Asymptotic properties of random quantum states and channels

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Outline

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States

Mixed states

1. Hermitian, $\rho = \rho^\dagger$,
2. positive, $\rho \geq 0$,
3. unit trace, $\mathrm{tr}\rho = 1$

We will denote the set of density mixed states of size d by Ω_d

Quantum channel

Definition

A quantum channel is a linear mapping $\Phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ that satisfies the following restrictions:

1. Φ is trace-preserving, i.e. $\forall A \in M_{d_1}(\mathbb{C})\ \mathrm{tr}(\Phi(A)) = \mathrm{tr}(A)$,
2. Φ is completely positive, that is for every finite s the product of Φ and an identity mapping on $M_s(\mathbb{C})$ is a non-negativity preserving operation, i.e.

$$\forall \mathcal{Z}\ \forall A \in M_{d_1}(\mathbb{C}) \otimes M_s(\mathbb{C}),\ A \geq 0\ (\Phi \otimes \mathbb{1})(A) \geq 0. \tag{1}$$

Choi-Jamiołkowski isomorphism

Choi matrix

Given a linear $\Phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$, we associate with it a Choi-Jamiołkowski, $J_\Phi \in M_{d_2}(\mathbb{C}) \otimes M_{d_1}(\mathbb{C})$:

$$J_\Phi = \sum_{i,j} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|. \tag{2}$$

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Equivalent definition

$$J_\Phi = d_1 \; (\Phi \otimes \mathbb{1})(|\phi^+\rangle\langle\phi^+|) \tag{3}$$

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Properties

1. If Φ is CP, then $J_\Phi \geq 0$
2. If Φ is TP, then $\mathrm{tr}_1 J_\Phi = \mathbb{1}$

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Random states in Ω_d

From pure states

1. Consider a random pure state $|\phi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$.
2. Trace out one of the systems $\rho = \mathrm{tr}_2 |\phi\rangle\langle\phi|$.
3. If $d_1 = d_2$, we get the Hilbert-Schmidt distribution of ρ.

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From Ginibre matrices

Let $G \in M_{d_1 \times s}(\mathbb{C})$ be a Ginibre matrix (independent normal complex entries). Then, the matrix

$$\rho = \frac{GG^\dagger}{\mathrm{tr} GG^\dagger}, \tag{4}$$

is a random mixed state. If $d_1 = s$ we recover the flat Hilbert-Schmidt distribution.

Random quantum channels

From Ginibre matrices

Let $G \in M_{d_2 d_1 \times s}(\mathbb{C})$ Ginibre matrix (independent normal complex entries). Then, the matrix

$$M_{d_2}(\mathbb{C}) \otimes M_{d_1}(\mathbb{C}) \ni J_\Phi = \left(\mathbb{1}_{d_2} \otimes \frac{1}{\sqrt{\mathrm{tr}_1 GG^\dagger}} \right) GG^\dagger \left(\mathbb{1}_{d_2} \otimes \frac{1}{\sqrt{\mathrm{tr}_1 GG^\dagger}} \right) \quad (5)$$

is a random Choi matrix for some channel $\Phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$.

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Random state vs random channel

Consider a Choi-Jamiołkowski matrix of quantum channel and a quantum state ρ.

Random quantum channels

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Random state vs random channel

Consider a Choi-Jamiołkowski matrix of quantum channel and a quantum state ρ.

- $\rho = \rho^\dagger$,
- $\mathrm{tr}\rho = 1$.

- $J_\Phi = J_\Phi^\dagger$,
- $\mathrm{tr} J_\Phi = d_1$,
- $\mathrm{tr}_1 J_\Phi = \mathbb{1}$.

Probability distributions on a set of quantum channels

Definition

The image measure of the Gaussian standard measure through the map $G \mapsto \Phi_G$ is called partially normalized Wishart measure and is denoted by $\gamma_{d_1,d_2,s}$.

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Marčenko-Pastur distribution

Definition (Marčenko-Pastur distribution)

Distribution of parameter $x > 0$ has density given by

$$d\mathcal{MP}_x = \max(1 - x, 0)\delta_0 + \frac{\sqrt{4x - (u - 1 - x)^2}}{2\pi u} 1_{[a,b]}(u)\, du,$$

where $a = (\sqrt{x} - 1)^2$ and $b = (\sqrt{x} + 1)^2$.

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where $a = (\sqrt{x}-1)^2$ and $b = (\sqrt{x}+1)^2$.

Consider matrices $G \in M_{d\times(xd)}$ such that $G_{ij} \sim \mathcal{N}_{\mathbb{C}}(0,1)$. We define Wishart matrix $W = GG^\dagger \in M_d$ and its empirical eigenvalue distribution

$$\mu_d(A) = \frac{1}{d}\#(\lambda(M/d) \in A).$$

We have almost surely convergence with $d \to \infty$

$$\lim_{d\to\infty} \mu_d(A) = \mathcal{MP}_x(A).$$

Subtracted Marčenko-Pastur distribution

Definition (Subtracted Marčenko-Pastur distribution)

Let a, b be two free random variables having Marčenko-Pastur distributions with respective parameters x and y. The distribution of the random variable $a/x - b/y$ is called the *subtracted Marčenko-Pastur distribution* with parameters x, y and is denoted by $\mathcal{SMP}_{x,y}$.

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Consider matrices $G_1 \in M_{d\times(xd)}$ and $G_2 \in M_{d\times(yd)}$ We define Wishart matrices $W_i = G_i G^\dagger \in M_d$ and its empirical eigenvalue distribution

$$\mu_d(A) = \frac{1}{d}\#\left(\lambda\Big((xd)^{-1}W_1 - (yd)^{-1}W_2\Big) \in A\right).$$

We have almost surely convergence with $d \to \infty$

$$\lim_{d\to\infty} \mu_d(A) = \mathcal{SMP}_{x,y}(A).$$

Subtracted Marčenko-Pastur distribution

Proposition

Let W_x (resp. W_y) be two Wishart matrices of parameters (d, s_x) (resp (d, s_y)). Assuming that $s_x/d \to x$ and $s_y/d \to y$ for some constants $x, y > 0$, then, almost surely as $d \to \infty$, we have

$$\lim_{d\to\infty} \|(xd^2)^{-1} W_x - (yd^2)^{-1} W_y\|_1 = \int |u| \, d\mathcal{SMP}_{x,y}(u) =: \Delta(x, y).$$

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Free convolution

We obtain the subtracted Marčenko Pastur distribution using free additive convolution $\mathcal{SMP}_{x,y}(u) = x\mathcal{MP}_x(ux) \boxplus y\mathcal{MP}_y(-uy)$.

Subtracted Marčenko-Pastur distribution

Proposition

Let $x, y > 0$. Then,

1. If $x + y < 1$, then the probability measure $\mathcal{SMP}_{x,y}$ has exactly one atom, located at 0, of mass $1 - (x + y)$. If $x + y \geq 1$, then $\mathcal{SMP}_{x,y}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.
2. Define

$$\begin{aligned} a_{x,y} &= (x-y)(2x+y)(x+2y) \\ b_{x,y} &= 2x^3 + 2y^3 + (x+y)^2 + xy(x+y+2) \\ c_{x,y} &= (x-y)(x+y+1-2(x+y)^2) \\ U_{x,y}(u) &= -u^3 a_{x,y} + 3u^2 b_{x,y} + 3u c_{x,y} + 2(x+y-1)^3 \\ T_{x,y}(u) &= (x+y-1-u(x-y))^2 + 3u(y-x+uxy) \\ Y_{x,y}(u) &= U_{x,y}(u) + \sqrt{[U_{x,y}(u)]^2 - 4\,[T_{x,y}(u)]^3}. \end{aligned} \tag{6}$$

Proposition

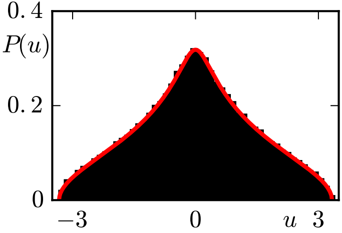
The support of the absolutely continuous part of $\mathcal{SMP}_{x,y}$ is the set

$$\{u : [U_{x,y}(u)]^2 - 4[T_{x,y}(u)]^3 \geq 0\}. \tag{7}$$

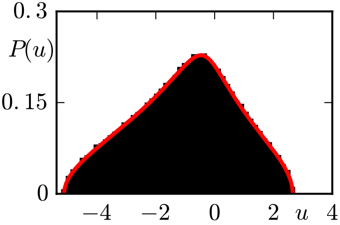
3 On its support, the density of $\mathcal{SMP}_{x,y}$ is given by

$$\frac{d\mathcal{SMP}_{x,y}}{du} = \left| \frac{[Y_{x,y}(u)]^{\frac{2}{3}} - 2^{\frac{2}{3}} T_{x,y}(u)}{2^{\frac{4}{3}}\sqrt{3}\pi u [Y_{x,y}(u)]^{\frac{1}{3}}} \right|. \tag{8}$$

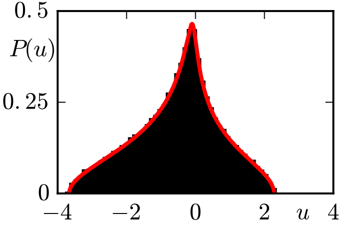
Subtracted Marčenko-Pastur distribution



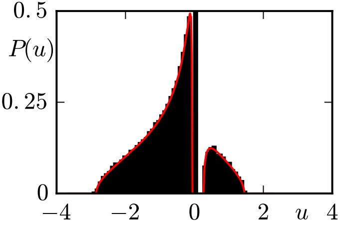
(a) $x = 1$, $y = 1$



(b) $x = 1$, $y = 2$



(c) $x = 0.5$, $y = 1$



(d) $x = 0.2$, $y = 0.5$

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Average distances between 2 random states

Take ρ and σ sampled from the flat (HS) measure, $s = 1$.
As $d \to \infty$, the trace distance tends to an integral over the symmetrized Marchenko-Pastur distribution:

$$D_{\mathrm{tr}} \to \frac{1}{2} \int \mathcal{SMP}_{1,1}(x)|x|\mathrm{d}x = \tilde{D} = \frac{1}{4} + \frac{1}{\pi} \approx 0.5683 \tag{9}$$

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Average distances of random state ρ to

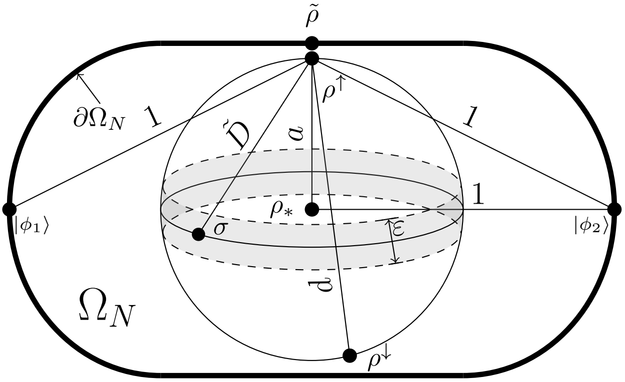
- the maximally mixed state ρ_*

$$D_{\mathrm{tr}}(\rho, \rho_*) = \frac{1}{2} \int |t-1|\mathcal{MP}_1(t)\mathrm{d}t = \frac{3\sqrt{3}}{4\pi} \approx 0.4135 \tag{10}$$

- the closest pure state, $D_{\mathrm{tr}}(\rho, |\phi\rangle\langle\phi|) \to 1$
- the closest boundary state $\tilde{\rho}$, $D_{\mathrm{tr}}(\rho, \tilde{\rho}) \to 0$

Te set Ω_d for large d

The HS measure is concentrated in an ε neighborhood of the unitary orbit, $U\rho U^\dagger$, where U a Haar unitary and ρ is a random state with spectrum distributed according to $\mathcal{MP}$. Here, d is the diameter given by the distance between two diagonal matrices with opposite order of the eigenvalues $d = D_{\mathrm{Tr}}(p^\uparrow, p^\downarrow) = \int_0^4 x\,\mathrm{sign}(x-M)\mathcal{MP}_1(x)\mathrm{d}x \simeq 0.7875$, where M denotes the median, $\int_0^M \mathcal{MP}_1(x)\mathrm{d}x = 1/2$.



Helstrom theorem

Theorem

Given two states ρ and σ, the probability p of discriminating between these two in a single-shot experiment is bounded by $p \leq \frac{1}{2} + \frac{1}{2} D_{\mathrm{tr}}(\rho, \sigma)$.

Distinguishing generic quantum states

Two random states ρ and σ of dimension $N \gg 1$ can be distinguished in a single-shot experiment with probability bounded by

$$p \leq \frac{1}{2} + \frac{1}{2}\tilde{D} = \frac{5}{8} + \frac{1}{2\pi} = 0.7842. \qquad (11)$$

Asymptotic distances

Given two random states ρ, σ of dimension d.
For large d ($d \gg 1$), we have:

- relative entropy $S(\rho||\sigma) = \mathrm{tr}\rho \log \rho - \rho \log \sigma$
  $S(\rho||\sigma) \to \int \mathrm{d}t \int \mathrm{d}s(t \log t - t \log s)\mathcal{MP}(t)\mathcal{MP}(s) = \frac{3}{2}$
- quantum Sanov theorem: Performing n measurements on ρ, we obtain result compatible with σ with probability $p \sim \exp(\frac{-3n}{2})$.
- Chernoff information $Q(\rho, \sigma) = \min_{s\in[0,1]} \mathrm{tr}\rho^s\sigma^{1-s}$. We get the Chernoff bound for generic quantum states:
  $Q(\rho, \sigma) = \langle \mathrm{tr}\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\rangle \to \int \sqrt{t}\mathcal{MP}(t)\mathrm{d}t = \left(\frac{8}{3\pi}\right)^2 = 0.72 = Q_*$

Performing n measurements on ρ and σ we get the probability of error $p \sim \exp(-Q_* n)$.

Even more distances

Some more asymptotic results:

1. root fidelity:

$$\sqrt{F(\rho,\sigma)} = \sum_i \sqrt{\lambda(\rho\sigma)} \to \int \sqrt{x}\mathcal{FC}(x)\mathrm{d}x = \frac{3}{4}, \tag{12}$$

   where

$$\mathcal{FC}(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi} \frac{\left[\sqrt[3]{2}\left(27+3\sqrt{81-12x}\right)^{\frac{2}{3}} - 6\sqrt[3]{x}\right]}{x^{\frac{2}{3}}\left(27+3\sqrt{81-12x}\right)^{\frac{1}{3}}}, \tag{13}$$

   is the Fuss-Catalan distribution, $\mathcal{FC}(x) = \mathcal{MP}(x) \boxtimes \mathcal{MP}(x)$

2. Bures didstance

$$D_B = \sqrt{2(1-\sqrt{F(\rho,\sigma)}} \to \frac{\sqrt{2}}{2}, \tag{14}$$

3. Hellinger distance

$$D_H = \sqrt{2 - 2\mathrm{tr}\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}} \to \sqrt{2 - \frac{128}{9\pi^2}} \approx 0.746 \tag{15}$$

Rate of convergence

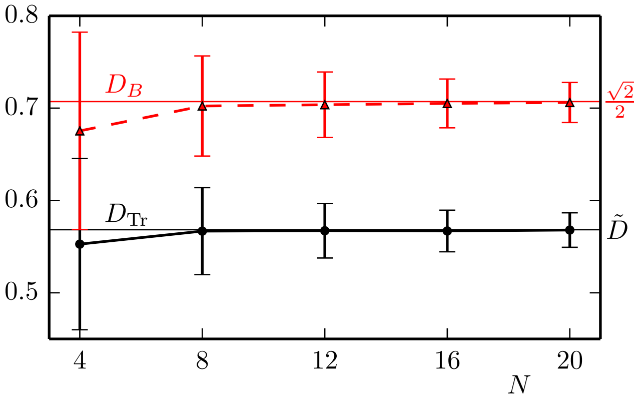


Figure: Dependence of average distance between two generic states on the dimension N. Dashed (red) line shows the Bures distance and solid (black) line shows the trace distance. The horizontal lines mark the asymptotic values.

Asymptotic entanglement

Consider $|\phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ and $\rho = \mathrm{tr}_1|\phi\rangle\langle\phi|$.
For a partially transposed matrix, ρ^{T_A}, its eigenvalues have the shifted semicircle as the limiting distribution (Aubrun 2012),

$$\lambda(\rho^{T_A}) \sim \frac{1}{2\pi}\sqrt{4-(x-1)^2}. \tag{16}$$

We get:

1. the fraction of negative eigenvalues tends to

$$\int_{-1}^{0} \frac{1}{2\pi}\sqrt{4-(x-1)^2}\mathrm{d}x = \frac{1}{3} - \frac{\sqrt{3}}{4\pi}, \tag{17}$$

2. the average negativity tends to

$$\mathcal{N} \to \int_{-1}^{0} \frac{|x|}{2\pi}\sqrt{4-(x-1)^2}\mathrm{d}x \approx 0.080. \tag{18}$$

The G-concurrence of a state $G(|\phi\rangle) = d(\det\rho)^{\frac{1}{d}}$, converges:

$$G(|\phi\rangle) \to \exp\left(\int_{0}^{4} \log t\mathcal{MP}(t)\mathrm{d}t\right) = \frac{1}{\mathrm{e}} \approx 0.368 \tag{19}$$

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Diamond norm

Induced trace norm

Given a mapping $\Phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ the induced trace norm is defined as:

$$\|\Phi\|_1 = \max\{\|\Phi(A)\|_1 : A \in M_{d_1}(\mathbb{C}), \|A\|_1 \leq 1\} \tag{20}$$

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Given a superoperator $\Phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ the diamond norm is defined as:

$$\|\Phi\|_\diamond = \|\Phi \otimes \mathbb{1}\|_1 \tag{21}$$

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Theorem

Given a Hermiticity-preserving mapping $\Phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$, it holds that

$$\|\Phi\|_\diamond = \max\{\|(\Phi \otimes \mathbb{1}(|\phi\rangle\langle\phi|)\|_1, |\phi\rangle \in \mathbb{C}^{d_1^2}\} \quad (22)$$

Bounds for the diamond norm

Lower bound for diamond norm

$$\|\Phi\|_\diamond \geq \frac{1}{d_1}\|J_\Phi\|_1. \tag{23}$$

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Upper bound for Hermiticity preserving mappings

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General upper bound

$$\|\Phi\|_\diamond \leq \frac{\left\| \mathrm{tr}_2 \sqrt{J_\Phi J_\Phi^\dagger} \right\| + \left\| \mathrm{tr}_2 \sqrt{J_\Phi^\dagger J_\Phi} \right\|}{2} \tag{25}$$

J. Watrous *Simpler semidefinite programs for completely bounded norms.* Chicago Journal of Theoretical Computer Science 8 1–19 (2013).

Distinguishing quantum channels

Theorem

Given two quantum channels $\Phi, \Psi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$. The probability of distinguishing these channels is upper bounded by:

$$p \leq \frac{1}{2} + \frac{1}{4}\|\Phi - \Psi\|_\diamond \tag{26}$$

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Asymptotic value of a diamond norm

Theorem

Let Φ, resp. Ψ, be two independent random quantum channels from $\Theta(d_1, d_2)$ having γ^W distribution with parameters (d_1, d_2, s_x), resp. (d_1, d_2, s_y). Then, almost surely as $d_{1,2} \to \infty$ in such a way that $s_x/(d_1 d_2) \to x$, $s_y/(d_1 d_2) \to y$ (for some positive constants x, y), and $d_1 \ll d_2^2$,

$$\lim_{d_{1,2}\to\infty} \|\Phi - \Psi\|_\diamond = \Delta(x, y) = \int |u| \, d\mathcal{SMP}_{x,y}(u).$$

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$$\lim_{d_{1,2}\to\infty} \|\Phi - \Psi\|_\diamond = \Delta(x, y) = \int |u| \, d\mathcal{SMP}_{x,y}(u).$$

In the case of flat Hilbert Schmidt distribution on quantum channels we obtain

$$\lim_{d\to\infty} \|\Phi - \Psi\|_\diamond = \frac{1}{2} + \frac{2}{\pi}.$$

The lower bound

Proposition

$$\lim_{d_{1,2}\to\infty} \frac{1}{d_1}\|J_\Phi - J_\Psi\|_1 = \Delta(x,y) = \int |u|\, d\mathcal{SMP}_{x,y}(u).$$

Proof

The result follows easily by approximating the partially normalized Wishart matrices with scalar normalizations. By the triangle inequality, with $D_x := J_\Phi$ and $D_y := J_\Psi$, we have

$$\begin{aligned}
&\left| \frac{1}{d_1}\|D_x - D_y\|_1 - \frac{1}{d_1}\|(xd_1d_2^2)^{-1}W_x - (yd_1d_2^2)^{-1}W_y\|_1 \right| \\
&\quad \le \frac{1}{d_1}\|D_x - (xd_1d_2^2)^{-1}W_x\|_1 + \frac{1}{d_1}\|D_y - (yd_1d_2^2)^{-1}W_y\|_1 \\
&\quad \le d_2\|D_x - (xd_1d_2^2)^{-1}W_x\|_\infty + d_2\|D_y - (yd_1d_2^2)^{-1}W_y\|_\infty.
\end{aligned}$$

The lower bound

Proposition

With the above assumptions almost surely as $d_{1,2} \to \infty$ in such a way that $s \sim t d_1 d_2$ for a fixed parameter $t > 0$,

$$\left\| D - (t d_1 d_2^2)^{-1} W \right\| = O(d_2^{-2}).$$

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The case of Wishart matrices was derived earlier:

$$\frac{1}{d_1} \|(x d_1 d_2^2)^{-1} W_x - (y d_1 d_2^2)^{-1} W_y\|_1 \to \int |u| \, d\mathcal{SMP}_{x,y}(u) = \Delta(x, y).$$

The upper bound

The core technical result of this work consists of deriving the asymptotic value of the upper bound for diamond norm.

Theorem

Let Φ, resp. Ψ, be two independent random quantum channels from $\Theta(d_1, d_2)$ having γ^W distribution with parameters (d_1, d_2, s_x), resp. (d_1, d_2, s_y). Then, **almost surely** *as $d_{1,2} \to \infty$ in such a way that $s_x/(d_1 d_2) \to x$, $s_y/(d_1 d_2) \to y$ (for some positive constants x, y), and $d_1/d_2^2 \to 0$,*

$$\lim_{d_{1,2} \to \infty} \| \operatorname{Tr}_2 |J_\Phi - J_\Psi| \| = \int |u| \, d\mathcal{SMP}_{x,y}(u) = \Delta(x, y).$$

The upper bound – proof

Using the triangle inequality we first show an approximation result (as before, we write $D_x := J_\Phi$ and $D_y := J_\Psi$):

$$\left| \, \| \operatorname{tr}_2 |D_x - D_y| \, \| - \| \operatorname{tr}_2 |(x d_1 d_2^2)^{-1} W_x - (y d_1 d_2^2)^{-1} W_y| \, \| \, \right| \leq \frac{\log(d_1 d_2)}{d_2} O(1) \to 0,$$

[1] E.B. Davies, *Lipschitz continuity of operators in the Schatten classes.* J. London Math. Soc., 37, pp. 148—157 (1988).

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We have used the following lemma[1]

Lemma

For any matrices A, B of size d, the following holds:

$$\| \, |A| - |B| \, \| \leq C \log d \, \|A - B\|,$$

for a universal constant C which does not depend on the dimension d.

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Convergence

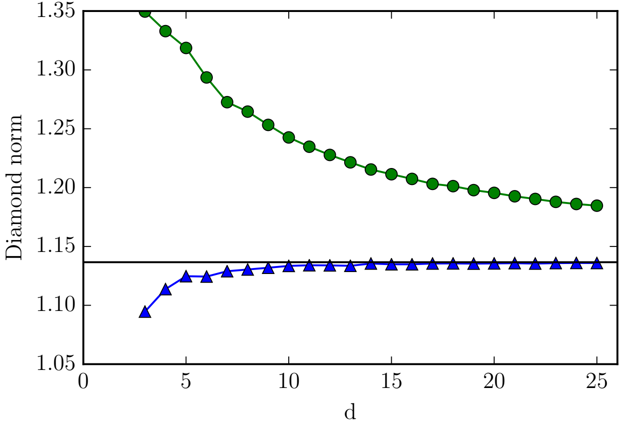
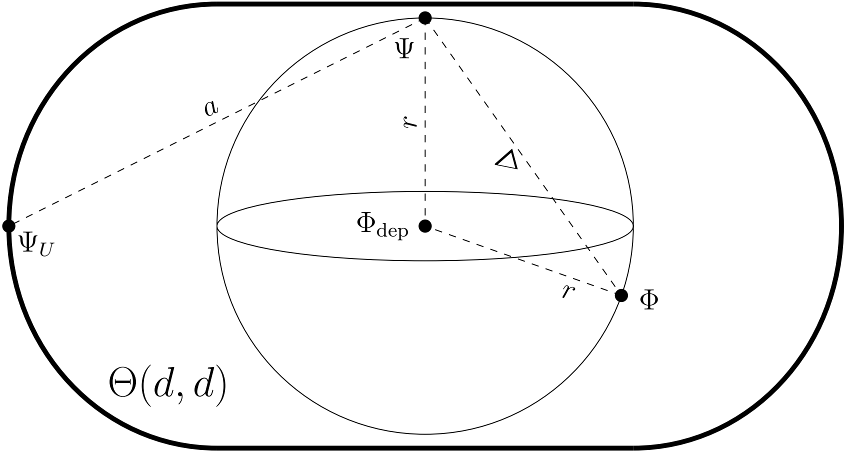


Figure: The convergence of upper (green circles) and lower (blue triangles) bounds on the distance between two random quantum channels sampled from the Hilbert-Schmidt distribution ($d_1 = d_2 = d$). The results were obtained via Monte Carlo simulation with 100 samples for each data point.

Sketch of the set of quantum channels

Sketch of the set $\Theta(d, d)$ of all channels acting on d-dimensional states. A generic channel Φ belongs to a sphere of radius $r = 3\sqrt{3}/2\pi$, centered at the maximally depolarizing channel, Φ_{dep}, in the metric induced by the diamond norm. The distance between generic channels, Φ, Ψ is $\Delta = 1/2 + 2/\pi$, while the distance to the nearest unitary channel reads as $a = 2$.



Partial traces of unitarily invariant random matrices

Theorem

Consider a sequence of Hermitian random matrices $A_d \in M_{d_1(d)}(\mathbb{C}) \otimes M_{d_2(d)}(\mathbb{C})$ *and assume that*

1. *Both functions* $d_{1,2}(d)$ *grow to infinity, in such a way that* $d_1/d_2^2 \to 0$.
2. *The matrices* A_d *are unitarily invariant.*
3. *The family* (A_d) *has almost surely limit distribution* μ*, for some compactly supported probability measure* μ.

Then, the normalized partial traces $B_d := d_2^{-1}[\mathrm{id} \otimes \mathrm{Tr}](A_d)$ *converge* **almost surely** *to multiple of the identity matrix:*

$$a.s. - \lim_{d\to\infty} \|B_d - aI_{d_1(d)}\| = 0,$$

where a *is the average of* μ*:*

$$a := \int x d\mu(x).$$

Partial traces of unitarily invariant random matrices

We define

$$b := \frac{1}{d_1} \sum_{i=1}^{d_1} \lambda_i(B)$$

$$v := \frac{1}{d_1} \sum_{i=1}^{d_1} (\lambda_i(B) - b)^2$$

the average and the variance of the eigenvalues of B; these are real random variables (actually, sequences of random variables indexed by d).

By Chebyshev's inequality, we have

$$\lambda_{\max}(B) \leq b + \sqrt{v}\sqrt{d_1}.$$

We proved that $b \to a$ almost surely and later that $d_1 v \to 0$ almost surely, which is what we need to conclude.

Partial traces of unitarily invariant random matrices

Average

The a.s. convergence $b \to a$ is straightforward.

Partial traces of unitarily invariant random matrices

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Variance

In order to show, that $d_1 v \to 0$ almost surely, we have calculated the mean and the variance of v.

Partial traces of unitarily invariant random matrices

Average

The a.s. convergence $b \to a$ is straightforward.

Variance

In order to show, that $d_1 v \to 0$ almost surely, we have calculated the mean and the variance of v.
We are able to compute the variance of v with the usage of symmetry arguments and obtain

$$\begin{aligned} \mathbb{E}v &= (1 + o(1))\; d_2^{-2}\,\mathrm{Var}(\mu) \\ \mathrm{Var}(v) &= (1 + o(1))\; 2d_1^{-2}d_2^{-4}\,\mathrm{Var}(\mu)^2, \end{aligned}$$

where $\mathrm{Var}(\mu) = \int x^2\, d\mu(x) - (\int x\, d\mu(x))^2$.

Partial traces of unitarily invariant random matrices

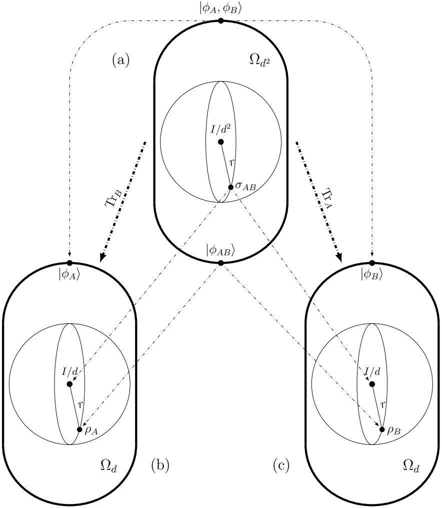
$$\mathbb{P}(\sqrt{d_1}\sqrt{v} \geq \varepsilon) = \mathbb{P}(v \geq \varepsilon^2 d_1^{-1}) \leq \frac{\mathrm{Var}(v)}{[\varepsilon^2 d_1^{-1} - \mathbb{E}v]^2} \sim \frac{C d_1^{-2} d_2^{-4}}{[\varepsilon^2 d_1^{-1} - (1+o(1))C' d_2^{-2}]^2},$$

Using $d_1 \ll d_2^2$,

$$\mathbb{P}(\sqrt{d_1}\sqrt{v} \geq \varepsilon) \lesssim C\varepsilon^{-4} d_2^{-4}.$$

Since the series $\sum d_2^{-4}$ is summable, we obtain the announced almost sure convergence.

Set of all bipartite quantum states of dimension d^2, Ω_{d^2} (a) and its partial traces (b) and (c) containing states of dimension d. A generic bipartite state σ_{AB}, distant $r = 3\sqrt{3}/4\pi$ from the maximally mixed state $\mathbb{1}/d^2$, is mapped into $\sigma_A \approx \sigma_B \approx \mathbb{1}/d$, while a typical pure state $|\phi_{AB}\rangle$ is sent into a generic mixed state $\rho_A \equiv \rho_B$ distant r from $\mathbb{1}/d$.



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