ARITHMETIC CHERN–SIMONS ACTIONS OF MASSEY TYPE AND TRIPLE SYMBOLS

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Abstract. We introduce and investigate a kind of arithmetic Chern–Simons action which we call the Massey type. We evaluate certain values of the Massey type action in terms of triple symbol. This relation is reminescent of the evaluation of the abelian type action in terms of Legendre symbols. Using the evaluation in terms of triple symbols, we also deduce from the genus theory that the Massey type action vanishes identically for an infinite family of quadratic fields.

1. Introduction

In 2015 [11], Minhyong Kim introduced the arithmetic Chern–Simons theory. Its key object is the arithmetic Chern–Simons action, whose definition – to be recalled soon – which is interpreted as the arithmetic analogue of its counterpart in differential topology. The subject evolved to include the arithmetic BF-theory [3, 2, 16] and an overview is given in [10]. A treatment focusing on the arithemtic Chern–Simons theory and how it incarnates a topological quantum field theory is given in [15].

Our goal in this article is to introduce a kind of the arithmetic Chern–Simons action which we call the Massey type and investigate it in terms of triple symbols [9]. To proceed, we need to recall the general definition of arithmetic Chern–Simons action. The description we give below is due to [7, 13], where the definition of the arithmetic Chern–Simons action as stated in [11] was extended to allow number fields with real places. This extension is relevant for us since we allow such real places throughout.

Let $n \geq 2$ be an integer, F a number field with a fixed primitive n-th root of unity $\zeta \in F$. We further choose an algebraic closure $\bar{F}/F$ and denote $\tilde{F}/F$ by the maximal unramified subextension – we adopt the convention that a real place is required to remain real in an unramified extension. Putting $\pi_1 = \mathrm{Gal}(\tilde{F}/F)$, π_1 is identified with the fundamental group of X_F, by which we denote the Artin–Verdier site associated to F. The choice of ζ determines an embedding $\mathbb{Z}/n\mathbb{Z} \hookrightarrow F^\times$ which sends $1 \in \mathbb{Z}/n\mathbb{Z}$ to ζ which uniquely extends to a map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{G}_m$ between sheaves. It gives rise to a map

$$j\colon H^3(\pi_1, \mathbb{Z}/n\mathbb{Z}) \to H^3(X_F, \mathbb{G}_m)$$

from the continuous group cohomology into the Artin–Verdier cohomology group with coefficients in the multiplicative group $\mathbb{G}_m$. The Artin–Verdier duality yields a canonical

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isomorphism

$$\mathrm{inv}\colon H^3(X_F, \mathbb{G}_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

Composition yields a map $\mathrm{inv} \circ j\colon H^3(\pi_1, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. Based on the above cohomological construction, we proceed as follows. Let A be an arbitrary finite group and

$$c \in Z^3(A, \mathbb{Z}/n\mathbb{Z})$$

be a cocycle. For each pair A and c, we define

$$\mathrm{CS}_c\colon \mathrm{Hom}(\pi_1, A) \to \mathbb{Q}/\mathbb{Z},$$

called the arithmetic Chern–Simons action, by

$$\mathrm{CS}_c(\rho) = \mathrm{inv} \circ j \circ \rho^*(c) \tag{1}$$

for every $\rho \in \mathrm{Hom}(\pi_1, A)$. Of course, CS_c only depends on the cohomology class represented by c, but we prefer the decoration by the cocycle c because much of the forthcoming analysis will depend on it.

A great amount of flexibility is available since CS_c makes sense for an arbitrary choice of the pair (A, c). It is a feature of the theory which allows us to access the information of π_1 that is not easily made available in terms of the class field theory describing its abelian quotient. However, the same flexibility makes implausible that a neat formula exists for $\mathrm{CS}_c(\rho)$ in the full generality.

For the sake of an explicit formula, a relatively accessible family arises when $A = \mathbb{Z}/n\mathbb{Z}$ and c is chosen to be $c = I_A \cup \delta I_A$, where $I_A\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is the identity map and δ is the Bockstein map associated to the short exact sequence $0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$. This case is often referred to as the abelian case because of its analogy with $U(1)$, the only abelian member in the family of unitary groups $U(m)$ with $m \geq 1$. The abelian case has been investigated in [1, 4, 13, 7] and the resulting partition function in [5, 6]. Here, we recall without proof the case treated in [4].

Proposition 1 (Prop. 5.14 of [4])**.** *Take $n = 2$ and $c = I_A \cup \delta I_A$. Let $F = \mathbb{Q}(\sqrt{-pt})$ where $p \equiv 1$ modulo 4 and t is a positive squarefree integer relatively priem to p. Consider $\rho\colon \pi_1 \to A$ corresponding to the extension $F(\sqrt{p})/F$. Then,*

$$\exp\left(2\pi\sqrt{-1}\,\mathrm{CS}_c(\rho)\right) = \left(\frac{t}{p}\right). \tag{2}$$

It is natural to ask if the arithmetic Chern–Simons action admits a similar interpretation apart from the abelian case. In this paper, we address this question by investigating what we call the Massey type by which we refer to the following choice of (A, c). Let $A = (\mathbb{Z}/n\mathbb{Z})^3$. For each $i = 1, 2, 3$, denote by $\mathrm{pr}_i\colon A \to \mathbb{Z}/n\mathbb{Z}$ the projection onto the i-th component. Then, we take $c = \mathrm{pr}_1 \cup \mathrm{pr}_2 \cup \mathrm{pr}_3$.

Remark 2. We call it the Massey type because in some cases it can be identified with Massey product in Galois cohomology. See [9] and [14] for details.

In the Massey case, a homomorphism $\rho\colon \pi_1 \to A$ is necessarily of the form $\rho = (\chi_1, \chi_2, \chi_3)$ where $\chi_i\colon \pi_1 \to \mathbb{Z}/n\mathbb{Z}$ is an unramified character for each i. Such a triple (χ_1, χ_2, χ_3) of characters determint their symbol $[\chi_1, \chi_2, \chi_3]_n$, as introduced in [9]. Our first result is the relation between the trple symbols and the Massey type arithmetic

Chern–Simons action. It is convenient to interpret triple symbols as complex numbers and for this purpose we identify $\zeta = \exp(2\pi\sqrt{-1}/n)$.

Theorem 3. *Suppose $\rho = (\chi_1, \chi_2, \chi_3)$ where χ_i is an unramified character for each $i = 1, 2, 3$. The Massey type arithmetic Chern–Simons action satisfies*

$$\exp\left(2\pi\sqrt{-1}\,\mathrm{CS}_c(\rho)\right) = [\chi_1, \chi_2, \chi_3]_n. \tag{3}$$

Proof. See § 4. □

Remark 4. In [12], generalized Rédei symbols were introduced based on certain axioms for Galois modules. The triple symbol introduced [9] generalized Rédeis symbols and satisfies the axiomatic conditions in [12].

Comparing the formula (2) in the abelian case and its counterpart (8) in the Massey case, we see that the role played by Legendre symbols is taken by triple symbols. From the point of view from the arithmetic topology, the Legendre symbols are interpreted as a double linking invariant while the Rédei symbols – a special kind of triple symbols – are intrepreted as a triple linking invariant. From this perspective, the Massey type arithmetic Chern–Simons action is a natural object to investigate after the abelian case was handled in [2, 1].

In the abelian case, (2) and its variants allow one to analyze the function CS_c in terms of the Legendre symbols [7, 6]. It is natural to ask if (8) has consequences on the behaviour of the Massey type action. This question motivates our second result. To state it, we need some notation; for an odd prime p, let $p^* = (-1)^{\frac{p-1}{2}}p$.

Theorem 5. *Let F be a quadratic field and $n = 2$. Assume that the discriminant of F is odd and that $\left(\frac{p^*}{q}\right) = 1$ for every odd distinct primes p and q ramified in F. Then, the Massey type arithmetic Chern–Simons action is identically zero; $\mathrm{CS}_c(\rho) = 0$ for every $\rho\colon \pi_1 \to A$.*

Proof. See § 7. □

Remark 6. Applying the Kunneth theorem, it is easy to see that the cocycle c in the Massey case is not a coboundary. Thm. 5 seems to be the first non-abelian case where one can show that the CS_c vanishes identically for a class c which is not a coboundary.

Remark 7. It remains to describe the Massey type action for quadratic fields when we relax the assumption on Legendre symbols in Thm. 5.

2. THE GLUING FORMULA

While the functional $\rho \mapsto \mathrm{CS}_c(\rho)$ is succintly defined in (1), an alternative description, which we call the gluing formula, will play a major role in the sequel. The formula expresses $\mathrm{CS}_c(\rho)$ in Galois-theoretic data which will depend on certain choices. Of course, the choices to be made will not affect $\mathrm{CS}_c(\rho)$. This indepndence is a consequence of the global reciprocity. One might regard the necessity to make such choices as unwanted complications, but it is precisely those choices which will allow us an access to $\mathrm{CS}_c(\rho)$.

The gluing formula will be based on several cochain complexes so we begin by setting up the notation for them. For a profinite group G acting continuously on a finite abelian group M, denote by

$$C^\bullet(G, M)$$

the bar complex. We denote the differential by d, the subgroup of cocycles by $Z^\bullet(G, M)$ and that of coboundaries by $B^\bullet(G, M)$. For a place v of F, we put

$$\pi_v = \mathrm{Gal}(\overline{F}_v/F_v)$$

where $\overline{F}_v$ is an algebraic closure of F_v. The complex $C^\bullet(\pi_v, M)$ is equipped with a subgroup consisting of unramified cochains, which we define to be those lying in the image of

$$C^\bullet(\pi_v/I_v, M^{I_v}) \to C^\bullet(\pi_v, M)$$

where $I_v \subset \pi_v$ denotes the inertia subgroup. The subgroup of unramified cochains will be denoted by

$$C^\bullet_{\mathrm{ur}}(\pi_v, M) \subset C^\bullet(\pi_v, M).$$

For a finite set S of finite places of F, put

$$\pi_S = \mathrm{Gal}(F_S/F)$$

where F_S denotes the maximal subextension of $\overline{F}/F$ unramified outside of S.

We are ready to formulate the gluing formula [4, 13, 8], also known as the decomposition formula. Suppose that we are given $c \in Z^3(A, \mathbb{Z}/n\mathbb{Z})$ and $\rho \in \mathrm{Hom}(\pi, A)$. It determines a 3-cocycle

$$\rho^* c \in Z^3(\pi_1, \mathbb{Z}/n\mathbb{Z}).$$

For each $v \in S$, let

$$(\rho^* c)_v \in Z^3(\pi_v, \mathbb{Z}/n\mathbb{Z})$$

be the image of $\rho^* c$ under the map

$$C^3(\pi_1, \mathbb{Z}/n\mathbb{Z}) \to C^3(\pi_v, \mathbb{Z}/n\mathbb{Z}).$$

We further denote by

$$x_v(\rho^* c) \in C^2_{\mathrm{ur}}(\pi_v, \mathbb{Z}/n\mathbb{Z})$$

any unramified cochain satisfying $dx_v(\rho^* c) = (\rho^* c)_v$. On the other hand, consider the image $(\rho^* c)_S$ of $\rho^* c$ under the map

$$C^\bullet(\pi_1, \mathbb{Z}/n\mathbb{Z}) \to C^\bullet(\pi_S, \mathbb{Z}/n\mathbb{Z})$$

which is induced by the natural surjection $\pi_S \to \pi_1$. We denote by

$$\beta_\rho \in C^2(\pi_S, \mathbb{Z}/n\mathbb{Z})$$

any cochain satisfying $d\beta_\rho = (\rho^* c)_S$. Note that we suppress c from the notation although β_ρ depends on c.

For each $v \in S$, let $\mathrm{loc}_v \colon \pi_v \to \pi_S$ be the natural map induced by restriction. Denote by loc_v^* the associated map on cochain complexes. In particular,

$$\mathrm{loc}_v^*(\beta_\rho) = C^2(\pi_v, \mathbb{Z}/n\mathbb{Z})$$

for each $v \in S$ and it satisfies $d\,\mathrm{loc}_v^*(\beta_\rho) = (\rho^* c)_v$.

Theorem 8 ([4, 13, 8]). *Suppose that S contains all places of F dividing n. Then,*

$$\mathrm{CS}_c(\rho) = \sum_{v \in S} \mathrm{inv}_v \left(\mathrm{loc}_v^*(\beta_\rho) - x_v(\rho^* c)\right) \tag{4}$$

holds for any choices of β_ρ and $x_v(\rho^* c)$'s. Here, $\mathrm{inv}_v \colon H^2(F_v, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ denotes the canonical isomorphism from the local class field theory determined by our choice of $\zeta \in F$.

Proof. See Thm. 2.2.14 of [8]. □

The implicit in the above theorem is that β_ρ and $x_v(\rho^* c)$'s exist. Once they are found, the computation of $\mathrm{CS}_c(\rho)$ is reduced to the calculation of local invariant maps.

3. Triple symbols

We recall the notion of triple symbols following [9]. For $i = 1, 2, 3$, consider a character

$$\chi_i \colon \mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}/n\mathbb{Z}.$$

In order to define the triple symbol associated to (χ_1, χ_2, χ_3) we impose some conditions.

Definition 9. A triple (χ_1, χ_2, χ_3) is called admissible if the following conditions are satisfied.

1. For every $i = 1, 2, 3$, χ_i is tamely ramified.
2. Denote by S_i the set of ramified places for χ_i. The sets S_1, S_2, S_3 are pairwise disjoint.
3. For every $v \in S_1 \cup S_2 \cup S_3$ and every $i, j = 1, 2, 3$, $\mathrm{loc}_v^*(\chi_i \cup \chi_j) = 0$ in $H^2(\pi_v, \mathbb{Z}/n\mathbb{Z})$.

According to [9, Thm. 1], for an admissible (χ_1, χ_2, χ_3) one can define its triple symbol

$$[\chi_1, \chi_2, \chi_3]_n \in \mu_n \tag{5}$$

where μ_n detotes the subgroup of $\overline{F}$ consisting of all n-th roots of unity. Since every element of μ_n is a power of $\exp(2\pi\sqrt{-1}/n)$, we have

$$[\chi_1, \chi_2, \chi_3]_n = \exp\left(2\pi\sqrt{-1}\mathfrak{d}(\chi_1, \chi_2, \chi_3)\right) \tag{6}$$

for a unique exponent $\mathfrak{d}(\chi_1, \chi_2, \chi_3) \in \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

We recall the definition of $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ which is reminiscent of the gluing formula. Choose any finite set S of places of F which contains all of S_i's. Then, for the Galois group π_S one has a cocycle

$$\chi_1 \cup \chi_2 \cup \chi_3 \in Z^3(\pi_S, \mathbb{Z}/n\mathbb{Z}).$$

We may assume, by enlarging S if necessary in order to ensure $H^2(\pi_S, \mathbb{Z}/n\mathbb{Z}) = 0$, that there exists $\eta \in C^2(\pi_S, \mathbb{Z}/n\mathbb{Z})$ such that

$$d\eta = \chi_1 \cup \chi_2 \cup \chi_3.$$

On the other hand, it is shown in [9] that the admissibility of (χ_1, χ_2, χ_3) implies the existence of an unramified cochain $\eta_v \in C^2_{\mathrm{ur}}(\pi_v, \mathbb{Z}/n\mathbb{Z})$ such that

$$d\eta_v = \mathrm{loc}_v^*\left(\chi_1 \cup \chi_2 \cup \chi_3\right).$$

Now we combine them to form the expression

$$\mathfrak{d}(\chi_1, \chi_2, \chi_3) := \sum_{v \in S} \mathrm{inv}_v \left(\mathrm{loc}_v^* \eta - \eta_v \right) \tag{7}$$

which is well-defined independently of the choices we made for η and η_v's, as shown in [9].

4. Proof of Thm. 3

For readers' convenience, we repeat the statement.

Theorem (Thm. 3)**.** *Suppose $\rho = (\chi_1, \chi_2, \chi_3)$ where χ_i is an unramified character for each $i = 1, 2, 3$. The Massey type arithmetic Chern–Simons action satisfies*

$$\exp\left(2\pi\sqrt{-1}\,\mathrm{CS}_c(\rho)\right) = [\chi_1, \chi_2, \chi_3]_n. \tag{8}$$

Proof. The similarity between (7) and (4) is manifest. It will indeed motivate our proof, which will equate the gluing formula and the right-hand-side of (7) term-by-term.

We review the choice of A and c we made in the introduction for the Massey case. Put $A = (\mathbb{Z}/n\mathbb{Z})^3$. For each $i = 1, 2, 3$, let $\mathrm{pr}_i \colon A \to \mathbb{Z}/n\mathbb{Z}$ be the projection onto the i-th component. Then, we take $c = \mathrm{pr}_1 \cup \mathrm{pr}_2 \cup \mathrm{pr}_3$ which is an element of $Z^3(A, \mathbb{Z}/n\mathbb{Z})$.

Consider homomorphisms

$$\chi_i \colon \mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}/n\mathbb{Z}$$

for each $i = 1, 2, 3$. We will assume throughout that each χ_i is unramified. Note that this implies that the triple is admissible in the sense of Def. 9. On the other hand, the assumption implies that

$$\begin{aligned} \mathrm{Gal}(\overline{F}/F) &\to A \\ g &\mapsto (\chi_1(g), \chi_2(g), \chi_3(g)) \end{aligned}$$

factors through $\mathrm{Gal}(\overline{F}/F) \twoheadrightarrow \pi_1$. Denote the result by $\rho : \pi_1 \to A$.

Recall the relation (6) between $[\chi_1, \chi_2, \chi_3]_n$ and $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$. The proof of Thm. 3 reduces to

$$\mathrm{CS}_c(\rho) = \mathfrak{d}(\chi_1, \chi_2, \chi_3)$$

by comparing the exponents.

By (7), choose a sufficiently large S and express

$$\mathfrak{d}(\chi_1, \chi_2, \chi_3) = \sum_{v \in S} \mathrm{inv}_v \left(\mathrm{loc}_v^* \eta - \eta_v \right).$$

Then, taking $\beta_\rho = \eta$ and $x_v(\rho^* c) = \eta_v$, which satisfy the requirements for the gluing formula, we obtain from (4) that

$$\mathrm{CS}_c(\rho) = \mathfrak{d}(\chi_1, \chi_2, \chi_3).$$

This completes the proof. □

5. TRIPLE SYMBOLS FOR RELATIVE GENUS CHARACTERS

Here we state and prove a proposition to be used in the sequel. Let K/F be a finite extension.

Definition 10. An unramified character $\psi\colon \mathrm{Gal}(\overline{F}/K) \to \mathbb{Z}/n\mathbb{Z}$ is called a relative genus character with respect to K/F if there is a possibly ramified character $\chi\colon \mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}/n\mathbb{Z}$ such that $\psi = \chi \circ r$ where $r\colon \mathrm{Gal}(\overline{F}/K) \to \mathrm{Gal}(\overline{F}/F)$ denotes the natural inclusion.

Proposition 11. *Let ψ_i be a relative genus character of the form $\psi_i = \chi_i \circ r$. Put $\tau = (\psi_1, \psi_2, \psi_3)$. Suppose that (χ_1, χ_2, χ_3) is admissible in the sense of Def. 9. Then, $\mathrm{CS}_c(\tau) = [K : F][\chi_1, \chi_2, \chi_3]_n$. Equivalently, we have $[\psi_1, \psi_2, \psi_3]_n = [K : F][\chi_1, \chi_2, \chi_3]_n$.*

Proof. Using (4), express $\mathfrak{d}(\chi_1, \chi_2, \chi_3)$ as

$$\mathfrak{d}(\chi_1, \chi_2, \chi_3) = \sum_{v\in S} \mathrm{inv}_v \left(\mathrm{loc}_v^* \, \eta - \eta_v\right).$$

Here, $\eta \in C^2(\pi_S, \mathbb{Z}/n\mathbb{Z})$ and $\eta_v \in C^2_{\mathrm{ur}}(\pi_v, \mathbb{Z}/n\mathbb{Z})$. Let T be the set of places of K whose elements lie over a place in S. Then, $K_T \supset F_S$ and the restriction induces a homomorphism $\pi_T \to \pi_S$. In turn, it induces a cochain map

$$C^2(\pi_S, \mathbb{Z}/n\mathbb{Z}) \to C^2(\pi_T, \mathbb{Z}/n\mathbb{Z}).$$

Denote by η_K the image of η under the above map. Similarly, for each $w \in T$ lying over $v \in F$, we have a map

$$C^2_{\mathrm{ur}}(\pi_v, \mathbb{Z}/n\mathbb{Z}) \to C^2_{\mathrm{ur}}(\pi_w, \mathbb{Z}/n\mathbb{Z}).$$

Denote by η_w the image of η_v under the above map. Then, (4) implies

$$\mathrm{CS}_c(\tau) = \sum_{v\in S}\sum_{w|v} \mathrm{inv}_w \left(\mathrm{loc}_w^* \, \eta_K - \eta_w\right). \tag{9}$$

We claim that

$$\sum_{w|v} \mathrm{inv}_w \left(\mathrm{loc}_w^* \, \eta_K - \eta_w\right) = [K : F] \, \mathrm{inv}_v \left(\mathrm{loc}_v^* \, \eta - \eta_v\right). \tag{10}$$

Indeed, we first note the numerical identity $[K : F] = \sum_{w|v} [K_w : F_v]$. The asserted equality follows from the commutation relation $[K_w : F_v] \, \mathrm{inv}_v = \mathrm{inv}_w \circ r_{v,w}$ from the local class field theory, where we temporarily denoty by $r_{v,w}$ the restriction map $H^2(F_v, \mathbb{Z}/n\mathbb{Z}) \to H^2(K_w, \mathbb{Z}/n\mathbb{Z})$ between the Galois cohomology groups.

Applying (10) to each inner summation of (9), we obtain the assertion of the proposition. □

6. Basis properties

We record three lemmas which will be used later. All of the arithmetic Chern–Simons actions are of the Massey type.

Lemma 12. *Suppose that $\rho = (\chi_1, \chi_2, \chi_3)$ is a homomorphism $\rho\colon \pi_1 \to A$ and that $u\colon \{1, 2, 3\} \to \{1, 2, 3\}$ is a permutation with sign $|u|$. Put $u^*\rho = \big(\chi_{u(1)}, \chi_{u(2)}, \chi_{u(3)}\big)$. Then,*

$$\mathrm{CS}_c(\rho) = |u| \, \mathrm{CS}_c(u^*\rho).$$

Proof. It follows from the fact that the cup product is graded-commutative. □

Lemma 13. *Suppose that $\rho = (\chi_1, \chi_2, \chi_3)$ is as above and that $\chi_1 = \chi_1' + \chi_1''$ for some $\chi_1', \chi_1'' \colon \pi_1 \to \mathbb{Z}/n\mathbb{Z}$. Put $\rho' = (\chi_1', \chi_2, \chi_3)$ and $\rho'' = (\chi_1'', \chi_2, \chi_3)$. Then,*

$$\mathrm{CS}_c(\rho) = \mathrm{CS}_c(\rho') + \mathrm{CS}_c(\rho'').$$

Proof. First observe

$$\rho^* c = \chi_1 \cup \chi_2 \cup \chi_3$$

and similarly

$$(\rho')^* c = \chi_1' \cup \chi_2 \cup \chi_3$$
$$(\rho'')^* c = \chi_1'' \cup \chi_2 \cup \chi_3.$$

The assertion of the lemma follows from the fact that cup product defines an associative ring structure on cohomology. □

Lemma 14. *Suppose that $\rho = (\chi_1, \chi_2, 0)$. Then,* $\mathrm{CS}_c(\rho) = 0$

Proof. An immediate consequence of Lem. 12 and Lem. 13. □

7. Proof of Thm 5

For readers' convenience, we repeat the statement.

Theorem (Thm. 5)**.** *Let F be a quadratic field and $n = 2$. Assume that the discriminant of F is odd and that $\left(\frac{p^*}{q}\right) = 1$ for every odd distinct primes p and q ramified in F. Then, the Massey type arithmetic Chern–Simons action is identically zero;* $\mathrm{CS}_c(\rho) = 0$ *for every $\rho \colon \pi_1 \to A$.*

Proof. First, we recall set up some notation and recall the genus theory. By the assumption, F is a quadratic field and its discriminant D is necessarily of the form $D = (-1)^{\frac{D-1}{2}} p_1 \cdots p_m$ where $p_1, \cdots, p_m$ are distinct odd primes. In fact, $\mathbb{Q}(\sqrt{D})$ is the unique such extension. Recall $p_i^* = (-1)^{\frac{p_i - 1}{2}} p_i$ and denote by $E_i := \mathbb{Q}(\sqrt{p_i^*})$ the unique quadratic extension of discriminant p_i^*. Then, an elementary consideration shows that FE_i/F is an unramified quadratic extension. Moreover, by the genus theory, if we denote by $\sigma_i \colon \pi_1 \to A$ the character corresponding to FE_i/F, then $\mathrm{Hom}(\pi_1, A)$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$ with a basis $\sigma_1, \cdots, \sigma_m$. For a subset $I \subset \{1, 2, \cdots, m\}$, put

$$\sigma(I) := \prod_{i \in I} \sigma_i.$$

The assertion that σ_i's form a basis is equivalent to: an element of $\mathrm{Hom}(\pi_1, A)$ can be uniquely written as $\sigma(I)$ for some I.

Put $\rho(I_1, I_2, I_3) := (\sigma(I_1), \sigma(I_2), \sigma(I_3))$. Now the goal of the Thm. 5 can be rephrased as action's vanishing

$$\mathrm{CS}_c\left(\rho(I_1, I_2, I_3)\right) = 0$$

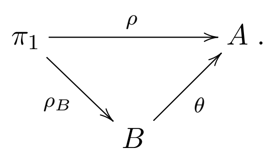
for all subsets $I_1, I_2, I_3 \subset \{1, 2, \cdots, m\}$. To prove the vanishing, by Lem. 14, it suffices to consider the case when none of I_1, I_2, I_3 are empty. For the cases when non of the I_1, I_2, I_3 are empty, Lem. 12 and Lem. 13 imply that it suffices to consider the case $|I_1| = |I_2| = |I_3| = 1$.

Assume $|I_1| = |I_2| = |I_3| = 1$ for the rest of the proof. There are two cases consider. In the first case, I_1, I_2, and I_3 are disjoint. Reordering the prime divisors of D if necessary, we may assume that $I_i = \{i\}$. By Thm. 3, we have $\mathrm{CS}_c(\rho) = [\sigma_1, \sigma_2, \sigma_3]_2$. Each σ_i is a relative genus character for $F/\mathbb{Q}$. Precisely, $\sigma_i = \chi_i \circ r$ and where χ_i is the Kummer character associated to p_i^*. By Prop. 5, $[\sigma_1, \sigma_2, \sigma_3]_2 = 2 \cdot [\chi_1, \chi_2, \chi_3]_2$. Since $[\chi_1, \chi_2, \chi_3]_2 \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, we conclude that $[\sigma_1, \sigma_2, \sigma_3]_2 = 0$.

We proceed to the second case when two of I_1, I_2, I_3 are equal. We claim that $\rho^* c = 0$ in $H^3(\pi_1, \mathbb{Z}/n\mathbb{Z})$, from which it formally follows that $\mathrm{CS}_c(\rho) = 0$. By Lem 12, we may assume without loss of generality that $I_1 = I_2 = \{1\}$. Letting $I_3 = \{j\}$, $\rho = (\sigma_1, \sigma_1, \sigma_j)$ is a homomorphism $\pi_1 \xrightarrow{\rho} A$. Recalling $A = (\mathbb{Z}/2\mathbb{Z})^3$, put $B = (\mathbb{Z}/2\mathbb{Z})^2$. Define a homomorphism $\theta \colon B \to A$ by

$$\theta(b_1, b_2) = (b_1, b_1, b_2).$$

Then, by construction, we have a commutative diagram



It follows that

$$\rho^* c = \rho_B^* \theta^* c$$

from which we conclude that to show $\rho^* c = 0$ it suffices to show $\theta^* c = 0$ in $H^3(B, \mathbb{Z}/2\mathbb{Z})$. To see $\theta^* c$ is null-homologous, begin by observing that $\theta^* c = \tau_1 \cup \tau_1 \cup \tau$ where $\tau_1 = \sigma_1 \circ \theta$ and $\tau = \sigma \circ \theta$. It remains to show that the product $\tau_1 \cup \tau_1 \cup \tau = 0$. In fact, we will show $\tau_1 \cup \tau_1 = 0$. Observe that τ_1 is in the image

$$\mathrm{pr}_1{}^* \colon H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to H^1(B, \mathbb{Z}/2\mathbb{Z})$$

where $\mathrm{pr}_1 \colon B \to \mathbb{Z}/2\mathbb{Z}$ denotes the projection onto the first component. For any prime p the cohomology ring $H^\bullet(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ is well understood and in particular the cup product

$$\cup \colon H^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \otimes H^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \to H^2(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$$

is identically zero. From this we conclude $\tau_1 \cup \tau_1 = 0$. This completes the proof. □

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