BERTRAND'S AND RODRIGUEZ VILLEGAS' CONJECTURE FOR REAL QUARTIC GALOIS EXTENSIONS OF THE RATIONALS

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ABSTRACT. The conjecture due to Bertrand and Rodriguez Villegas asserts that the 1-norm of the nonzero element in an exterior power of the units of a number field has a certain lower bound. For the exterior square case of totally real quartic extensions of the rationals, Costa and Friedman gave a lower bound of 0.802. We prove that the bound can be improved to 1.203 when the extension is further assumed to be Galois.

1. INTRODUCTION

While Dirichlet's theorem determines the rank of the unit group in a given algebraic number field in terms of its Archimedean places, less is known about its geometry when viewed as a lattice in a Euclidean space via the associated logarithm embedding. The resulting lattice and its exterior powers are conjecturally subject to certain numerical constraints [1, 2]. We begin by introducing notation to clarify the nature of these numerical constraints.

Definition 1.1. Let L be a number field and $\mathcal{A}_L$ be the set of its Archimedean places. Then the logarithmic embedding of the units $\mathrm{LOG} : \mathcal{O}_L^* \to \mathbb{R}^{\mathcal{A}_L}$ into a Eucliedean space is defined by

$$(\mathrm{LOG}(\gamma))_v := e_v \log|\gamma|_v \quad \text{where} \quad e_v := \begin{cases} 1 & \text{if } v \text{ is real,} \\ 2 & \text{if } v \text{ is complex} \end{cases}$$

for $\gamma \in \mathcal{O}_L^*$ and $v \in \mathcal{A}_L$. Here, $|\cdot|_v$ is the absolute value associated to v extending the absolute value on $\mathbb{Q}$.

Consider the orthonormal basis $\{\delta^v\}_{v\in\mathcal{A}_L}$ of $\mathbb{R}^{\mathcal{A}_L}$ given by

$$\delta^v_w := \begin{cases} 1 & \text{if } \ w = v, \\ 0 & \text{if } \ w \neq v \end{cases}$$

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for $w \in \mathcal{A}_L$. Let $\mathcal{A}_L^{[j]}$ be the set of subsets of $\mathcal{A}_L$ having cardinality j. For each $I \in \mathcal{A}_L^{[j]}$, fix an ordering $\{v_1, v_2, ..., v_j\}$ of elements of I. Define

$$\delta^I := \delta^{v_1} \wedge \delta^{v_2} \wedge \cdots \wedge \delta^{v_j}$$

to get an orthonormal basis $\{\delta^I\}_{I \in \mathcal{A}_L^{[j]}}$ of $\bigwedge^j \mathbb{R}^{\mathcal{A}_L}$. For $w = \sum_{I \in \mathcal{A}_L^{[j]}} c_I \delta^I \in \bigwedge^j \mathbb{R}^{\mathcal{A}_L}$, define its 1-norm as

$$\|w\|_1 := \sum_{I \in \mathcal{A}_L^{[j]}} |c_I|.$$

Our aim in this paper is to investigate the lower bound of $\|w\|_1$ for nonzero element w of $\bigwedge^j \mathrm{LOG}(\mathcal{O}_L^*)$. For a positive integer n, let L_n denote the set of subfields of $\mathbb{R}$ that are Galois extension of $\mathbb{Q}$ of degree n. For such fields, $\mathrm{rank}(\mathcal{O}_L^*) = n-1$ by Dirichlet's unit theorem. For $1 \le j \le n-1$, define a real number $A_{n,j}$ as

$$A_{n,j} := \inf_{\substack{L \in L_n \\ w \in \bigwedge^j \mathrm{LOG}(\mathcal{O}_L^*) \setminus \{0\}}} \{\|w\|_1\}. \tag{1.1}$$

This constant is directly related to the real Galois extension case of the Bertrand's and Rodriguez Villegas' conjecture. The conjecture stated in [2] is as follows.

Conjecture (Bertrand–Rodriguez Villegas)**.** There exists two absolute constants $c_0 > 0$ and $c_1 > 1$ such that for any number field L and any $j \in \mathbb{Z}_{>0}$, the following inequality holds

$$\|w\|_1 \ge c_0 c_1^j$$

for any nonzero $w \in \bigwedge^j \mathrm{LOG}(\mathcal{O}_L^*) \subset \bigwedge^j \mathbb{R}^{\mathcal{A}_L}$.

Only considering the number fields that are real Galois extensions of $\mathbb{Q}$, a consequence of the conjecture is that

$$A_{n,j} \ge c_0 c_1^j$$

holds, where the c_0, c_1 are the constants satisfying the conjecture. For the specific case of $n = 4, j = 2$, the work of Costa and Friedman [3] implies that

$$A_{4,2} > 2\sqrt{3} \log \left(\frac{1+\sqrt{5}}{2} \right)^2 \approx 0.802,$$

as explained in § 2. To the best of the author's knowledge, this was the largest known lower bound for $A_{4,2}$.

In this paper, we give a larger lower bound for $A_{4,2}$.

Theorem 1.2. *Let L be a real quartic Galois extension of $\mathbb{Q}$. Then*

$$\|w\|_1 > 3\sqrt{3} \log \left(\frac{1+\sqrt{5}}{2} \right)^2 \approx 1.203$$

for any nonzero $w \in \bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$. *In other words,*

$$A_{4,2} > 3\sqrt{3}\log\left(\frac{1+\sqrt{5}}{2}\right)^2 \approx 1.203.$$

As demonstrated in [2], Bertrand–Rodriguez Villegas conjecture is equivalent to Lehmer's conjecture [6] when $j = 1$, and to Zimmert's theorem on regulators [8] when $j = \mathrm{rank}(\mathcal{O}_L^*)$. Hence the case of $n = 4, j = 2$ is neither covered by Lehmer's conjecture nor Zimmert's theorem. While our result only covers this specific case, we obtained a better lower bound, which is 1.5 times the previous known lower bound. In fact, for $L = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ and $w = \pm\mathrm{LOG}(\frac{1+\sqrt{5}}{2}) \wedge \mathrm{LOG}(1+\sqrt{2})$, we have $\|w\|_1 = 8\log(\frac{1+\sqrt{5}}{2})\log(1+\sqrt{2})$ and thus

$$A_{4,2} \leq 8\log(\frac{1+\sqrt{5}}{2})\log(1+\sqrt{2}) \approx 3.3930.$$

Hence our lower bound is no less than a third of an upper bound of $A_{4,2}$.

The proof of the theorem is done separately for the case when $\mathrm{Gal}(L/\mathbb{Q})$ is Klein four-group, and the case when $\mathrm{Gal}(L/\mathbb{Q})$ is cyclic. For the Klein four-group case, we define a subgroup E generated by the units of the quadratic subfields of L. Using the Galois module structure of E, we give a lower bound of $\|w\|_1$ for nonzero element w of $\bigwedge^2 \mathrm{LOG}(E)$ and extend this result to $\mathcal{O}_L^*$. For the cyclic group case, we use the structure of $\mathcal{O}_L^*$ given by Hasse [5] to obtain a basis of $\bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$. We then use the result of Pohst [7] to compute the lower bound.

In § 2 we derive the best lower bound of $A_{4,2}$ known so far, from the work of Costa and Friedman [3]. In the remaining sections, we prove the main theorem to give the better lower bound for $A_{4,2}$. The case when $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the Klein four-group is covered in § 3, and the case when $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the cyclic group is covered in § 4.

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2. The lower bound due to Costa and Friedman

In this section, we derive the largest lower bound of $A_{4,2}$ known so far, to the best of the author's knowledge. Using Pohst's result [7], Costa and Friedman [3] showed that for independent elements $\epsilon_1, ..., \epsilon_j$ of $\mathcal{O}_L^*$, the inequality

$$\|\mathrm{LOG}(\epsilon_1) \wedge \cdots \wedge \mathrm{LOG}(\epsilon_j)\|_2 > \left(\frac{n}{\gamma_j}\log\left(\frac{1+\sqrt{5}}{2}\right)^2\right)^{j/2}$$

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holds for $1 \le j < n$, where $n = [L : \mathbb{Q}]$ and γ_j is Hermite's constant in dimension j. If $j = n - 2 = \mathrm{rank}(\mathcal{O}_L^*) - 1$, every element of $\bigwedge^{n-2} \mathrm{LOG}(\mathcal{O}_L^*)$ can be written as $d \cdot \mathrm{LOG}(\epsilon_1) \wedge \cdots \wedge \mathrm{LOG}(\epsilon_{n-2})$ where $d \in \mathbb{Z}$ and $\mathrm{LOG}(\epsilon_1), ..., \mathrm{LOG}(\epsilon_{n-1})$ are the basis of $\mathrm{LOG}(\mathcal{O}_L^*)$, as shown in the Lemma 28 of [2]. Therefore, for any nonzero element w of $\bigwedge^{n-2} \mathrm{LOG}(\mathcal{O}_L^*)$, the inequality

$$\|w\|_1 \ge \|w\|_2 > \left(\frac{n}{\gamma_j} \log \left(\frac{1+\sqrt{5}}{2} \right)^2 \right)^{j/2}$$

holds. Restricting to the Galois extensions of $\mathbb{Q}$, we have

$$A_{n,n-2} > \left(\frac{n}{\gamma_{n-2}} \log \left(\frac{1+\sqrt{5}}{2} \right)^2 \right)^{\frac{n-2}{2}}$$

and in particular, since $\gamma_2 = \frac{2}{\sqrt{3}}$, we obtain

$$A_{4,2} > 2\sqrt{3} \log \left(\frac{1+\sqrt{5}}{2} \right)^2 \approx 0.802.$$

3. The Klein four-group Case

In this section, we prove the main theorem for the case when $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the Klein four-group. Throughout this section, let L be a real Galois extension of $\mathbb{Q}$ with $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $\mathbb{Q}(\sqrt{d_1}), \mathbb{Q}(\sqrt{d_2}), \mathbb{Q}(\sqrt{d_3})$ be three quadratic subfields of L, where d_1, d_2, d_3 are square-free integers. Then we can write $L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. For $i = 1, 2, 3$, let $\mathrm{Gal}(L/\mathbb{Q}(\sqrt{d_i})) = \{1, \sigma_i\}$. See Figure 1. Let $u_i > 1$ be the fundamental unit of $\mathbb{Q}(\sqrt{d_i})$. Without loss of generality, let $1 < u_1 < u_2 < u_3$. If i and j are distinct integers in $\{1, 2, 3\}$, since $\sqrt{d_j} \notin \mathbb{Q}(\sqrt{d_i})$, $\sigma_i \in \mathrm{Gal}(L/\mathbb{Q}(\sqrt{d_i}))$ sends $\sqrt{d_j}$ to its conjugate $-\sqrt{d_j}$. Thus

$$u_j \sigma_i(u_j) = \mathrm{N}_{\mathbb{Q}(\sqrt{d_j})/\mathbb{Q}}(u_j) = \pm 1. \tag{3.1}$$

Lemma 3.1. *u_1, u_2, u_3 are multiplicatively independent. In other words, if integers m_1, m_2, m_3 satisfy $u_1^{m_1} u_2^{m_2} u_3^{m_3} = 1$, then $m_1 = m_2 = m_3 = 0$.*

Proof. Assume that

$$u_1^{m_1} u_2^{m_2} u_3^{m_3} = 1$$

for some $m_1, m_2, m_3 \in \mathbb{Z}$. Then we have

$$\sigma_1(u_1)^{m_1} \sigma_1(u_2)^{m_2} \sigma_1(u_3)^{m_3} = 1$$

and thus

$$(u_1\sigma_1(u_1))^{m_1} (u_2\sigma_1(u_2))^{m_2} (u_3\sigma_1(u_3))^{m_3} = 1.$$

By (3.1), $u_2\sigma_1(u_2) = \pm 1$ and $u_3\sigma_1(u_3) = \pm 1$. Therefore

$$u_1^{2m_1} = \pm 1$$

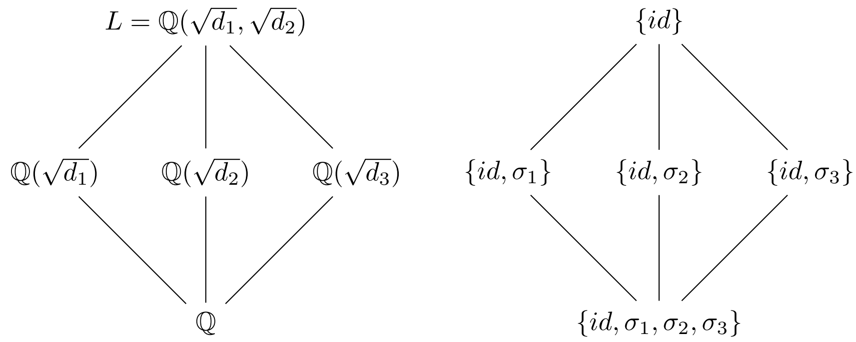

FIGURE 1. Subfields of non-cyclic extension L and corresponding Galois groups

and thus $m_1 = 0$. Similarly, $m_2 = m_3 = 0$. Therefore, u_1, u_2, u_3 are multiplicatively independent. □

Let $E = \{u_1^{m_1} u_2^{m_2} u_3^{m_3} | m_1, m_2, m_3 \in \mathbb{Z}\}$, then we have the following lemma.

Lemma 3.2. *If $u \in \mathcal{O}_L^*$, then $u^2 \in E$.*

Proof. From $u \in \mathcal{O}_L^*$, we have $|N_{L/\mathbb{Q}}(u)| = 1$. The unit u of L and its conjugate $\sigma_1(u)$ are units of L, so $u\sigma_1(u)$ is a unit of L. Since $u\sigma_1(u) = N_{L/\mathbb{Q}(\sqrt{d_1})}(u) \in \mathbb{Q}(\sqrt{d_1})$, it is also a unit of $\mathbb{Q}(\sqrt{d_1})$. Hence $u\sigma_1(u) = \pm u_1^{m_1}$ for some $m_1 \in \mathbb{Z}$. Similarly, $u\sigma_2(u) = \pm u_2^{m_2}$ and $u\sigma_1(u) = \pm u_3^{m_3}$ for some $m_2, m_3 \in \mathbb{Z}$. Then we have

$$u^2 = |u^2 \cdot N_{L/\mathbb{Q}}(u)| = |u^3 \sigma_1(u)\sigma_2(u)\sigma_3(u)| = |\pm u_1^{m_1} u_2^{m_2} u_3^{m_3}| = u_1^{m_1} u_2^{m_2} u_3^{m_3}$$

and therefore $u^2 \in E$. □

We first compute the 1-norm of the elements of $\bigwedge^2 \mathrm{LOG}(E)$. $\mathrm{LOG}(E)$ is generated by $\mathrm{LOG}(u_i)$'s. Using (3.1), we can compute $\mathrm{LOG}(u_i)$'s in terms of u_i's as follows:

$$\begin{aligned} \mathrm{LOG}(u_1) &= (\log(u_1), \ \ \log(u_1), -\log(u_1), -\log(u_1)) \\ \mathrm{LOG}(u_2) &= (\log(u_2), -\log(u_2), \ \ \log(u_2), -\log(u_2)) \\ \mathrm{LOG}(u_3) &= (\log(u_3), -\log(u_3), -\log(u_3), \ \ \log(u_3)). \end{aligned}$$

Here, the order of the basis of $\mathbb{R}^{\mathcal{A}_L}$ is $\delta^{id}, \delta^{\sigma_1}, \delta^{\sigma_2}, \delta^{\sigma_3}$. Then the wedge product of these $\mathrm{LOG}(u_i)$'s are:

$$\begin{aligned} \mathrm{LOG}(u_2) \wedge \mathrm{LOG}(u_3) &= (\ \ \ 0, \ \ \ 0, \ \ 2X_1, \ \ 2X_1, -2X_1, -2X_1) \\ \mathrm{LOG}(u_1) \wedge \mathrm{LOG}(u_3) &= (-2X_2, -2X_2, \ \ 2X_2, -2X_2, \ \ \ 0, \ \ \ 0) \\ \mathrm{LOG}(u_1) \wedge \mathrm{LOG}(u_2) &= (-2X_3, \ \ 2X_3, \ \ \ 0, \ \ \ 0, \ \ 2X_3, -2X_3) \end{aligned}$$

where $X_1 = \log(u_2)\log(u_3), X_2 = \log(u_1)\log(u_3), X_3 = \log(u_1)\log(u_2)$ and the order of the basis of $\bigwedge^j \mathbb{R}^{\mathcal{A}_L}$ is $\delta^{id}\wedge\delta^{\sigma_1}, \delta^{\sigma_2}\wedge\delta^{\sigma_3}, \delta^{id}\wedge\delta^{\sigma_3}, \delta^{\sigma_1}\wedge\delta^{\sigma_2}, \delta^{id}\wedge\delta^{\sigma_2}, \delta^{\sigma_1}\wedge\delta^{\sigma_3}$. Note that $X_1 > X_2 > X_3 > 0$.

For $w \in \bigwedge^2 \mathrm{LOG}(E)$, w can be written as

$$w = n_1 \cdot \mathrm{LOG}(u_2) \wedge \mathrm{LOG}(u_3) + n_2 \cdot \mathrm{LOG}(u_1) \wedge \mathrm{LOG}(u_3) + n_3 \cdot \mathrm{LOG}(u_1) \wedge \mathrm{LOG}(u_2)$$

where n_i's are integers. Its 1-norm is

$$\begin{aligned}\|w\|_1 =& 2(|n_2X_2 + n_3X_3| + |n_2X_2 - n_3X_3| + |n_1X_1 + n_2X_2| + |n_1X_1 - n_2X_2| \\ &+ |n_1X_1 + n_3X_3| + |n_1X_1 - n_3X_3|) \\ =& 4(\max\{|n_2X_2|, |n_3X_3|\} + \max\{|n_1X_1|, |n_2X_2|\} + \max\{|n_1X_1|, |n_3X_3|\})\end{aligned}$$

where the second equality is from the following equation.

$$|X + Y| + |X - Y| = 2\max\{|X|, |Y|\} \quad \forall X, Y \in \mathbb{R}. \tag{3.2}$$

We have

$$\max\{|n_1X_1|, |n_2X_2|\} + \max\{|n_1X_1|, |n_3X_3|\} \geq 2|n_1X_1|$$

and thus

$$4(\max\{|n_2X_2|, |n_3X_3|\} + \max\{|n_1X_1|, |n_2X_2|\} + \max\{|n_1X_1|, |n_3X_3|\}) \geq 8|n_1X_1|$$

holds. Similarly, the above inequality also holds when the right hand side is $8|n_2X_2|$ or $8|n_3X_3|$. Therefore

$$\begin{aligned}\|w\|_1 &= 4(\max\{|n_2X_2|, |n_3X_3|\} + \max\{|n_1X_1|, |n_2X_2|\} + \max\{|n_1X_1|, |n_3X_3|\}) \\ &\geq 8\max\{|n_1X_1|, |n_2X_2|, |n_3X_3|\}\end{aligned}$$

and if $(n_1, n_2, n_3) \neq 0$, then

$$\|w\|_1 \geq 8|X_3| = 8\log(u_1)\log(u_2). \tag{3.3}$$

For the case $L = \mathbb{Q}(\sqrt{2}, \sqrt{5})$, we have $u_1 = \frac{1+\sqrt{5}}{2}$, $u_2 = 1 + \sqrt{2}$, and $u_3 = 3 + \sqrt{10}$. A simple calculation shows that $\sqrt{u_1u_2u_3}$ is in $\mathcal{O}_L^*$, and it is easy to verify that $u_1, u_2, \sqrt{u_1u_2u_3}$ generate $\mathcal{O}_L^*$. Then the group $\bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$ is generated by $\mathrm{LOG}(u_1) \wedge \mathrm{LOG}(u_2)$, $\frac{1}{2}(\mathrm{LOG}(u_1) \wedge \mathrm{LOG}(u_2) + \mathrm{LOG}(u_1) \wedge \mathrm{LOG}(u_3))$ and $\frac{1}{2}(\mathrm{LOG}(u_1) \wedge \mathrm{LOG}(u_2) + \mathrm{LOG}(u_2) \wedge \mathrm{LOG}(u_3))$. Thus for any element w of $\bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$, $2w \in \bigwedge^2 \mathrm{LOG}(E)$. Hence we have

$$\|w\|_1 = \frac{1}{2}\|2w\|_1 \geq 4\log\left(\frac{1+\sqrt{5}}{2}\right)\log(1+\sqrt{2}) \approx 1.697. \tag{3.4}$$

For the case $L = \mathbb{Q}(\sqrt{5}, \sqrt{13})$, $u_1 = \frac{1+\sqrt{5}}{2}, u_2 = \frac{3+\sqrt{13}}{2}, u_3 = 8 + \sqrt{65}$. Similarly, $u_1, u_2, \sqrt{u_1u_2u_3}$ generate $\mathcal{O}_L^*$ and therefore

$$\|w\|_1 = \frac{1}{2}\|2w\|_1 \geq 4\log\left(\frac{1+\sqrt{5}}{2}\right)\log(\frac{3+\sqrt{13}}{2}) \approx 2.300 \tag{3.5}$$

for any nonzero element w of $\bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$.

Now consider the case where $L \neq \mathbb{Q}(\sqrt{2}, \sqrt{5})$ and $L \neq \mathbb{Q}(\sqrt{5}, \sqrt{13})$. By Lemma 3.2, for any $u \in \mathcal{O}_L^*$, $u^2 \in E$ and thus $2\,\mathrm{LOG}(u) \in \mathrm{LOG}(E)$. Hence if nonzero w is in $\bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$, then $4w \in \bigwedge^2 \mathrm{LOG}(E)$. We have

$$\|w\|_1 = \frac{1}{4}||4w||_1 \geq 2\log(u_1)\log(u_2) \tag{3.6}$$

by (3.3). Excluding $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ and $\mathbb{Q}(\sqrt{5}, \sqrt{13})$, $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ has the smallest possible value of $\log(u_1)\log(u_2)$, which is $\log\left(\frac{1+\sqrt{5}}{2}\right)\log(2+\sqrt{3})$. This is from the Lemma 3.3 below and the fact that $\log\left(\frac{1+\sqrt{5}}{2}\right)\log(2+\sqrt{3}) < \log(1+\sqrt{2})\log(\frac{3+\sqrt{13}}{2})$. Hence from (3.6), we see that

$$\|w\|_1 \geq 2\log\left(\frac{1+\sqrt{5}}{2}\right)\log(2+\sqrt{3}) \approx 1.267.$$

With (3.4) and (3.5), we conclude that if $\mathrm{Gal}(L/\mathbb{Q})$ is Klein four-group,

$$\|w\|_1 \geq 2\log\left(\frac{1+\sqrt{5}}{2}\right)\log(2+\sqrt{3}) > 3\sqrt{3}\log\left(\frac{1+\sqrt{5}}{2}\right)^2$$

for any nonzero $w \in \bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$.

Lemma 3.3. *For square-free integer m, let $v_m > 1$ be the fundamental unit of $\mathbb{Q}(\sqrt{m})$. Then $\frac{1+\sqrt{5}}{2}, 1+\sqrt{2}, \frac{3+\sqrt{13}}{2}, 2+\sqrt{3}$ are the four smallest possible values of v_m.*

Proof. First, consider the case $m \equiv 2, 3 \pmod 4$. The fundamental unit v_m is of the form $a + b\sqrt{m}$ where a, b are the smallest positive integers satisfying $a^2 - mb^2 = \pm 1$. If $m \geq 6$, then

$$v_m \geq \sqrt{mb^2 - 1} + b\sqrt{m} \geq \sqrt{m-1} + \sqrt{m} \geq \sqrt{5} + \sqrt{6} > 2 + \sqrt{3}.$$

Now consider the case $m \equiv 1 \pmod 4$. Then v_m is of the form $\frac{a+b\sqrt{m}}{2}$ where a, b are the smallest positive integers satisfying $a^2 - mb^2 = \pm 4$. If $m \geq 17$,

$$v_m \geq \frac{\sqrt{mb^2 - 4} + b\sqrt{m}}{2} \geq \frac{\sqrt{m-4} + \sqrt{m}}{2} \geq \frac{\sqrt{13} + \sqrt{17}}{2} > 2 + \sqrt{3}.$$

Hence $v_m \leq 2 + \sqrt{3}$ only for $m = 2, 3, 5, 13$. For those m, we have $v_5 < v_2 < v_{13} < v_3$. □

4. The Cyclic Group Case

In this section, we prove the main theorem for the case when $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the cyclic group. Throughout this section, let L be a real Galois extension of $\mathbb{Q}$ with $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$. Denote the Galois group as G. Let $\mathrm{Gal}(L/\mathbb{Q})$ be generated by σ. Let l be the unique quadratic subfield of L. See Figure 2. Let $u_l > 1$

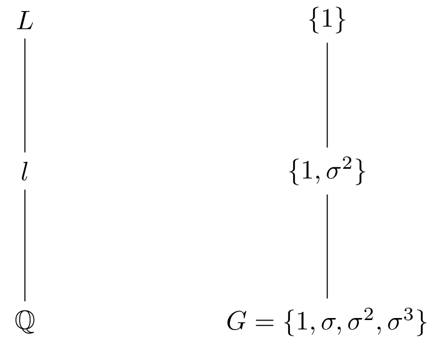

FIGURE 2. Subfields of cyclic extension L and corresponding Galois groups

be the fundamental unit of l. Then $\mathcal{O}_l^* = \{\pm u_l^m \mid m \in \mathbb{Z}\}$. First we define relative units as in [5].

Definition 4.1. If $w \in \mathcal{O}_L^*$ satisfies $N_{L/l}(w) = \pm 1$, we say that w is a *relative unit of* L.

Let $E_{L/l}$ be a group consisting of all relative units of L. As in [4], let E^L be a sub G-module of $\mathcal{O}_L^*$ generated by elements of $\mathcal{O}_l^*$ and $E_{L/l}$. Let $Q = [\mathcal{O}_L^* : E^L]$. The following lemmas are from Hasse [5].

Lemma 4.2. *The index Q is either 1 or 2. Furthermore, $Q = 2$ if and only if there exists $u_* \in \mathcal{O}_L^*$ such that $N_{L/l}(u_*) = \pm u_l$.*

Proof. See Satz 16, p.37 of Hasse [5]. □

Lemma 4.3. *There exists an element u_0 of $E_{L/l}$ such that $-1, u_0$ and its conjugate $\sigma(u_0)$ generates $E_{L/l}$.*

Proof. See Satz 22, p.40 of Hasse [5]. □

Define u_0 as in the lemma. We have

$$E^L = \{\pm u_l^{m_1} u_0^{m_2} \sigma(u_0)^{m_3} \mid (m_1, m_2, m_3) \in \mathbb{Z}^3\}.$$

We first compute the 1-norm of the elements of $\bigwedge^2 \mathrm{LOG}(E^L)$. Note that since u_0 is relative unit of L, $\sigma^2(u_0) = \pm\frac{1}{u_0}$. We can compute LOG's as follows:

$$\begin{aligned} \mathrm{LOG}(u_l) &= (W_1, -W_1, W_1, -W_1) \\ \mathrm{LOG}(u_0) &= (W_2, W_3, -W_2, -W_3) \\ \mathrm{LOG}(\sigma(u_0)) &= (W_3, -W_2, -W_3, W_2) \end{aligned} \tag{4.1}$$

where $W_1 = \log(u_l), W_2 = \log(|u_0|), W_3 = \log(|\sigma(u_0)|)$. Here, the order of the basis of $\mathbb{R}^{\mathcal{A}_L}$ is $\delta^{id}, \delta^{\sigma}, \delta^{\sigma^2}, \delta^{\sigma^3}$. Note that W_1, W_2, W_3 are nonzero since $u_l, u_0, \sigma(u_0)$ are not ± 1. Then the wedge product of these terms are:

$$\begin{aligned}\mathrm{LOG}(u_l)\wedge\mathrm{LOG}(u_0) &= (\ \ Y_4, -Y_4, Y_5, \ \ Y_5, -Y_2, \ \ Y_3)\\ \mathrm{LOG}(u_l)\wedge\mathrm{LOG}(\sigma(u_0)) &= (-Y_5, \ \ Y_5, Y_4, \ \ Y_4, -Y_3, -Y_2)\\ \mathrm{LOG}(u_0)\wedge\mathrm{LOG}(\sigma(u_0)) &= (-Y_1, -Y_1, Y_1, -Y_1, \ \ 0, \ \ 0)\end{aligned}$$

where $Y_1 = W_2^2 + W_3^2, Y_2 = 2W_1W_2, Y_3 = 2W_1W_3, Y_4 = W_1W_2 + W_1W_3, Y_5 = W_1W_2 - W_1W_3$. Here the order of the basis of $\bigwedge^2 \mathbb{R}^{\mathcal{A}_L}$ is $\delta^{id}\wedge\delta^{\sigma}, \delta^{\sigma^2}\wedge\delta^{\sigma^3}, \delta^{id}\wedge\delta^{\sigma^3}, \delta^{\sigma}\wedge\delta^{\sigma^2}, \delta^{id}\wedge\delta^{\sigma^2}, \delta^{\sigma}\wedge\delta^{\sigma^3}$.

For any nonzero element $w \in \bigwedge^2 \mathrm{LOG}(E^L)$, it can be written as

$$\begin{aligned}w =& n_1\cdot\mathrm{LOG}(u_l)\wedge\mathrm{LOG}(u_0) + n_2\cdot\mathrm{LOG}(u_l)\wedge\mathrm{LOG}(\sigma(u_0))\\ &+ n_3\cdot\mathrm{LOG}(u_0)\wedge\mathrm{LOG}(\sigma(u_0))\end{aligned}$$

where n_1, n_2, n_3 are integers. Then $\|w\|_1 = f(n_1, n_2, n_3)$ where the map $f : \mathbb{Z}^3 \to \mathbb{R}$ is given as

$$\begin{aligned}f(n_1,n_2,n_3) =& |n_1Y_4 - n_2Y_5 - n_3Y_1| + |-n_1Y_4 + n_2Y_5 - n_3Y_1|\\ &+ |n_1Y_5 + n_2Y_4 + n_3Y_1| + |n_1Y_5 + n_2Y_4 - n_3Y_1|\\ &+ |-n_1Y_2 - n_2Y_3| + |n_1Y_3 - n_2Y_2|.\end{aligned}$$

Applying (3.2), above equation becomes

$$\begin{aligned}f(n_1,n_2,n_3) =& 2\max\{|n_1Y_4 - n_2Y_5|, |n_3Y_1|\} + 2\max\{|n_1Y_5 + n_2Y_4|, |n_3Y_1|\}\\ &+ |-n_1Y_2 - n_2Y_3| + |n_1Y_3 - n_2Y_2|.\end{aligned}$$

In particular, we have

$$f(n_1,n_2,n_3) \geq 4|n_3Y_3| = 4|n_3|(W_2^2 + W_3^2) \tag{4.2}$$

and

$$\begin{aligned}f(n_1,n_2,n_3) \geq& 2|n_1Y_4 - n_2Y_5| + 2|n_1Y_5 + n_2Y_4|\\ &+ |-n_1Y_2 - n_2Y_3| + |n_1Y_3 - n_2Y_2|.\end{aligned}$$

If $(n_1, n_2) \neq (0, 0)$, then by Lemma 4.4 below,

$$\begin{aligned}f(n_1,n_2,n_3) &\geq 2|Y_4| + 2|Y_5| + |Y_2| + |Y_3|\\ &= 2|W_1W_2 + W_1W_3| + 2|W_1W_2 - W_1W_3| + 2|W_1W_2| + 2|W_1W_3|\end{aligned}$$

and applying (3.2) once more, we have

$$f(n_1,n_2,n_3) \geq 2W_1(2\max\{|W_2|, |W_3|\} + |W_2| + |W_3|) \tag{4.3}$$

Using these results, we prove the Theorem 1.2 for each case $Q = 1$ and $Q = 2$.

Lemma 4.4. *Let $X, Y \in \mathbb{R}$. For any $(m, n) \in \mathbb{Z}^2\backslash\{(0, 0)\}$, the following inequality*

$$|mX + nY| + |nX - mY| \geq |X| + |Y|$$

holds.

Proof. The inequality is trivial if $X = 0$ or $Y = 0$. Assume that they are both nonzero. We may further assume that they are positive and $X \geq Y$. Let $Z := Y/X \leq 1$. Since

$$|mX + nY| + |nX - mY| = |X|(|m + nZ| + |n - mZ|),$$

we only need to show the inequality

$$|m + nZ| + |n - mZ| \geq 1 + Z.$$

Without loss of generality, assume $m \geq 0$. If $m = 0$, $|nZ| + |n| \geq 1 + Z$ for any nonzero integer n. If $m \geq 2$,

$$\begin{aligned} |m + nZ| + |n - mZ| &\geq |m + nZ| + |Z(n - mZ)| \\ &\geq |(m + nZ) - Z(n - mZ)| \\ &= |m(1 + Z^2)| \\ &\geq m \\ &\geq 2 \\ &\geq 1 + Z. \end{aligned}$$

Now consider the case $m = 1$. The case $n = 0$ is trivial. If $n \leq -1$, then

$$|1 + nZ| + |n - Z| \geq n - Z \geq 1 + Z$$

and if $n \geq 1$,

$$|1 + nZ| + |n - Z| \geq 1 + nZ \geq 1 + Z.$$

□

4.1. **The Case Q=1.** Since $\mathcal{O}_L^* = E^L$, for any nonzero $w \in \bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$, its norm is of the form $\|w\|_1 = f(n_1, n_2, n_3)$ where $(n_1, n_2, n_3) \neq (0, 0, 0)$. If $n_3 \neq 0$, then (4.2) gives

$$\|w\|_1 \geq 4(W_2^2 + W_3^2)$$

and if $n_3 = 0$, then $(n_1, n_2) \neq (0, 0)$, so (4.3) gives

$$\|w\|_1 \geq 2W_1(2 \max\{|W_2|, |W_3|\} + |W_2| + |W_3|).$$

Therefore we have

$$\|w\|_1 \geq \min\{4(W_2^2 + W_3^2), 2W_1(2 \max\{|W_2|, |W_3|\} + |W_2| + |W_3|)\} \tag{4.4}$$

for any nonzero $w \in \bigwedge \mathrm{LOG}(\mathcal{O}_L^*)$.

We use the following theorem from Pohst [7] to give a lower bound for (4.4).

Theorem 4.5. *Let L be totally real number field and $u \in \mathcal{O}_L^*$ with $u \neq \pm 1$. Then*

$$\|\mathrm{LOG}(u)\|_2^2 \geq [L:\mathbb{Q}] \log\left(\frac{1+\sqrt{5}}{2}\right)^2$$

holds.

Proof. Bemerkung in p.102 of Pohst [7] states that the Lemma in p.98 applies for M of the eqaution (10) in p.97. Then the inequality (11) of the Lemma gives the required result. □

Then from the above theorem and (4.1),

$$W_2^2 + W_3^2 = \frac{1}{2}\|\mathrm{LOG}(u_0)\|_2^2 \geq 2\log\left(\frac{1+\sqrt{5}}{2}\right)^2. \tag{4.5}$$

Under this constraint and the fact that W_2, W_3 are nonzero, an elementary consideration yields

$$2\max\{|W_2|, |W_3|\} + |W_2| + |W_3| > 3\sqrt{2}\log\left(\frac{1+\sqrt{5}}{2}\right).$$

Since u_l is a fundamental unit of quadratic field, by Lemma 3.3,

$$W_1 = \log(u_l) \geq \log\left(\frac{1+\sqrt{5}}{2}\right) \tag{4.6}$$

holds, and with previous inequality,

$$2W_1(2\max\{|W_2|, |W_3|\} + |W_2| + |W_3|) > 6\sqrt{2}\log\left(\frac{1+\sqrt{5}}{2}\right)^2.$$

With (4.5), the inequality (4.4) becomes

$$\|w\|_1 \geq \min\left\{8\log\left(\frac{1+\sqrt{5}}{2}\right)^2, 6\sqrt{2}\log\left(\frac{1+\sqrt{5}}{2}\right)^2\right\} = 8\log\left(\frac{1+\sqrt{5}}{2}\right)^2 \tag{4.7}$$

for any nonzero $w \in \bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$. Thus Theorem 1.2 holds for the case $Q = 1$.

4.2. **The Case Q=2.** We use the following lemma from Hasse [5].

Lemma 4.6. *If $Q = 2$, there exists a unique unit $u_* > 0$ of L that satisfy*

$$N_{L/l}(u_*) = u_*\sigma^2(u_*) = \pm u_l, \quad u_*\sigma(u_*) = \pm u_0. \tag{4.8}$$

Proof. See Satz 28, p.45 of Hasse [5]. □

Define u_* as in the lemma. Then

$$u_*^2 = \frac{u_*\sigma^2(u_*) \cdot u_*\sigma(u_*)}{\sigma(u_*\sigma(u_*))} = \pm\frac{u_l u_0}{\sigma(u_0)} \tag{4.9}$$

and therefore $\sigma(u_0) = \pm u_l u_0 u_*^{-2}$. The proof of Lemma 4.2 shows that when $Q = 2$, every unit $u \in \mathcal{O}_L^*$ can be written uniquely as the form $\pm u_*^{n_0} u_l^{n_1} u_0^{n_2} \sigma(u_0)^{n_3}$ where n_0 is 0 or 1 and n_1, n_2, n_3 are integers. Thus $\mathcal{O}_L^*$ is generated by $-1, u_l, u_0, u_*$. Hence $\bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$ is generated by $\mathrm{LOG}(u_l) \wedge \mathrm{LOG}(u_0), \mathrm{LOG}(u_l) \wedge \mathrm{LOG}(u_*)$ and $\mathrm{LOG}(u_0) \wedge \mathrm{LOG}(u_*)$. From (4.9), we also have

$$\mathrm{LOG}(u_*) = \frac{1}{2}(\mathrm{LOG}(u_l) + \mathrm{LOG}(u_0) - \mathrm{LOG}(\sigma(u_0))) \tag{4.10}$$

and thus

$$\mathrm{LOG}(u_l) \wedge \mathrm{LOG}(u_*) = \frac{1}{2}(\mathrm{LOG}(u_l) \wedge \mathrm{LOG}(u_0) - \mathrm{LOG}(u_l) \wedge \mathrm{LOG}(\sigma(u_0))),$$
$$\mathrm{LOG}(u_0) \wedge \mathrm{LOG}(u_*) = -\frac{1}{2}(\mathrm{LOG}(u_l) \wedge \mathrm{LOG}(u_0) + \mathrm{LOG}(u_0) \wedge \mathrm{LOG}(\sigma(u_0))).$$

Hence for any $w \in \bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$,

$$\begin{aligned} 2w = {} & n_1 \cdot \mathrm{LOG}(u_l) \wedge \mathrm{LOG}(u_0) + n_2 \cdot \mathrm{LOG}(u_l) \wedge \mathrm{LOG}(\sigma(u_0)) \\ & + n_3 \cdot \mathrm{LOG}(u_0) \wedge \mathrm{LOG}(\sigma(u_0)) \end{aligned}$$

where n_1, n_2, n_3 are integers and $n_1 + n_2 + n_3$ is even and thus

$$\|w\|_1 = \frac{1}{2} f(n_1, n_2, n_3). \tag{4.11}$$

We now proceed to give a lower bound for $f(n_1, n_2, n_3)$ where $n_1 + n_2 + n_3$ is even and $(n_1, n_2, n_3) \neq (0, 0, 0)$. If $(n_1, n_2) = 0$, then $|n_3| \geq 2$, so (4.2) gives

$$f(n_1, n_2, n_3) \geq 8(W_2^2 + W_3^2).$$

Since (4.5) still holds in the $Q = 2$ case, we have

$$f(n_1, n_2, n_3) \geq 8 \log \left(\frac{1 + \sqrt{5}}{2} \right)^2. \tag{4.12}$$

Note that (4.6) also still holds.

Now consider the case $(n_1, n_2) \neq (0, 0)$. Since $\mathrm{LOG}(u_l), \mathrm{LOG}(u_0), \mathrm{LOG}(\sigma(u_0))$ are perpendicular as seen in (4.1), we have

$$\begin{aligned} \|\mathrm{LOG}(u_*)\|_2^2 &= \frac{1}{4}(\|\mathrm{LOG}(u_l)\|_2^2 + \|\mathrm{LOG}(u_0)\|_2^2 + \|\mathrm{LOG}(\sigma(u_0))\|_2^2) \\ &= W_1^2 + W_2^2 + W_3^2 \end{aligned}$$

from (4.10). Then Theorem 4.5 gives

$$W_1^2 + W_2^2 + W_3^2 \geq 4 \log \left(\frac{1 + \sqrt{5}}{2} \right)^2.$$

Under the constraint of above inequality, (4.5), (4.6), and the fact that W_1, W_2, W_3 are nonzero, an elementary consideration yields

$$2W_1(2\max\{|W_2|, |W_3|\} + |W_2| + |W_3|) > 6\sqrt{3}\log\left(\frac{1+\sqrt{5}}{2}\right)^2.$$

With (4.3), we have

$$f(n_1, n_2, n_3) > 6\sqrt{3}\log\left(\frac{1+\sqrt{5}}{2}\right)^2.$$

Since $8\log\left(\frac{1+\sqrt{5}}{2}\right)^2 > 6\sqrt{3}\log\left(\frac{1+\sqrt{5}}{2}\right)^2$, by (4.12), the above inequality holds for any $(n_1, n_2, n_3) \neq (0, 0, 0)$ with $n_1 + n_2 + n_3$ even. Therefore, with (4.11),

$$\|w\|_1 > 3\sqrt{3}\log\left(\frac{1+\sqrt{5}}{2}\right)^2$$

for any nonzero $w \in \bigwedge^2 \mathrm{LOG}(\mathcal{O}_L^*)$. This concludes the proof of Theorem 1.2.

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