Mathematical and Numerical Optimization, 3341.454
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Homework 9

Problem 1: *Parallel PAPC/PDFP^{2}O.* Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad h(x) + \frac{1}{m}\sum_{i=1}^{\ell} g_i(A_i x).$$

where $A_1, \dots, A_\ell \in \mathbb{R}^{m\times n}$, $g_1, \dots, g_\ell$ are CCP, and h is differentiable and CCP. Assume $\mathcal{F}[x \mapsto \nabla h] = C_h$ flops and computing $\mathcal{F}[y \mapsto \mathrm{Prox}_{\alpha g_i}(y)] \le C_g$ for $i = 1, \dots, \ell$. Find a method that solves this problem using $\mathcal{O}(\ell mn + \ell C_g + C_h)$ flops per iteration. Is this method parallelizable?

Problem 2: *Logistic regression with MISO/Finito.* Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \sum_{j=1}^{m} \log(1+\exp(-y_j a_j^\intercal x)),$$

where $a_1, \dots, a_m \in \mathbb{R}^n$ and $y_1, \dots, y_m \in \{-1, +1\}$. Describe gradient descent and MISO/Finito applied to this problem. What are their flop counts per iteration?

Problem 3: Consider the operators

$$\mathbb{A} = \mathbb{N}_{\mathbb{R}^2_+}, \qquad \mathbb{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \ge 0,\, x_2 \ge 0\}$. Show

(a) $\mathrm{Zer}\,(\mathbb{A}+\mathbb{B}) = \{(x_1, 0) \in \mathbb{R}^2 \,|\, x_1 \ge 0\}$

(b) $\mathrm{Zer}\,(\mathbb{A}^{-\circledvee} + \mathbb{B}^{-1}) = \{(0, u_2) \in \mathbb{R}^2 \,|\, u_2 \ge 0\}$

(c) $\mathrm{Fix}\,(\mathbb{R}_\mathbb{A}\mathbb{R}_\mathbb{B}) = \{(z, z) \in \mathbb{R}^2 \,|\, z \ge 0\}$

and conclude that

$$\mathrm{Fix}\,(\mathbb{R}_\mathbb{A}\mathbb{R}_\mathbb{B}) \ne \mathrm{Zer}\,(\mathbb{A}+\mathbb{B}) + \mathrm{Zer}\,(\mathbb{A}^{-\circledvee} + \mathbb{B}^{-1}).$$

Hint. Use

$$\mathbb{B}^{-1} = -\mathbb{B}, \qquad \mathbb{J}_\mathbb{B} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbb{R}_\mathbb{B} = -\mathbb{B}$$

and

$$\mathbb{N}_{\mathbb{R}^2_+}(x) = \begin{cases} \{y \in \mathbb{R}^2 \,|\, y_1 \le 0,\, y_2 \le 0,\, \langle x, y\rangle = 0\} & \text{if } x \in \mathbb{R}^2_+ \\ \emptyset & \text{if } x \notin \mathbb{R}^2_+. \end{cases}$$

Problem 4: Consider the Fenchel dual setup with primal and dual problems

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) + g(x), \qquad \underset{u\in\mathbb{R}^n}{\text{maximize}} \quad -f^*(-u) - g^*(u),$$

where f and g are CCP functions on $\mathbb{R}^n$, generated by

$$\mathbf{L}(x, u) = f(x) + \langle x, u \rangle - g^*(u).$$

Assume total duality holds. Write $X^\star$ and $U^\star$ for the sets of primal and dual solutions. Show that

$$\text{Fix}\ (\mathbb{R}_{\alpha\partial f}\mathbb{R}_{\alpha\partial g}) = X^\star + \alpha U^\star$$

Hint. Use the fact that [$x^\star \in X^\star$ and $u^\star \in U^\star$] if and only if [$(x^\star, u^\star)$ is a saddle point of $\mathbf{L}$].