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On a weak form of Ennola's conjecture about certain cubic number fields

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Abstract

We establish a weak form of Ennola's conjecture. We achieve this by showing that two main assumptions Louboutin made in his previous work hold true. These assumptions are about Laurent polynomials over the rationals, and we prove them by using polynomial relations reminiscent of Newton identities.

Keywords: Ennola's conjecture, Fundamental units, Unit index, Newton identities

1 Introduction

An algebraic number ϵ is called *exceptional* if both ϵ and $\epsilon - 1$ are units. For example, let $l \geq 3$ be an integer, and consider a non-Galois totally real cubic number field $\mathbb{Q}(\epsilon_l)$, where the minimal polynomial of ϵ_l is

$$X^3 + (l-1)X^2 - lX - 1 \in \mathbb{Z}[X]. \tag{1}$$

One can easily observe that ϵ_l and $\epsilon_l - 1$ are both units. i.e., ϵ_l is an exceptional unit.

By Dirichlet's unit theorem, the unit group $\mathbb{U}_l$ of the ring of integers of $\mathbb{Q}(\epsilon_l)$ has rank 2. Hence, one can now naturally ask the following: is $\{\epsilon_l, \epsilon_l - 1\}$ a pair of fundamental units? That is, $\mathbb{U}_l = \langle -1, \epsilon_l, \epsilon_l - 1\rangle$?

This is still an open problem. Ennola conjectured in [1] that $\{\epsilon_l, \epsilon_l - 1\}$ above is a pair of fundamental units for the ring of integers $\mathcal{O}$ of $\mathbb{Q}(\epsilon_l)$. We call this *Ennola's conjecture*. He showed that his conjecture is true for $3 \leq l \leq 500$ and that the *unit index*

$$j_l := (\mathbb{U}_l : \langle -1, \epsilon_l, \epsilon_l - 1\rangle)$$

of the group of units generated by -1, ϵ_l, and $\epsilon_l - 1$ in the group of units $\mathbb{U}_l$ is always coprime to 2, 3, and 5. He also showed that $\{\epsilon_l, \epsilon_l - 1\}$ is a pair of fundamental units for $\mathcal{O}$ if $(\mathcal{O} : \mathbb{Z}[\epsilon_l]) \leq l/3$ in [2].

To show that Ennola's conjecture holds true is equivalent to prove $j_l = 1$ for $\forall l \geq 3$. S. Louboutin obtained several results on this conjecture. For example, in [3], he showed that for $l \geq 3$, the unit index j_l is coprime to 19!, and $j_l = 1$ for $3 \leq l \leq 5 \cdot 10^7$. He also proved in [4] that if we assume ABC conjecture is true, then Ennola's conjecture is true except for finitely many l. i.e., $j_l = 1$ for any sufficiently large l. Another significant result in [4] is a conditional proof of a weak form of Ennola's conjecture that for any given integer $N \geq 2$, we have $\gcd(j_l, N!) = 1$ for $l \geq l_N$ effectively large enough.

Assuming Conjectures 1 and 2 below, S. Louboutin deduced a weak form of Ennola's conjecture. Even though Louboutin checked the validity of the conjectures by computation for finitely many case in [4], the Conjectures 1 and 2 are still unsolved in [4].

We provide proofs of Conjectures 14 and 20 of [4] and consequently establish Theorem 1 below.

Theorem 1 (Weak form of Ennola's conjecture, Theorem 2 and Proposition 15 of [4]) *For any given prime $p \geq 3$, there are only finitely many $l \geq 3$ for which p divides the unit index j_l. Hence, for any given integer $N \geq 2$ we have $\gcd(j_l, N!) = 1$ for $l \geq l_N$ effectively large enough.*

To prove the above theorem, we will formulate and verify Conjectures 1 and 2, which are aforementioned conjectures in [4]. Both conjectures are about certain Laurent polynomials. Our proof of Conjecture 1 is reminiscent of Newton identities, and that of Conjecture 2 is based on the validity of Conjecture 1 combined with an argument involving elementary manipulations of binomial coefficients.

Theorem 1 is a statement about divisors of a unit index j_l. A similar work was done by Louboutin and Lee for another family of cubic number fields. In [5], they showed that the unit index $(\mathbb{U}_a : \langle -1, \epsilon_a, \epsilon'_a \rangle)$ of the group of units generated by -1, ϵ_a, and ϵ'_a in the group of units $\mathbb{U}_a$ of the ring of integers of $\mathbb{Q}(\alpha)$ is coprime to 3 for $a \geq 1$ and coprime to 6 for $1 \leq a \equiv 2, 3 \pmod 4$, where α is a root of $X^3 - 4a^2X + 2 \in \mathbb{Z}[X]$, and ϵ_a, ϵ'_a are two distinct units in the ring of integers of $\mathbb{Q}(\alpha)$. Our result is similar to that of Louboutin and Lee, and we are concerned with a family of cubic polynomials (1) with $l \geq 3$. On the other hand, this family (1) attracts interests both of number theorists and of topologists. For topological context, see [6].

This paper is organized as follows. In §2, we introduce Conjectures 1 and 2, which are used in the proof of Theorem 1. In §3 and §4, we give proofs of conjectures.

2 Louboutin's conjectures

Here, we introduce Louboutin's conjectures. They are concerned with the family of following polynomials.

Definition 1 For $d \geq 1$, we define the polynomial

$$P_d(X, Y) = d \sum_{\substack{k,l \geq 0 \\ 0 \leq 2k+3l \leq d}} (-1)^{k-1} \binom{k+l}{k} \binom{d-k-2l}{k+l} \frac{X^k Y^{d-2k-3l}}{d-k-2l} \in \mathbb{Z}[X, Y].$$

A precursor of the polynomial $P_d(X, Y)$ first appeared in [7, Lemma 8] and it was defined in this form in [4].

Example 1 For small d's, $P_d(X, Y)$'s are: $P_1(X, Y) = -Y$, $P_2(X, Y) = -Y^2 + 2X$, $P_3(X, Y) = -Y^3 + 3XY - 3$, and $P_4(X, Y) = -Y^4 + 4XY^2 - 2X^2 - 4Y$.

Conjecture 1 (Conjecture 14 of [4]) Let $a, b \in \mathbb{Z}$ be nonzero such that $c := a + b \neq 0$. Set

$$S_{a,b}(T) = \frac{1}{T^a} + \frac{1}{T^b} + T^{a+b}.$$

Then for $d \in \{a, b, c\}$ and $P_d(X, Y)$ as in the Definition 1, we have

$$P_{|d|}(S_{a,b}(T), S_{a,b}(1/T)) = -S_{a,b}(1/T^{|d|}). \tag{2}$$

Moreover, if a is even and b is odd, then with

$$R_{a,b}(T) = \frac{1}{T^a} + \frac{(-1)^{a+b}}{T^b} + T^{a+b},$$

we have

$$P_{|d|}(-R_{a,b}(T), -R_{a,b}(1/T)) = \begin{cases} -S_{a,b}(1/T^{|d|}), & \text{if } d = a \\ R_{a,b}(1/T^{|d|}), & \text{if } d \in \{b, c\}. \end{cases} \tag{3}$$

Conjecture 1 is used to deduce the following theorem in [4].

Theorem 2 [4, Theorem 18] *Let $p \equiv 5 \pmod 6$ be a given prime. Assume that Conjecture 1 holds true. Then p does not divide j_l for l sufficiently large enough.*

There is other conjecture we have to prove to establish Theorem 1. Conjecture 20 in [4] was necessary to apply Proposition 19 of [4] in the proof of Theorem 1, which is stated below.

Proposition 1 (Proposition 19 of [4]) *Let $a, b \in \mathbb{Z}$ not both equal to 0 be given. Let $m \geq 3$ be odd. Set $G_{a,b}(T) := F_{a,b}(R_{a,b}(T), R_{-a,-b}(T))$, where $0 \neq F_{a,b}(X, Y) = \sum_{u,v} f_{u,v} X^u Y^v \in \mathbb{Z}[X, Y]$ is given as Table 1 below. We also define*

$$\begin{aligned} s &= \max(a + b, -a, -b),\ t = \max(-a - b, a, b), \\ M_{a,b} &= \max\{us + vt : f_{u,v} \neq 0\}, \\ R_{a,b,m}(T) &= R_{a,b}(T) + \frac{b-a}{m} T^{-a-m} + \frac{(-1)^{a+b}(a-2b)}{m} T^{-b-m} + \frac{b}{m} T^{a+b-m}. \end{aligned}$$

Assume that

$$G_{a,b,m}(T) = F_{a,b}(R_{a,b,m}(T), R_{-a,-b,m}(T)) \in \mathbb{Q}[T, T^{-1}]$$

is of negative degree.
Set $N_{a,b,m} = -\deg G_{a,b,m}(T) \geq 1$ and $B_{a,b,m} := (M_{a,b} + N_{a,b,m} + 1)/2$. If $B_{a,b,m} \leq m$, then the unit $\epsilon_{a,b} = (-1)^{a+b} \epsilon_l^a (\epsilon_l - 1)^b$ is not a m-th power in $\mathbb{Q}(\epsilon_l)$ for $l \geq l_m$ effectively large.

Table 1 Cases of $F_{a,b}(X, Y)$

Cases	$F_{a,b}(X, Y)$
Case 1: $a \geq 1$ odd and $b \geq 1$ odd	$F_{a,b}(X, Y) = -P_a(Y, X) - P_b(Y, X) + P_c(X, Y)$
Case 2: $a \geq 1$ odd and $b \geq 1$ even	$F_{a,b}(X, Y) = -P_a(-Y, -X) - P_b(-Y, -X) + P_c(-X, -Y)$
Case 3: $a \geq 2$ even and $b \geq 1$ odd	$F_{a,b}(X, Y) = -P_a(-Y, -X) - P_b(-Y, -X) - P_c(-X, -Y)$

Table 1 gives all cases that will be dealt with in this paper ($c := a + b \neq 0$):

Conjecture 2 (Conjecture 20 of [4]) *With notations as in Proposition 1, let $(a, b) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z}_{\geq 1}$ be such that $m_{a,b} = a^2 + ab + b^2$ is odd and $m_{a,b} \geq 5$. Assume that the pair (a, b) is not of the form $(-2b, b)$ with $b \geq 1$ odd, (b, b) with $b \geq 1$ odd, $(-b/2, b)$ with $b \geq 2$ even. Then the assumptions in Proposition 1 are satisfied for $m = m_{a,b}$ with $M_{a,b} = (a + b) \max(a, b)$ and $N_{a,b,m_{a,b}} = \min(a, b)^2$. Namely, $\deg G_{a,b,m_{a,b}}(T) < 0$, and $B_{a,b,m_{a,b}} \leq m_{a,b}$.*

Conjectures 1 and 2 yield the following theorem in [4].

Theorem 3 [4, Theorem 21] *Let $p \equiv 1 \pmod 6$ be a given prime. Assume that Conjectures 1 and 2 are true. Then p does not divide j_l for l sufficiently large enough.*

Clearly, Theorem 1 can be obtained by combining Theorem 2 with Theorem 3.

If $M_{a,b}$ and $N_{a,b,m_{a,b}}$ are as in the Conjecture 2, then $B_{a,b,m_{a,b}} \leq m_{a,b}$, since $B_{a,b,m_{a,b}} = (a^2 + ab + b^2 + 1)/2 = (m_{a,b} + 1)/2 \leq m_{a,b}$ as [4, p. 15]. That is, the assumption of Prop. 1 is satisfied.

In summary, to prove Conjecture 2, we have to compute $M_{a,b}$, $N_{a,b,m_{a,b}}$ and verify $M_{a,b} = (a + b) \max(a, b)$, $N_{a,b,m_{a,b}} = \min(a, b)^2$ in each case of Table 1.

3 Proof of Conjecture 1

As an intermediate step towards the proof of Conjecture 1, we show the following proposition, which is reminiscent of Newton identities.

Proposition 2 *For any $d \geq 4$, we have*

$$P_d(X, Y) = YP_{d-1}(X, Y) - XP_{d-2}(X, Y) + P_{d-3}(X, Y). \tag{4}$$

Proof To prove (4), we show that the coefficients of $X^k Y^{d-2k-3l}$ in the LHS and RHS of (4) coincide. Set

$$k = A,\ d - 2k - 3l = B,\ M = \frac{d + A - B}{3},\ N = \frac{d + A + 2B}{3}.$$

Then $l = (d - 2A - B)/3$, $k + l = M$, and $d - k - 2l = N$. Now we determine the coefficients of $X^A Y^B$ in both sides of (4). First, we assume $A > 0$ (if so, it is clear that $M, N > 0$).

In $P_d(X, Y)$, the coefficient of $X^k Y^{d-2k-3l} = X^A Y^B$ is

$$(-1)^{k-1} \binom{k + l}{k} \binom{d - k - 2l}{k + l} \frac{d}{d - k - 2l} = (-1)^{A-1} \binom{M}{A} \binom{N}{M} \frac{d}{N}. \tag{5}$$

In $YP_{d-1}(X, Y)$, the term $X^k Y^{d-2k-3l} = X^A Y^B$ occurs when $A = k$ and $B = d - 2k - 3l$, where the coefficient of the monomial is

$$(-1)^{k-1} \binom{k + l}{k} \binom{d - 1 - k - 2l}{k + l} \frac{d - 1}{d - 1 - k - 2l} = (-1)^{A-1} \binom{M}{A} \binom{N - 1}{M} \frac{d - 1}{N - 1}. \tag{6}$$

In $XP_{d-2}(X, Y)$, the term $X^kY^{d-2k-3l} = X^AY^B$ occurs when $A = k+1$ and $B = d-2-2k-3l$, where the coefficient of the monomial is

$$(-1)^{k-1}\binom{k+l}{k}\binom{d-2-k-2l}{k+l}\frac{d-2}{d-2-k-2l} = (-1)^A\binom{M-1}{A-1}\binom{N-1}{M-1}\frac{d-2}{N-1}. \tag{7}$$

In $P_{d-3}(X, Y)$, the term $X^kY^{d-2k-3l} = X^AY^B$ occurs when $A = k$ and $B = d-3-2k-3l$, where the coefficient of the monomial is

$$(-1)^{k-1}\binom{k+l}{k}\binom{d-3-k-2l}{k+l}\frac{d-3}{d-3-k-2l} = (-1)^{A-1}\binom{M-1}{A}\binom{N-1}{M-1}\frac{d-3}{N-1}. \tag{8}$$

Now we verify that

$$(\text{RHS of } (5)) = (\text{that of } (6)) - (\text{that of } (7)) + (\text{that of } (8)). \tag{9}$$

(9) is equivalent to

$$\binom{M}{A}\binom{N}{M}\frac{d}{N} = \binom{M}{A}\binom{N-1}{M}\frac{d-1}{N-1} + \binom{M-1}{A-1}\binom{N-1}{M-1}\frac{d-2}{N-1} + \binom{M-1}{A}\binom{N-1}{M-1}\frac{d-3}{N-1}.$$

Note that

$$\binom{N-1}{M} = \frac{N-M}{N}\binom{N}{M}, \quad \binom{M-1}{A-1} = \frac{A}{M}\binom{M}{A}, \tag{10}$$

$$\binom{M-1}{A} = \frac{M-A}{M}\binom{M}{A}, \quad \binom{N-1}{M-1} = \frac{M}{N}\binom{N}{M}. \tag{11}$$

So, by using (10) and (11) to simplify (9), it suffices to check whether the equality

$$\frac{d}{N} = \frac{N-M}{N}\frac{d-1}{N-1} + \frac{A}{M}\frac{M}{N}\frac{d-2}{N-1} + \frac{M-A}{M}\frac{M}{N}\frac{d-3}{N-1} \tag{12}$$

holds. Multiplying $N(N-1)$ on both sides of (12), we know

$$(N-1)d = (N-M)(d-1) + A(d-2) + (M-A)(d-3) = Nd - N - 2M + A.$$

Since $2M + N = d + A$ by definition of A, B, M, and N, we are done.

Suppose $A = 0$. Then $k = 0 = A$, $B = d - 3l$, $M = l$, and $N = d - 2l$. If $A = M = 0$, then $X^AY^B = Y^d$, and since Y^d solely comes from (6), there is nothing to show. So we assume $A = 0$, and $M > 0$. In this case, (9) becomes

$$(\text{RHS of } (5)) = (\text{that of } (6)) + (\text{that of } (8)). \tag{13}$$

From (5), (6), and (8), it is easy to observe that (13) is equivalent to

$$\binom{N}{M}\frac{d}{N} = \binom{N-1}{M}\frac{d-1}{N-1} + \binom{N-1}{M-1}\frac{d-3}{N-1}. \tag{14}$$

Since $2M + N = d$, the RHS of (14) is

$$\begin{aligned}
&\frac{1}{N-1}\left((d-1)\binom{N-1}{M-1}\frac{N-M}{M} + (d-3)\binom{N-1}{M-1}\right) \\
&= \frac{1}{N-1}\binom{N-1}{M-1}\left(\frac{Nd-N-2M}{M}\right) = \frac{d}{M}\binom{N-1}{M-1} = \frac{d}{N}\binom{N}{M}
\end{aligned}$$

and the proof is complete. □

Now we are ready to prove Conjecture 1 by using Proposition 2. First, a corollary:

Corollary 1 *For $d \geq 1$, we have*

$$P_d\left(\frac{1}{X} + \frac{1}{Y} + XY, X + Y + \frac{1}{XY}\right) = -\left(X^d + Y^d + \frac{1}{X^d Y^d}\right).$$

Proof We proceed by induction on d. From the expression of $P_1(X, Y)$, $P_2(X, Y)$, and $P_3(X, Y)$ in Example 1, it is easy to observe that the result holds for $d = 1, 2, 3$. Suppose $d \geq 4$. By Proposition 2 and the inductive hypothesis,

$$\begin{aligned}
&P_d\left(\frac{1}{X} + \frac{1}{Y} + XY, X + Y + \frac{1}{XY}\right) \\
&= \left(X + Y + \frac{1}{XY}\right) P_{d-1}\left(\frac{1}{X} + \frac{1}{Y} + XY, X + Y + \frac{1}{XY}\right) \\
&\quad - \left(\frac{1}{X} + \frac{1}{Y} + XY\right) P_{d-2}\left(\frac{1}{X} + \frac{1}{Y} + XY, X + Y + \frac{1}{XY}\right) \\
&\quad + P_{d-3}\left(\frac{1}{X} + \frac{1}{Y} + XY, X + Y + \frac{1}{XY}\right) \\
&= -\left(X + Y + \frac{1}{XY}\right)\left(X^{d-1} + Y^{d-1} + \frac{1}{X^{d-1}Y^{d-1}}\right) \\
&\quad + \left(\frac{1}{X} + \frac{1}{Y} + XY\right)\left(X^{d-2} + Y^{d-2} + \frac{1}{X^{d-2}Y^{d-2}}\right) \\
&\quad - \left(X^{d-3} + Y^{d-3} + \frac{1}{X^{d-3}Y^{d-3}}\right) \\
&= -\left(X^d + Y^d + \frac{1}{X^d Y^d}\right).
\end{aligned}$$

□

Conjecture 1 naturally follows from Corollary 1.

Proof of Conjecture 1 First, we verify (2). Put $X = T^a$, $Y = T^b$ in Corollary 1. Then

$$S_{a,b}(T) = \frac{1}{X} + \frac{1}{Y} + XY, S_{a,b}(1/T) = X + Y + \frac{1}{XY}.$$

Table 2 $G_{a,b}(T)$ in each case

Cases	$G_{a,b}(T)$
Case 1	$G_{a,b}(T) = T^{-a^2} + T^{-b^2} - T^{-c^2} + 2T^{-ab}$
Case 2	$G_{a,b}(T) = -T^{-a^2} + T^{-b^2} + T^{-c^2} + 2T^{-ab}$
Case 3	$G_{a,b}(T) = T^{-a^2} + T^{-b^2} - T^{-c^2}$

From Corollary 1,

$$-S_{a,b}(1/T^{|d|}) = -\left(T^{a|d|} + T^{b|d|} + \frac{1}{T^{(a+b)|d|}}\right) = -\left(X^d + Y^d + \frac{1}{X^dY^d}\right)$$
$$= P_d\left(\frac{1}{X} + \frac{1}{Y} + XY, X + Y + \frac{1}{XY}\right) = P_{|d|}(S_{a,b}(T), S_{a,b}(1/T)).$$

The proof of (3) directly follows from the same method, by replacing $X \to -X$, $Y \to Y$, and $XY \to -XY$. □

4 Proof of Conjecture 2

We need the following lemma to prove Conjecture 2:

Lemma 1 *Define, following* [4, Conjecture 12],

$$G_{a,b}(T) = F_{a,b}(R_{a,b}(T), R_{a,b}(1/T))$$

with $F_{a,b}(X, Y)$ as in Table 1. Then $G_{a,b}(T)$'s for each case in Table 1 are given as the Table 2:

Proof This is a direct consequence of [4, Proposition 15] and [4, Conjecture 14] which has just been proved in §3. □

Before we give a proof of the cases in Table 1, we set

$$E_{a,b}(T) = \frac{b-a}{T^a} + \frac{(-1)^{a+b}(a-2b)}{T^b} + bT^{a+b}.$$

For notational convenience, we let $m_{a,b} = m$ from now on. Then

$$R_{a,b,m}(T) = R_{a,b}(T) + \frac{1}{mT^m}E_{a,b}(T),$$
$$R_{-a,-b,m}(T) = R_{-a,-b}(T) + \frac{1}{mT^m}E_{-a,-b}(T) = R_{a,b}\left(\frac{1}{T}\right) - \frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right).$$

4.1 Case 1: $a \geq 1$ odd and $b \geq 1$ odd

In §4.1, we show $M_{a,b} = (a+b)\max(a,b)$, $N_{a,b,m} = \min(a,b)^2$ from direct computation. By Table 1, $F_{a,b}(X,Y) = -P_a(Y,X) - P_b(Y,X) + P_c(X,Y)$ for Case 1, and we get

$$G_{a,b,m}(T) = Q_{1,a,b,m}(T) + Q_{2,a,b,m}(T) + Q_{3,a,b,m}(T),$$

where

$$Q_{1,a,b,m}(T) = -P_a\left(R_{a,b}\left(\frac{1}{T}\right) - \frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T) + \frac{1}{mT^m}E_{a,b}(T)\right), \tag{15}$$

$$Q_{2,a,b,m}(T) = -P_b\left(R_{a,b}\left(\frac{1}{T}\right) - \frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T) + \frac{1}{mT^m}E_{a,b}(T)\right), \tag{16}$$

$$Q_{3,a,b,m}(T) = P_c\left(R_{a,b}(T) + \frac{1}{mT^m}E_{a,b}(T), R_{a,b}\left(\frac{1}{T}\right) - \frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right)\right). \tag{17}$$

Using Definition 1, (15) is

$$Q_{1,a,b,m}(T) = a\sum_{\substack{k,l\geq 0\\ 0\leq 2k+3l\leq a}}(-1)^k\binom{k+l}{k}\binom{a-k-2l}{k+l}\frac{1}{a-k-2l}\sum_{\substack{0\leq i\leq k\\ 0\leq j\leq a-2k-3l}}A_{i,j,k,l},$$

where

$$\begin{aligned}A_{i,j,k,l} =&\binom{k}{i}\left(\frac{-1}{mT^m}E_{a,b}\left(\frac{1}{T}\right)\right)^i\left(R_{a,b}\left(\frac{1}{T}\right)\right)^{k-i}\\ &\cdot\binom{a-2k-3l}{j}\left(\frac{1}{mT^m}E_{a,b}(T)\right)^j(R_{a,b}(T))^{a-2k-3l-j}.\end{aligned}$$

Note that

$$A_{0,0,k,l} = \left(R_{a,b}\left(\frac{1}{T}\right)\right)^k(R_{a,b}(T))^{a-2k-3l}.$$

Hence,

$$\begin{aligned}Q_{1,a,b,m}(T) =& -P_a\left(R_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T)\right)\\ &\underbrace{+a\sum_{\substack{k,l\geq 0\\ 0\leq 2k+3l\leq a}}(-1)^k\binom{k+l}{k}\binom{a-k-2l}{k+l}\frac{1}{a-k-2l}\sum_{\substack{0\leq i\leq k\\ 0\leq j\leq a-2k-3l\\ (i,j)\neq(0,0)}}A_{i,j,k,l}}_{=A_1(T)}\end{aligned}$$

In the same way, we can modify (16) and (17) as

$$\begin{aligned}Q_{2,a,b,m}(T) =& -P_b\left(R_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T)\right)\\ &\underbrace{+b\sum_{\substack{k,l\geq 0\\ 0\leq 2k+3l\leq b}}(-1)^k\binom{k+l}{k}\binom{b-k-2l}{k+l}\frac{1}{b-k-2l}\sum_{\substack{0\leq i\leq k\\ 0\leq j\leq b-2k-3l\\ (i,j)\neq(0,0)}}B_{i,j,k,l}}_{=B_1(T)}\end{aligned}$$

and

$$\begin{aligned}Q_{3,a,b,m}(T) =&P_c\left(R_{a,b}(T), R_{a,b}\left(\frac{1}{T}\right)\right)\\ &\underbrace{+c\sum_{\substack{k,l\geq 0\\ 0\leq 2k+3l\leq c}}(-1)^{k-1}\binom{k+l}{k}\binom{c-k-2l}{k+l}\frac{1}{c-k-2l}\sum_{\substack{0\leq i\leq k\\ 0\leq j\leq c-2k-3l\\ (i,j)\neq(0,0)}}C_{i,j,k,l}}_{=C_1(T)}\end{aligned}$$

where

$$B_{i,j,k,l} = \binom{k}{i}\left(\frac{-1}{mT^m}E_{a,b}\left(\frac{1}{T}\right)\right)^i \left(R_{a,b}\left(\frac{1}{T}\right)\right)^{k-i} \cdot \binom{b-2k-3l}{j}\left(\frac{1}{mT^m}E_{a,b}(T)\right)^j (R_{a,b}(T))^{b-2k-3l-j},$$

and

$$C_{i,j,k,l} = \binom{k}{i}\left(\frac{1}{mT^m}E_{a,b}(T)\right)^i (R_{a,b}(T))^{k-i} \cdot \binom{c-2k-3l}{j}\left(\frac{-1}{mT^m}E_{a,b}\left(\frac{1}{T}\right)\right)^j \left(R_{a,b}\left(\frac{1}{T}\right)\right)^{c-2k-3l-j}.$$

Finally, noticing that by Lemma 1 that we have

$$\begin{aligned} &-P_a\left(R_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T)\right) - P_b\left(R_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T)\right) + P_c\left(R_{a,b}(T), R_{a,b}\left(\frac{1}{T}\right)\right) \\ &= F_{a,b}(R_{a,b}(T), R_{a,b}(1/T)) = G_{a,b}(T), \end{aligned}$$

we obtain:

Lemma 2 *For $a \geq 1$ odd and $b \geq 1$ odd, $m = m_{a,b}$ and $c = a+b$, we have*

$$G_{a,b,m}(T) = \frac{1}{T^{a^2}} + \frac{1}{T^{b^2}} - \frac{1}{T^{c^2}} + \frac{2}{T^{ab}} + A_1(T) + B_1(T) + C_1(T).$$

Hence for all possible pairs (i, j) $(\neq (0, 0))$ and (k, l), we have

$$\deg A_{i,j,k,l} = k\max(a,b) + (a-2k-3l)(a+b) - (i+j)(a^2+ab+b^2), \tag{18}$$

$$\deg B_{i,j,k,l} = k\max(a,b) + (b-2k-3l)(a+b) - (i+j)(a^2+ab+b^2), \tag{19}$$

$$\deg C_{i,j,k,l} = k(a+b) + (c-2k-3l)\max(a,b) - (i+j)(a^2+ab+b^2). \tag{20}$$

By looking at the two possibilities for $\max(a, b)$, we determine in the following two sections the maxima of (18), (19), and (20).

4.1.1 $\max(a, b) = a$

If $\max(a, b) = a$, then (18), (19), and (20) are

$$\begin{aligned} \text{The RHS of (18)} &= ka + (a-2k-3l)(a+b) - (i+j)(a^2+ab+b^2) \\ &= -k(a+2b) - 3(a+b)l + a(a+b) - (i+j)(a^2+ab+b^2), \\ \text{that of (19)} &= ka + (b-2k-3l)(a+b) - (i+j)(a^2+ab+b^2) \\ &= -k(a+2b) - 3(a+b)l + b(a+b) - (i+j)(a^2+ab+b^2), \\ \text{that of (20)} &= k(a+b) + (c-2k-3l)a - (i+j)(a^2+ab+b^2) \\ &= k(-a+b) - 3al + a(a+b) - (i+j)(a^2+ab+b^2). \end{aligned}$$

Note that $-a+b<0$ since $\max(a,b)=a$ with $a\neq b$, and $i+j\geq 1$. Now, for $k,l\in\mathbb{Z}_{\geq 0}$, we check

$$\max\{\deg A_{i,j,k,l} : 0\leq i\leq k, 0\leq j\leq a-2k-3l, (i,j)\neq(0,0), 0\leq 2k+3l\leq a\}, \quad (21)$$
$$\max\{\deg B_{i,j,k,l} : 0\leq i\leq k, 0\leq j\leq b-2k-3l, (i,j)\neq(0,0), 0\leq 2k+3l\leq b\}, \quad (22)$$
$$\max\{\deg C_{i,j,k,l} : 0\leq i\leq k, 0\leq j\leq c-2k-3l, (i,j)\neq(0,0), 0\leq 2k+3l\leq c\}. \quad (23)$$

For (18) to be maximum, we know $k=l=i=0, j=1$. In this case, (21) $=-b^2$. (19) and (20) also obtain maximum at $k=l=i=0, j=1$, and for such i,j,k,l, (22) $=-a^2$, and (23) $=-b^2$. We do not have to consider other cases of i,j,k,l, because as k,l,i, and j increase, the degree of $A_{i,j,k,l}$, $B_{i,j,k,l}$ and $C_{i,j,k,l}$ decreases, so they do not contribute to $\deg G_{a,b,m}(T)$.

From the expression of $G_{a,b,m}(T)$ in Lemma 2, we have $\deg G_{a,b,m}(T)\leq -b^2 = -\min(a,b)^2$. To make = hold, it suffices to show that the coefficient of $1/T^{b^2}$ in $G_{a,b,m}(T)$ is nonzero.

Let $\mathrm{lc}(Q)$ be the leading coefficient of a polynomial $Q\in\mathbb{Q}[T,T^{-1}]$, and $\alpha_k(Q)$ be the coefficient of the monomial T^k in $Q\in\mathbb{Q}[T,T^{-1}]$, $k\in\mathbb{Z}$. Then $\alpha_{-b^2}(G_{a,b,m})$ comes from $\mathrm{lc}(A_{0,1,0,0})$, $\mathrm{lc}(C_{0,1,0,0})$, and $1/T^{b^2}$ of $G_{a,b}(T)$. Namely,

$$\begin{aligned}\alpha_{-b^2}(G_{a,b,m}) = a\binom{0+0}{0}\binom{a-0-0}{0}\frac{1}{a-0-0}\cdot \mathrm{lc}(A_{0,1,0,0})\\ + c\binom{0+0}{0}\binom{c-0-0}{0}\frac{-1}{c-0-0}\cdot \mathrm{lc}(C_{0,1,0,0})+1.\end{aligned}$$

Carefully expanding $A_{0,1,0,0}$ and $C_{0,1,0,0}$, we know

$$\begin{aligned} a\binom{0+0}{0}\binom{a-0-0}{0}\frac{1}{a-0-0}\cdot \mathrm{lc}(A_{0,1,0,0}) &= \frac{ab}{a^2+ab+b^2},\\ c\binom{0+0}{0}\binom{c-0-0}{0}\frac{-1}{c-0-0}\cdot \mathrm{lc}(C_{0,1,0,0}) &= \frac{b^2-a^2}{a^2+ab+b^2}.\end{aligned}$$

Hence,

$$\begin{aligned}\alpha_{-b^2}(G_{a,b,m}) &= \frac{ab}{a^2+ab+b^2}+\frac{b^2-a^2}{a^2+ab+b^2}+1\\ &= \frac{2b(a+b)}{a^2+ab+b^2}=\frac{2c\min(a,b)}{a^2+ab+b^2}\neq 0,\end{aligned}$$

and we obtain $\alpha_{-b^2}(G_{a,b,m})=\mathrm{lc}(G_{a,b,m})\neq 0$.

The number $M_{a,b}$ can be obtained in a similar way. First, we determine s and t. From Case 1 to Case 3 in Table 1 it is easy to observe

$$\begin{aligned} s &= \max(a+b,-a,-b)=a+b,\\ t &= \max(-a-b,a,b)=\max(a,b).\end{aligned}$$

By definitions of $M_{a,b}$ in Proposition 1 and $F_{a,b}$ in Table 1, $M_{a,b}$ should be one of the followings ($c = a + b$):

$$\max\{(a+b)(a-2k-3l) + k\max(a,b) : 0 \le k, l,\ 0 \le 2k+3l \le a\}, \tag{24}$$

$$\max\{(a+b)(b-2k-3l) + k\max(a,b) : 0 \le k, l,\ 0 \le 2k+3l \le b\}, \tag{25}$$

$$\max\{(a+b)k + (c-2k-3l)\max(a,b) : 0 \le k, l,\ 0 \le 2k+3l \le c\}. \tag{26}$$

Since we assumed $\max(a, b) = a$,

$$(a+b)(a-2k-3l) + k\max(a,b) = a(a+b) - (a+2b)k - 3(a+b)l, \tag{27}$$

$$(a+b)(b-2k-3l) + k\max(a,b) = b(a+b) - (a+2b)k - 3(a+b)l, \tag{28}$$

$$(a+b)k + (c-2k-3l)\max(a,b) = a(a+b) - (a-b)k - 3al. \tag{29}$$

So, (24), (25), (26) can be obtained when $k = l = 0$. That is, (24) $= a(a+b)$, (25) $= b(a+b)$, (26) $= a(a+b)$ and clearly $M_{a,b} = a(a+b) = (a+b)\max(a,b)$.

Remark 1 While the numerator of $\mathrm{lc}(G_{a,b,m})$ and that of q_n in Table 2 of [4] coincide, the denominators are different. This happens in other cases below. But our result agrees with Louboutin's prediction on $N_{a,b,m}$.

4.1.2 $\max(a, b) = b$

We repeat what we did in 4.1.1. If $\max(a, b) = b$, then (18), (19), and (20) are

$$\begin{aligned}
\text{The RHS of (18)} &= kb + (a-2k-3l)(a+b) - (i+j)(a^2+ab+b^2)\\
&= -k(2a+b) - 3(a+b)l + a(a+b) - (i+j)(a^2+ab+b^2),\\
\text{That of (19)} &= kb + (b-2k-3l)(a+b) - (i+j)(a^2+ab+b^2)\\
&= -k(2a+b) - 3(a+b)l + b(a+b) - (i+j)(a^2+ab+b^2),\\
\text{That of (20)} &= k(a+b) + (c-2k-3l)b - (i+j)(a^2+ab+b^2)\\
&= k(a-b) - 3bl + b(a+b) - (i+j)(a^2+ab+b^2).
\end{aligned}$$

We also have the maxima of $\deg A_{i,j,k,l}$, $\deg B_{i,j,k,l}$, and $\deg C_{i,j,k,l}$ when $k = l = i = 0, j = 1$. Then $\deg A_{0,1,0,0} = -b^2$, $\deg B_{0,1,0,0} = -a^2$, and $\deg C_{0,1,0,0} = -a^2$. So, $\deg G_{a,b,m}(T) = -a^2$ provided $\alpha_{-a^2}(G_{a,b,m}) \neq 0$. By checking $\alpha_{-a^2}(G_{a,b,m})$ as

$$\begin{aligned}
\alpha_{-a^2}(G_{a,b,m}) =& b\binom{0+0}{0}\binom{b-0-0}{0}\frac{1}{b-0-0}\cdot \mathrm{lc}(B_{0,1,0,0})\\
&+ c\binom{0+0}{0}\binom{c-0-0}{0}\frac{-1}{c-0-0}\cdot \mathrm{lc}(C_{0,1,0,0}) + 1\\
&= \frac{b^2}{a^2+ab+b^2} + \frac{(b+a)(a-2b)}{a^2+ab+b^2} + 1\\
&= \frac{2a^2}{a^2+ab+b^2} = \frac{2\min(a,b)a}{a^2+ab+b^2} \neq 0,
\end{aligned}$$

we have $\alpha_{-a^2}(G_{a,b,m}) = \mathrm{lc}(G_{a,b,m}) \neq 0$ and conclude $\deg G_{a,b,m}(T) = -a^2 = -\min(a,b)^2 = -N_{a,b,m}$.

To determine $M_{a,b}$, we investigate (24), (25), and (26) as in §4.1.1. Since

$$(a+b)(a-2k-3l)+k\max(a,b)=a(a+b)-(2a+b)k-3(a+b)l, \tag{30}$$

$$(a+b)(b-2k-3l)+k\max(a,b)=b(a+b)-(2a+b)k-3(a+b)l, \tag{31}$$

$$(a+b)k+(c-2k-3l)\max(a,b)=b(a+b)-(b-a)k-3bl, \tag{32}$$

we know (24) $= a(a+b)$, (25) $= b(a+b)$, (26) $= b(a+b)$ by the same argument. Hence $M_{a,b} = b(a+b) = (a+b)\max(a,b)$.

Remark 2 One can completely verify Table 1 of [4] in the same way. Since (27), (28), (29) are still valid in §4.2.1 and §4.3.1, and (30), (31), (32) valid in §4.2.2 and §4.3.2, we will not repeat the proof $M_{a,b} = (a+b)\max(a,b)$ for remaining cases in following sections.

Remark 3 In the remaining cases, one can easily observe that (21) $= -b^2$, (22) $= -a^2$, and (23) $= -\min(a,b)^2$ as in §4.1 by the same argument. So, it suffices to observe whether $\alpha_{-b^2}(G_{a,b,m}) \neq 0$ in §4.2.1 and §4.3.1, and $\alpha_{-a^2}(G_{a,b,m}) \neq 0$ in §4.2.2 and §4.3.2.

4.2 Case 2: $a \geq 1$ odd and $b \geq 1$ even

The argument in this subsection is similar to §4.1. For Case 2, $F_{a,b}(X,Y)$ and $G_{a,b}(T)$ are given by $F_{a,b}(X,Y) = -P_a(-Y,-X) - P_b(-Y,-X) + P_c(-X,-Y)$, and $G_{a,b}(T) = -T^{-a^2} + T^{-b^2} + T^{-c^2} + 2T^{-ab}$. As in Case 1, we put $X = R_{a,b,m}(T)$ and $Y = R_{-a,-b,m}(T)$. Then $G_{a,b,m}(T)$ is

$$\begin{aligned}
G_{a,b,m}(T) = &-\frac{1}{T^{a^2}} + \frac{1}{T^{b^2}} + \frac{1}{T^{c^2}} + \frac{2}{T^{ab}} \\
&+ \underbrace{a \sum_{\substack{k,l\geq 0 \\ 0\leq 2k+3l\leq a}} (-1)^k \binom{k+l}{k}\binom{a-k-2l}{k+l} \frac{(-1)^{a-k-3l}}{a-k-2l} \sum_{\substack{0\leq i\leq k \\ 0\leq j\leq a-2k-3l \\ (i,j)\neq(0,0)}} A_{i,j,k,l}}_{=A_2(T)} \\
&+ \underbrace{b \sum_{\substack{k,l\geq 0 \\ 0\leq 2k+3l\leq b}} (-1)^k \binom{k+l}{k}\binom{b-k-2l}{k+l} \frac{(-1)^{b-k-3l}}{b-k-2l} \sum_{\substack{0\leq i\leq k \\ 0\leq j\leq b-2k-3l \\ (i,j)\neq(0,0)}} B_{i,j,k,l}}_{=B_2(T)} \\
&+ \underbrace{c \sum_{\substack{k,l\geq 0 \\ 0\leq 2k+3l\leq c}} (-1)^{k-1} \binom{k+l}{k}\binom{c-k-2l}{k+l} \frac{(-1)^{c-k-3l}}{c-k-2l} \sum_{\substack{0\leq i\leq k \\ 0\leq j\leq c-2k-3l \\ (i,j)\neq(0,0)}} C_{i,j,k,l}}_{=C_2(T)}.
\end{aligned}$$

We get

Lemma 3 *For $a \geq 1$ odd, $b \geq 1$ even, $m = m_{a,b}$, and $c = a+b$, we have*

$$G_{a,b,m}(T) = -\frac{1}{T^{a^2}} + \frac{1}{T^{b^2}} + \frac{1}{T^{c^2}} + \frac{2}{T^{ab}} + A_2(T) + B_2(T) + C_2(T).$$

4.2.1 max(*a*, *b*) = *a*

By Remark 3, we check $\alpha_{-b^2}(G_{a,b,m}) \neq 0$. It is

$$\begin{aligned} &a\binom{0+0}{0}\binom{a-0-0}{0}\frac{(-1)^a}{a-0-0}\cdot \mathrm{lc}(A_{0,1,0,0})\\ &\quad + c\binom{0+0}{0}\binom{c-0-0}{0}\frac{(-1)^{c+1}}{c-0-0}\cdot \mathrm{lc}(C_{0,1,0,0}) + 1\\ &= -\frac{ab}{a^2+ab+b^2} - \frac{b^2-a^2}{a^2+ab+b^2} + 1 = \frac{2a^2}{a^2+ab+b^2} \neq 0, \end{aligned}$$

we know $\alpha_{-b^2}(G_{a,b,m}) = \mathrm{lc}(G_{a,b,m}) \neq 0$ and $\deg G_{a,b,m}(T) = -b^2 = -\min(a,b)^2 = -N_{a,b,m}$.

4.2.2 max(*a*, *b*) = *b*

Observing $\alpha_{-a^2}(G_{a,b,m})$ by (with Remark 3 kept in mind)

$$\begin{aligned} &b\binom{0+0}{0}\binom{b-0-0}{0}\frac{(-1)^b}{b-0-0}\cdot \mathrm{lc}(B_{0,1,0,0})\\ &\quad + c\binom{0+0}{0}\binom{c-0-0}{0}\frac{(-1)^{c+1}}{c-0-0}\cdot \mathrm{lc}(C_{0,1,0,0}) - 1\\ &= \frac{b^2}{a^2+ab+b^2} + \frac{(b+a)(a-2b)}{a^2+ab+b^2} - 1 = \frac{-2b(a+b)}{a^2+ab+b^2} = \frac{-2bc}{a^2+ab+b^2} \neq 0, \end{aligned}$$

we conclude $\deg G_{a,b,m}(T) = -a^2 = -\min(a,b)^2 = -N_{a,b,m}$.

4.3 Case 3: *a* ≥ 2 even and *b* ≥ 1 odd

While explicitly describing $G_{a,b,m}(T)$ by putting $X = R_{a,b,m}(T)$ and $Y = R_{-a,-b,m}(T)$ at $F_{a,b}(X,Y) = -P_a(-Y,-X) - P_b(-Y,-X) - P_c(-X,-Y)$, we only have to take extra care of $-P_c(-X,-Y)$, with others same as in Case 2. From Lemma 1,

$$\begin{aligned} G_{a,b,m}(T) = &\frac{1}{T^{a^2}} + \frac{1}{T^{b^2}} - \frac{1}{T^{c^2}}\\ &+ \underbrace{a\sum_{\substack{k,l\geq 0\\ 0\leq 2k+3l\leq a}} (-1)^k\binom{k+l}{k}\binom{a-k-2l}{k+l}\frac{(-1)^{a-k-3l}}{a-k-2l}\sum_{\substack{0\leq i\leq k\\ 0\leq j\leq a-2k-3l\\ (i,j)\neq(0,0)}} A_{i,j,k,l}}_{=A_3(T)}\\ &+ \underbrace{b\sum_{\substack{k,l\geq 0\\ 0\leq 2k+3l\leq b}} (-1)^k\binom{k+l}{k}\binom{b-k-2l}{k+l}\frac{(-1)^{b-k-3l}}{b-k-2l}\sum_{\substack{0\leq i\leq k\\ 0\leq j\leq b-2k-3l\\ (i,j)\neq(0,0)}} B_{i,j,k,l}}_{=B_3(T)}\\ &+ \underbrace{c\sum_{\substack{k,l\geq 0\\ 0\leq 2k+3l\leq c}} (-1)^k\binom{k+l}{k}\binom{c-k-2l}{k+l}\frac{(-1)^{c-k-3l}}{c-k-2l}\sum_{\substack{0\leq i\leq k\\ 0\leq j\leq c-2k-3l\\ (i,j)\neq(0,0)}} C_{i,j,k,l}}_{=C_3(T)}. \end{aligned}$$

Lemma 4 *For $a \geq 2$ even, $b \geq 1$ odd, $m = m_{a,b}$, and $c = a + b$, we have*

$$G_{a,b,m}(T) = \frac{1}{T^{a^2}} + \frac{1}{T^{b^2}} - \frac{1}{T^{c^2}} + A_3(T) + B_3(T) + C_3(T).$$

4.3.1 max(*a*, *b*) = *a*

$\alpha_{-b^2}(G_{a,b,m})$ can be described as

$$\begin{aligned}
&a\binom{0+0}{0}\binom{a-0-0}{0}\frac{(-1)^a}{a-0-0}\cdot \mathrm{lc}(A_{0,1,0,0})\\
&\quad + c\binom{0+0}{0}\binom{c-0-0}{0}\frac{(-1)^c}{c-0-0}\cdot \mathrm{lc}(C_{0,1,0,0}) + 1\\
&= \frac{ab}{a^2+ab+b^2} + \frac{b^2-a^2}{a^2+ab+b^2} + 1 = \frac{2b(a+b)}{a^2+ab+b^2} = \frac{2bc}{a^2+ab+b^2} \neq 0.
\end{aligned}$$

So, we know $\alpha_{-b^2}(G_{a,b,m}) = \mathrm{lc}(G_{a,b,m}) \neq 0$ and $\deg G_{a,b,m}(T) = -b^2 = -\min(a,b)^2 = -N_{a,b,m}$.

4.3.2 max(*a*, *b*) = *b*

Since $\alpha_{-a^2}(G_{a,b,m})$ is

$$\begin{aligned}
&b\binom{0+0}{0}\binom{b-0-0}{0}\frac{(-1)^b}{b-0-0}\cdot \mathrm{lc}(B_{0,1,0,0})\\
&\quad + c\binom{0+0}{0}\binom{c-0-0}{0}\frac{(-1)^c}{c-0-0}\cdot \mathrm{lc}(C_{0,1,0,0}) - 1\\
&= -\frac{b^2}{a^2+ab+b^2} - \frac{(b+a)(a-2b)}{a^2+ab+b^2} + 1 = \frac{2b(a+b)}{a^2+ab+b^2} = \frac{2bc}{a^2+ab+b^2} \neq 0,
\end{aligned}$$

we conclude $\deg G_{a,b,m}(T) = -a^2 = -\min(a,b)^2 = -N_{a,b,m}$. Hence, the proof of Conjecture 2 is complete.

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Data availability No datasets were generated or analysed during the current study.

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