row indicates that the local unramified trivializations are taken to be zero whenever $\mathrm{loc}_v^3(\chi_1 \cup \chi_2 \cup \chi_3)$ is identically zero as a cochain. The fourth row indicates that the constraints in the first three columns forces $\mathrm{inv}_v(\mathrm{loc}_v^2(\eta) - \eta_v)$ to be zero except for $v = p_3$. So we obtain $[\chi_1, \chi_2, \chi_3]_2 = (-1)^m$.

To finish the proof, it suffices to express m in terms of the decomposition of p_3 in K_{p_1,p_2} and compare the result to the definition (7) of $[p_1, p_2, p_3]$. Since

v	p_1 or p_2	p_3	∞
$\mathrm{loc}_v^2(\eta)$	ramified	unramified	possibly ramified
$\mathrm{loc}_v^1 \circ \chi_3$	0	ramified	0
η_v	0	0	0
$\mathrm{inv}_v\left(\mathrm{loc}_v^2(\eta) - \eta_v\right)$	0	m	0

TABLE 1. local terms in the case $F = \mathbb{Q}$ and $n = 2$

$\eta = \epsilon \cup \chi_3$ and $\eta_{p_3} = 0$, we need to evaluate $\mathrm{inv}_{p_3}((\mathrm{loc}_{p_3}^1 \epsilon) \cup (\mathrm{loc}_{p_3}^1 \circ \chi_3))$. Put $\psi_3 = \mathrm{loc}_{p_3}^1 \epsilon$. Then ψ_3 is a character of $G_{p_3}^{\mathrm{ur}}$ whose order is at most two. Moreover, it is trivial if and only if p_3 splits completely in K_{p_1,p_2}. Since $\mathrm{loc}_{p_3}^1 \circ \chi_3$ is the Kummer character associated to p_3, the local class field theory implies $\mathrm{inv}_{p_3}(\psi_3 \cup (\mathrm{loc}_{p_3}^1 \circ \chi_3)) = \psi_3(p_3)$. So we conclude $m = 0$ if p_3 splits completely in K_{p_1,p_2}, and $m = 1$ if p_3 has degree two in K_{p_1,p_2}, as desired. $\square$

Cor.1 yields the following reciprocity law of Rédei cohomologically.

Corollary 2. *For a permulation σ on $\{1, 2, 3\}$, we have*

$$[p_1, p_2, p_3] = [p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}].$$

4. TRIPLE CUBIC RESIDUE SYMBOLS IN THE EISENSTEIN FIELD

In this section, let $F = \mathbb{Q}(\zeta_3)$ where ζ_3 is a fixed primitive third root of unity. We recall the cubic triple symbol introduced in [2].

Let $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$ be distinct maximal ideals of $\mathbb{Z}[\zeta_3]$ such that $\mathrm{N}\mathfrak{p}_i \equiv 1 \bmod 9$ $(i = 1, 2, 3)$. Note that $\mathrm{N}\mathfrak{p}_i \equiv 1 \bmod 9$ if and only if there is a prime element $\pi_i \in \mathbb{Z}[\zeta_3]$ such that $\mathfrak{p}_i = (\pi_i), \pi_i \equiv 1 \bmod (3\sqrt{-3})$. In the following, we fix such prime elements π_i's and $\left(\frac{\cdot}{\cdot}\right)_3$ denotes the cubic residue symbol in F. When

$$\left(\frac{\pi_i}{\pi_j}\right)_3 = 1$$

for all $1 \leq i < j \leq 3$, the triple cubic residue symbol

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 \in \mu_3$$

is defined as follows.

Definition 3. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be distinct maximal ideals of $\mathbb{Z}[\zeta_3]$ such that $\mathrm{N}\mathfrak{p}_1 \equiv \mathrm{N}\mathfrak{p}_2 \equiv 1 \pmod 9$. A Rédei type extension K of $F = \mathbb{Q}(\zeta_3)$ associated to $\mathfrak{p}_1, \mathfrak{p}_2$ is a Galois extension such that

1. it is unramified outside $\mathfrak{p}_1$ and $\mathfrak{p}_2$,
2. its Galois group $\mathrm{Gal}(K/F)$ is isomophic to D_3, and
3. its ramification indices in $F_{\mathfrak{p}_1,\mathfrak{p}_2}$ of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are exactly 3.

Theorem 3 (Amano, Mizusawa and Morishita)**.** *Notations being as in Definition 3, a Rédei type extension of F assocciated to $\mathfrak{p}_1, \mathfrak{p}_2$ is unique if it exists. Let $\mathfrak{p}_1 = (\pi_1), \mathfrak{p}_2 = (\pi_2)$ with $\pi_1 \equiv \pi_2 \equiv 1$ mod $(3\sqrt{-3})$. Suppose that $\left(\frac{\pi_1}{\pi_2}\right)_3 = 1$. Then, a Rédei extension exists if π_1 and π_2 are rational primes. It is given by the following form*

$$F_{\mathfrak{p}_1,\mathfrak{p}_2} := F(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta}) \tag{8}$$

where θ is an element of $F(\sqrt[3]{\pi_1})$ which is given explicitly.

Proof. See Theorem 4.1, Corollary 5.9, Theorem 5.7 and Theorem 5.11 of [2] □

Let $F_{\mathfrak{p}_1,\mathfrak{p}_2}$ be the Rédei type extension given by (8) in Proposition 5. Let $\mathfrak{p}_3 = (\pi_3)$ be a maximal ideal of $\mathbb{Z}[\zeta_3]$ such that it is prime to $\mathfrak{p}_1\mathfrak{p}_2$ and $\mathrm{N}\mathfrak{p}_3 \equiv 1 \bmod 9$, equivalently, $\pi_3 \equiv 1 \pmod{3\sqrt{-3}}$. We assume that $\left(\frac{\pi_1}{\pi_3}\right)_3 = \left(\frac{\pi_2}{\pi_3}\right) = 1$. Let $\mathfrak{P}_3$ be a place of $F_{\mathfrak{p}_1,\mathfrak{p}_2}$ lying over $\mathfrak{p}_3$. Since $\mathfrak{P}_3$ is unramified in $F_{\mathfrak{p}_1,\mathfrak{p}_2}/F$, the Artin symbol $\left(\frac{F_{\mathfrak{p}_1,\mathfrak{p}_2}/F}{\mathfrak{P}_3}\right) \in \mathrm{Gal}(F_{\mathfrak{p}_1,\mathfrak{p}_2}/F)$ is defined. Since $\mathfrak{p}_3$ is totally decomposed in $F(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2})$ by the assumption, $\left(\frac{F_{\mathfrak{p}_1,\mathfrak{p}_2}/F}{\mathfrak{P}_3}\right)$ is independent of a choice of $\mathfrak{P}_3$ over $\mathfrak{p}_3$ and so we denote it by $\left(\frac{F_{\mathfrak{p}_1,\mathfrak{p}_2}/F}{\mathfrak{p}_3}\right)$. Then, by Theorem 6.3 of [2], the triple cubic residue symbol is given by

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \left(\left(\frac{F_{\mathfrak{p}_1,\mathfrak{p}_2}/F}{\mathfrak{p}_3}\right)\sqrt[3]{\theta}\right)/\sqrt[3]{\theta}. \tag{9}$$

For our purpose, we adopt the right hand side of (9) as definition of $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3$.

Theorem 4. *Let $\mathfrak{p}_1 = (\pi_1), \mathfrak{p}_2 = (\pi_2)$ and $\mathfrak{p}_3 = (\pi_3)$ be distinct maximal ideals of $\mathbb{Z}[\zeta_3]$ such that $\mathrm{N}\mathfrak{p}_i \equiv 1 \bmod 9$ $(i = 1, 2, 3)$ and that $\left(\frac{\pi_i}{\pi_j}\right)_3 = 1$ $(1 \le i < j \le 3)$. Denote by $\chi_i : G \to \mathbb{Z}/3\mathbb{Z}$ the character defined by $\zeta_3^{\chi_i(g)} = g(\sqrt[3]{\pi_i})/\sqrt[3]{\pi_i}$ for $g \in G$. Then, $[\chi_1, \chi_2, \chi_3]_3$ is defined and satisfies*

$$[\chi_1, \chi_2, \chi_3]_3 = [\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3^{-1}.$$

Proof. Let $L := F(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2})$ and let $\mathfrak{p}'_3$ is a place of L lying below $\mathfrak{P}_3$ and let $\psi_a : \mathrm{Gal}((F_{\mathfrak{p}_1\mathfrak{p}_2})_{\mathfrak{P}_3}/L_{\mathfrak{p}'_3}) \to \mathbb{Z}/3\mathbb{Z}$ be the Kummer character associated to $a \in L^{\times}_{\mathfrak{p}'_3}$, namely, $\zeta_3^{\psi_a(g)} = g(\sqrt[3]{a})/\sqrt[3]{a}$ for $g \in \mathrm{Gal}((F_{\mathfrak{p}_1,\mathfrak{p}_2})_{\mathfrak{P}_3}/L_{\mathfrak{p}'_3})$. Let $\mathrm{inv}_{\mathfrak{p}'_3} : H^2(\mathrm{Gal}((F_{\mathfrak{p}_1,\mathfrak{p}_2})_{\mathfrak{P}_3}/L_{\mathfrak{p}'_3}), \mathbb{Z}/3\mathbb{Z}) \to \mathbb{Z}/3\mathbb{Z}$ be the invariant map of local class field theory given under the identification $\mathbb{Z}/3\mathbb{Z} \simeq \mu_3$ by $1 \mapsto \zeta$. By the known relation between the norm residue symbol and the cup product (Theorem 8.12 of [6]), we have

$$\left(\left(\frac{F_{\mathfrak{p}_1,\mathfrak{p}_2}/F}{\mathfrak{p}_3}\right)\sqrt[3]{\theta}\right)/\sqrt[3]{\theta} = \zeta^{-\mathrm{inv}_{\mathfrak{p}'_3}(\psi_\theta \cup \psi_{\pi_3})}. \tag{10}$$

Note that $\mathfrak{p}_3'$ is degree one prime unramified over $\mathfrak{p}_3$ so $(F_{\mathfrak{p}_1,\mathfrak{p}_2})_{\mathfrak{p}_3'} \simeq F_{\mathfrak{p}_3}$. From this we have

$$\mathrm{inv}_{\mathfrak{p}_3'}\left(\psi_\theta \cup \psi_{\pi_3}\right) = \mathrm{inv}_{\mathfrak{p}_3}\left(\psi_\theta \cup \psi_{\pi_3}\right).$$

We proceed to compare (9) with ours. The argument is similar to that for the Rédei case. The local computation for $[\chi_1, \chi_2, \chi_3]_3$ is given in Table 2. Here the extension $F_{\mathfrak{p}_1,\mathfrak{p}_2}$ gives rise to $\epsilon \in C^2(G, \mathbb{Z}/3\mathbb{Z})$ such that ϵ is unramified outside $\{\mathfrak{p}_1, \mathfrak{p}_2\}$ and that $d(\epsilon \cup \chi_3) = \chi_1 \cup \chi_2 \cup \chi_3$. Since there are no real places, we take $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$. Since $\eta = \epsilon \cup \chi_3$ is unramified at $\sqrt{-3}$, there are no contributions at $\sqrt{-3}$, in view of Remark 2.

v	$\mathfrak{p}_1$ or $\mathfrak{p}_2$	$\mathfrak{p}_3$
$\mathrm{loc}_v^2\left(\eta^{\mathrm{glob}}\right)$	ramified	unramified
χ_3	0	ramified
η_v	0	0
$\mathrm{loc}_v\left(\epsilon \cup \chi_3\right) - \eta_v$	0	$\psi_\theta \cup \psi_{\pi_3}$

Table 2. local terms in the case $F = \mathbb{Q}(\zeta_3)$

Then, except for the term at $\mathfrak{p}_3$, where we have $\mathrm{inv}_{\mathfrak{p}_3}(\psi_\theta \cup \psi_{\pi_3})$, the local terms vanish. In view of (10), we conclude that $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3^{-1} = [\chi_1, \chi_2, \chi_3]_3$. □

Corollary 3. *For a permulation σ on $\{1, 2, 3\}$ with signe $|\sigma|$, we have*

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3] = [\mathfrak{p}_{\sigma(1)}, \mathfrak{p}_{\sigma(2)}, \mathfrak{p}_{\sigma(3)}]^{|\sigma|}.$$

Proof. It is a consequence of our comparison and Cor. 1. □

5. COMPARISON TO THE TRIPLE MASSEY PRODUCTS

In this section, we show that our symbols $[\chi_1, \chi_2, \chi_3]_n$ are expressed in terms of the Massey products of the characters χ_1, χ_2, χ_3 in the cases discussed in the sections 4 and 5. We keep the same notations as in the previous sections.

Let χ_1, χ_2 and χ_3 be the characters $G = \mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}/n\mathbb{Z}$ in the sections 3 and 4. Namely, when $F = \mathbb{Q}$ and $n = 2$, they are the Kummer characters associated to distinct prime numbers p_1, p_2 and p_3 satisfying the conditions $p_i \equiv 1 \mod 4$ $(i = 1, 2, 3)$ and $\left(\frac{p_i}{p_j}\right) = 1$ $(1 \le i < j \le 3)$, which are $(-1)^{\chi_i(g)} = g(\sqrt{p_i})/\sqrt{p_i}$ $(i = 1, 2, 3)$. When when $F = \mathbb{Q}(\zeta_3)$ and $n = 3$, they are the Kummer characters associated to distinct maximal ideals $\mathfrak{p}_1 = (\pi_1), \mathfrak{p}_2 = (\pi_2)$ and $\mathfrak{p}_3 = (\pi_3)$ satisfying the conditions $\mathrm{N}\mathfrak{p}_i \equiv 1 \mod 9$ $(i = 1, 2, 3)$ and $\left(\frac{\pi_i}{\pi_j}\right)_3 = 1$ $(1 \le i < j \le 3)$, which are defined by $\zeta_3^{\chi_i(g)} = g(\sqrt[3]{\pi_i})/\sqrt[3]{\pi_j}$ $(i = 1, 2, 3)$.

As we see in the sections 3 and 4, we have

$$\mathrm{loc}_v^2(\chi_i \cup \chi_j) = 0 \quad (i \neq j)$$

for any place of F. By the global reciprocity,

$$\bigoplus_v \mathrm{loc}_v^2 : H^2(G, \mathbb{Z}/n\mathbb{Z}) \to \bigoplus_v H^2(G_v, \mathbb{Z}/n\mathbb{Z}),$$

where v runs over all places of F, is injective and hence

$$\chi_i \cup \chi_j = 0 \quad (i \neq j)$$

in $H^2(G, \mathbb{Z}/n\mathbb{Z})$. So there are $c_{12}, c_{23} \in C^1(G, \mathbb{Z}/n\mathbb{Z})$ such that

$$\chi_1 \cup \chi_{23} = dc_{12}, \ \ \chi_2 \cup \chi_3 = d\chi_{23}.$$

The Massey product $\langle \chi_1, \chi_2, \chi_3 \rangle$ of χ_1, χ_2, χ_3 is then defined by

$$\langle \chi_1, \chi_2, \chi_3 \rangle := [c_{12} \cup \chi_3 + \chi_1 \cup c_{23}],$$

which is independent of choices of c_{12} and c_{23}.

Theorem 5. *Let v denote p_3 (resp. $\mathfrak{p}_3$) for the case that $n = 2, F = \mathbb{Q}$ (resp. $n = 3, F = \mathbb{Q}(\zeta_3)$), and set $\langle \chi_1, \chi_2, \chi_3 \rangle_v := \mathrm{inv}_v(\mathrm{loc}_v^2(\langle \chi_1, \chi_2, \chi_3 \rangle))$ for simplicity. Then we have*

$$\mathfrak{d}(\chi_1, \chi_2, \chi_3) = \langle \chi_1, \chi_2, \chi_3 \rangle_v.$$

Consequently, we have

$$[p_1, p_2, p_3] = (-1)^{\langle \chi_1, \chi_2, \chi_3 \rangle_{p_3}}, \ \ [\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3] = \zeta_3^{-\langle \chi_1, \chi_2, \chi_3 \rangle_{\mathfrak{p}_3}}.$$

Proof. Since $\mathrm{loc}_v^1(\chi_1) = 0$ by the assumptions, we have

$$\mathrm{loc}_v^2(c_{12} \cup \chi_3 + \chi_1 \cup c_{23}) = \mathrm{loc}_v^1(c_{12} \cup \chi_3).$$

Observing that c_{12} can be taken to be the cochain ϵ in the proofs of Theorem 3 and Theorem 4, we obtain the first assertion. The second assertions follow from Theorem 2 and Theorem 3. □

Remark 3. In [2] and [8], the Rédei symbol and the triple cubic residue symbol were expressed by the Massey products, in a group-theoretic manner, using a (link group like) presentation of the maximal pro-ℓ quotient of G_S and the transgression map, where $\ell = 2$ and $S = \{p_1, p_2, p_3, \infty\}$ for the case of the Rédei symbol or $l = 3$ and $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_0\}$ for the case of the triple cubic residue symbol, where $\mathfrak{p}_0$ is a maximal ideal of $\mathbb{Z}[\zeta_3]$ such that $\mathrm{N}\mathfrak{p}_0 \equiv 4$ or $7 \bmod 9$. Theorem 6 gives a different cohomological approach to the Massey product expression of the triple symbols.

References

[1] Fumiya Amano. On Rédei's dihedral extension and triple reciprocity law. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 90(1):1 – 5, 2014.

[2] Fumiya Amano, Yasushi Mizusawa, and Masanori Morishita. On mod 3 triple Milnor invariants and triple cubic residue symbols in the Eisenstein number field. *Res. Number Theory*, 4(1):Paper No. 7, 29, 2018.

[3] Hee-Joong Chung, Dohyeong Kim, Minhyong Kim, Jeehoon Park, and Hwajong Yoo. Arithmetic Chern-Simons theory II. In *p-adic hodge theory*, Simons Symp., pages 81–128. Springer, Cham, [2020] ©2020.

[4] A. Fröhlich. A prime decomposition symbol for certain non Abelian number fields. *Acta Sci. Math. (Szeged)*, 21:229–246, 1960.

[5] Yoshiomi Furuta. A prime decomposition symbol for a nonabelian central extension which is abelian over a bicyclic biquadratic field. *Nagoya Math. J.*, 79:79–109, 1980.

[6] Helmut Koch. *Galois theory of p-extensions*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002. With a foreword by I. R. Shafarevich, Translated from the 1970 German original by Franz Lemmermeyer, With a postscript by the author and Lemmermeyer.

[7] John Milnor. Isotopy of links. In *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, pages 280–306. Princeton Univ. Press, Princeton, NJ, 1957.

[8] Masanori Morishita. Milnor invariants and Massey products for prime numbers. *Compos. Math.*, 140(1):69–83, 2004.

[9] Masanori Morishita. *Knots and primes: An Introduction to Arithmetic Topology*. Universitext. Springer, London, 2nd edition, 2024. An introduction to arithmetic topology.

[10] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2008.

[11] Ladislaus Rédei. Ein neues zahlentheoretisches Symbol mit Anwendungen auf die Theorie der quadratischen Zahlkörper. I. *J. Reine Angew. Math.*, 180:1–43, 1939.

[12] Jean-Pierre Serre. *Cohomologie Galoisienne*, volume 5 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, fifth edition, 1994.

[13] Hiroshi Suzuki. A tripling symbol for central extensions of algebraic number fields. In *Investigations in number theory*, volume 13 of *Adv. Stud. Pure Math.*, pages 9–24. Academic Press, Boston, MA, 1988.

[14] V. G. Turaev. The Milnor invariants and Massey products. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 66:189–203, 209–210, 1976. Studies in topology, II.

(D.K.) Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Gwanak-ro 1, Gwankak-gu, Seoul, South Korea 08826

Email address, D.K.: `dohyeongkim@snu.ac.kr`

(M.M.) Faculty of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka 819-0395, Japan

Email address, M.M.: `morishita.masanori.259@m.kyushu-u.ac.jp`