Mathematical and Numerical Optimization, 3341.454
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Homework 6

Problem 1: Show that if $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is β-cocoercive and $M \in \mathbb{R}^{n\times n}$ is symmetric positive definite, then $M^{-1/2}TM^{-1/2}$ is $(\beta/\|M^{-1}\|)$-cocoercive.

Problem 2: *Convergence of DRS.* Consider the FPI with DRS. Theorem 1 implies $z^k \to z^\star$ for any $\alpha > 0$, provided that a fixed point exists. Show that this implies $x^k \to x^\star$, and $x^{k+1/2} \to x^\star$. Is $\|x^{k+3/2} - x^{k+1/2}\| \to 0$ and $\|x^{k+1} - x^k\| \to 0$ true?

Problem 3: Let f be a CCP function on $\mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$. Assume $\mathcal{R}(A^\intercal) \cap \operatorname{ri}\operatorname{dom} f^* \neq 0$. Consider the optimization problem

$$\begin{array}{ll} \underset{\mu\in\mathbb{R}^m,\,\nu\in\mathbb{R}^n}{\text{minimize}} & f^*(\nu) - \mu^\intercal y + \frac{1}{2}\|\mu\|^2 \\ \text{subject to} & A^\intercal\mu - \nu = 0 \end{array}$$

generated by the Lagrangian

$$\mathbf{L}(\mu,\nu,x) = f^*(\nu) - \mu^\intercal y + \frac{1}{2}\|\mu\|^2 + \langle x, A^\intercal\mu - \nu\rangle.$$

Using Slater's constraint qualification, show

$$\underset{x\in\mathbb{R}^n}{\operatorname{argmin}} \left\{ f(x) + (1/2)\|Ax - y\|^2 \right\} \neq \emptyset.$$

Hint. Slater's constraint qualification states that if a "strictly feasible" point exists and if the optimal values are finite, then a dual solution exists.

Problem 4: *Prox of infimal postcomposition.* Let f be CCP. Show that if $\mathcal{R}(A^\intercal)\cap\operatorname{ri}\operatorname{dom} f^* \neq \emptyset$, then

$$\begin{array}{l} x \in \underset{x}{\operatorname{argmin}} \left\{ f(x) + (1/2)\|Ax - y\|^2 \right\} \\ z = Ax \end{array} \quad \Leftrightarrow \quad z = \operatorname{Prox}_{A\triangleright f}(y)$$

and the argmin of the left-hand side exists.

Hint. Use Exercise 3 and show

$$\underset{z}{\operatorname{argmin}} \left\{ \inf_{x\in\{x\,|\,Ax=z\}} f(x) + \frac{1}{2}\|Ax - y\|^2 \right\} = \operatorname{Prox}_{A\triangleright f}(y).$$

Problem 5: *Saddle points of augmented Lagrangians.* Let f be a CCP function on $\mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Show that the Lagrangian

$$\mathbf{L}(x,u) = f(x) + \langle u, Ax - b\rangle$$

and augmented Lagrangian

$$\mathbf{L}_\alpha(x,u) = f(x) + \langle u, Ax - b\rangle + \frac{\alpha}{2}\|Ax - b\|^2,$$

where $\alpha > 0$, share the same set of saddle points.

Problem 6: *Doubly linearized method of multipliers.* Consider the primal problem

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x)+h(x) \\ \text{subject to} & Ax=b, \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, f and h are CCP, and h is differentiable, generated by the Lagrangian

$$\mathbf{L}(x,u) = f(x) + h(x) + \langle u, Ax - b\rangle.$$

Show that the FPI with $(M+\mathbb{G})^{-1}(M-\mathbb{H})$ with $\mathbb{G}$ and $\mathbb{H}$ defined as in (3.15) of the textbook and

$$M = \begin{bmatrix} (1/\alpha)I - \beta A^\intercal A & 0 \\ 0 & (1/\beta)I \end{bmatrix}$$

gives us

$$\begin{aligned} x^{k+1} &= \mathrm{Prox}_{\alpha f}\left(x^k - \alpha\nabla h(x^k) - \alpha A^\intercal(u^k + \beta(Ax^k - b))\right) \\ u^{k+1} &= u^k + \beta(Ax^{k+1} - b). \end{aligned}$$

Under what conditions does this method converge? Note that Condat–Vũ and PD3O can be used to solve this problem. How do the algorithms and their convergence conditions compare?