SO_3

김연아
(2010년 동계올림픽 금메달 리스트)

(석창우 작, 2018 평창 동계올림픽 기념)

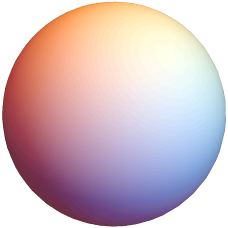
Kosmos Club, 2026. 5. 9. 김홍종

Motivation

Last month, we have studied the hyper-sphere $\mathbb{S}^3$,
which is the one-point-compactification of $\mathbb{R}^3$.

This month, we study its quotient.

$$\begin{tikzcd}\mathbb{S}^3 = \mathrm{Sp}(1) = \mathrm{Spin}(3) \arrow[d] \\ \mathrm{SO}_3 = \mathbb{P}^3\end{tikzcd}$$



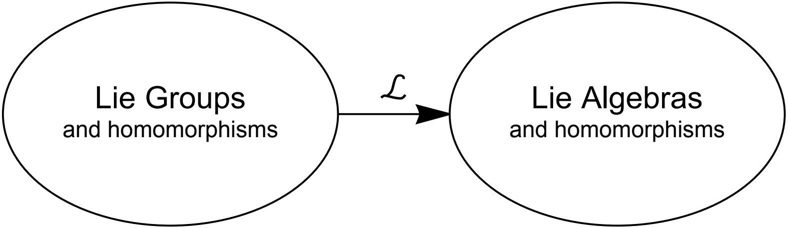
For every space M, there exists (unique up to isomorphism) a *simply connected space* $\tilde{M}$ which covers M.

$\mathbb{S}^3$ is the unique closed 3-manifold which is simply connected.

A connected space M is said to be **simply connected**, if any pair of paths $p, q : [0,1] \to M$ with the same initial and terminal positions (i.e., $p(0) = q(0)$ and $p(1) = q(1)$) are **homotopic**, i.e., p is a continuous deformation of q.

Some quantities (e.g., integration along a path of a conservative field) are invariants of a homotopy class of paths.

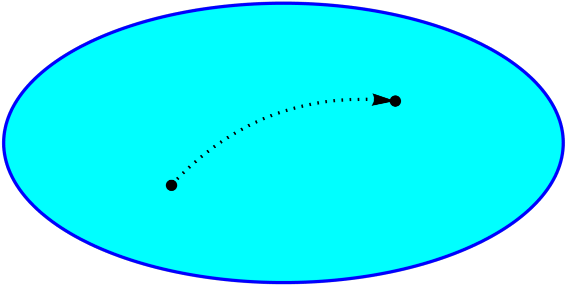
Categories, Functors (and natural transformations)



$$\begin{tikzcd}
\mathfrak{so}_3 \arrow[d, "\exp"'] \\
\mathrm{SO}_3
\end{tikzcd}$$

Motion

Let (M, d) be a *metric space*.



A *transformation* $T : M \to M$ is an **isometry** if

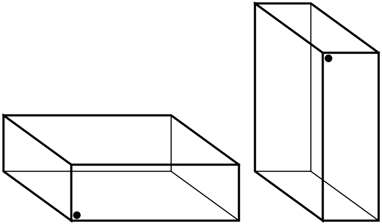
$$d(p, q) = d(Tp, Tq)$$

for any points p and q in M.
(Usually one assumes an isometry to be a bijection.)

Motion of Rigid Bodies

In a Euclidean space $\mathbb{E}$,
a continuous motion of a rigid body is described by a curve in the Lie group

$$\vec{\mathbb{E}} \rtimes \mathrm{SO}(\vec{\mathbb{E}}).$$



$\vec{\mathbb{E}}$ is the group of **translations** (or vectors), and
$\mathrm{SO}(\vec{\mathbb{E}})$ is the group of **rotations** about the center of the body.

These symmetries (of the space $\mathbb{E}$) generate
linear momentum and **angular momentum**.

Proper Motions

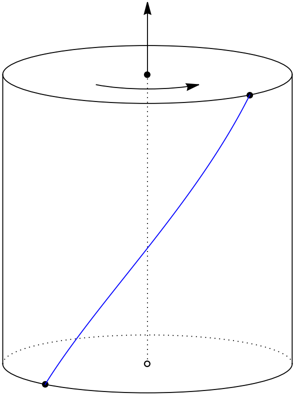
A transformation is said to be **proper** (or of *the first kind*) if it preserves the orientation.

Any proper motion $M : \mathbb{E}^3 \to \mathbb{E}^3$ is a **screw motion**:

$$M = R_{p,\mathbf{u}}(\theta) \circ T_{s\mathbf{u}}$$

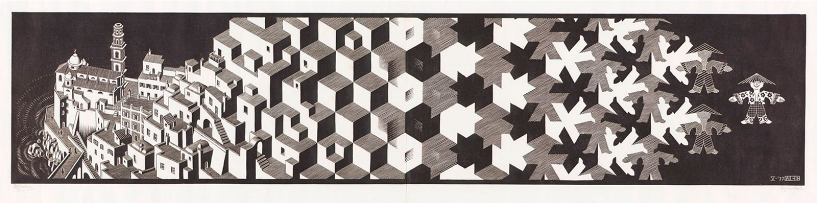
for some $p \in \mathbb{E}^3$, $\mathbf{u}$ being a unit vector, $s \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$, where $T_{s\mathbf{u}}$ denotes the translation by the vector $s\mathbf{u}$, and $R_{p,\mathbf{u}}(\theta)$ denotes the rotation by the angle θ along the line through p in the direction $\mathbf{u}$.

The motions $T_{s\mathbf{u}}$ and $R_{p,\mathbf{u}}(\theta)$ commute.



`screw motion`

Transformation (Metamorphoses)



M.C. Escher, *Metamorphosis I*, 1937

(https://www.escherinhetpaleis.nl/story-of-escher/metamorphosis-i-ii-iii/?lang=en)

진리를 아는 것보다 더 중요한 것은 진리에 대한 열정이다.

('누구나 다 깨달을 수 있다'고 말씀하셨지만, 왜 아무도 깨닫지 못합니까?)

— Osho (1931–1990) in *The Path of Meditation*, 1995

- Phase Transition (相轉移) — water, steel, crystal
- Brain, Mind — 점수돈오(漸修頓悟), 見性, 거듭남

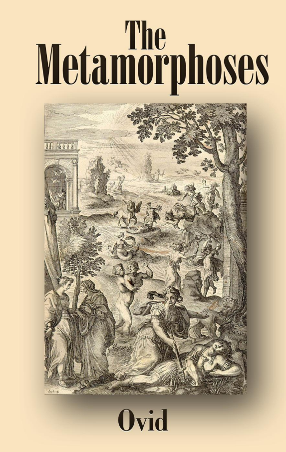
오름

높이 오르면 멀리 보인다.

다랑쉬 오름, 제주도

Metamorphoses, 諸行無常

고대 그리스의 에페소스에서 활동하던 **헤라클레이토스**는 변화를 이야기 하면서 "같은 강물에 두 번 발을 담글 수 없다"고 하며 이것이 변하여 저것이 되고 저것이 변하여 이것이 된다고 하였다. 하지만 그것은 변화를 강조하려는 것이라기보다 항상 있었고 지금도 있으며 앞으로도 있을 변화하지 않는 만물의 지배자, 영원불멸하는 우주 정신인 **Logos**와 하나가 되어야 함을 설명하고자 한 것이다. 그는 "만물은 하나다"라고 하였다. 현대 물리학이나 화학의 핵심에도 "에너지 보존법칙" 등 변하지 않는 것에 대한 믿음이 그 바탕에 깔려 있다.



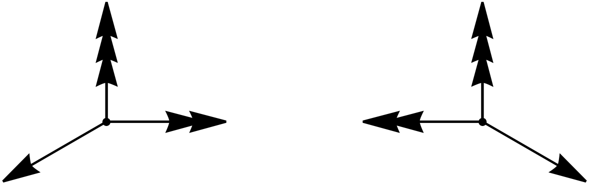
고대 로마 시대의 유명한 시인인 오비디우스는 "사랑의 기술" 등 인기 저서 외에도 그리스-로마 신화인 "Metamorphoses(변신이야기)"라는 저서로도 유명한데, 많은 서평가들이 이 신화는 변화무상한 것에 관한 이야기라고 잘못 평한다. 마치 불교의 용어 "제행무상(諸行無常)"을 "변하지 않는 것이 없다"고 해석하는 것처럼 잘못된 해석이다. 제행무상이라는 것은 현상계를 두고 말하는 것이고, 그 너머에는 영원불멸한 하나가 있음을 말하기 위함이다.
오비디우스는 메타모르포시스의 마지막 장에서 다음과 같이 말한다:

> 우리 로마인들을 깨우쳐 주신 피타고라스 선생님께서는 아틀라스라는 거인의 어깨에 서서 세상을 내려다 보며 말씀하시기를 세상에는 변하지 않는 진리가 있다고 하셨습니다. 하지만 세상사람들은 이 말씀을 잘 이해하지 못 하였습니다.

O_3, the orthogonal group for $\mathbb{R}^3$

$$\mathrm{O}_3 := \left\{ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathbf{M}_3 \,\middle|\, A^{-1} = A^{\mathrm{t}} \right\}$$

where $\mathbf{M}_3$ is the collection of all 3×3 matrices with real entries.

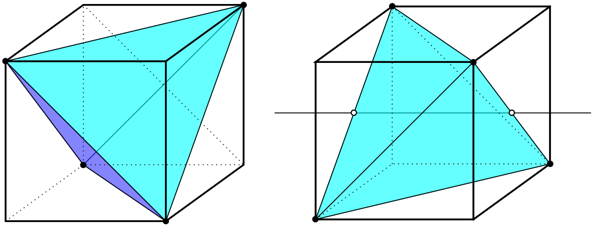


O_3 is the group of **orthonormal bases** for $\mathbb{R}^3$.

$$\dim \mathrm{O}_3 = 3$$

(O_3 has nothing to do with ozone.)

$\mathbf{T}$, Symmetries of a Tetrahedron



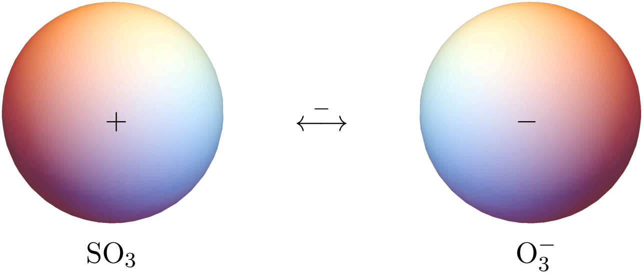
T, the 12 $(= |A_4|)$ proper symmetries of a tetrahedron:

- The identity symmetry
- Eight 3-fold symmetries (120° and 240° rotations)
- Three 2-fold symmetries (180° rotations, half-turns)

$\overline{\mathbf{T}}$, the 24 $(= |S_4|)$ full symmetries of a tetrahedron contains six reflections and six 90°-roto-reflections.

Remarks: The *alternating group* A_4 consists of even permutations in the *symmetric group* S_4. Discrete subgroups of SO_3 are realized by the cylinders, cones, and Platonic solids as described by Euclid in *Elements*.

O_3



SO_3 and O_3^- are topologically same
and isomorphic to the projective space $\mathbb{P}^3$.
They have no boundary.

The center of O_3 is $\{\mathrm{id}_3, -\mathrm{id}_3\}$.

O_3 has two components

$$\mathrm{O}_3 = \mathrm{SO}_3 \;\cup\; \mathrm{O}_3^-$$

$$\mathrm{SO}_3 = \{A \in \mathrm{O}_3 \mid \det A = 1\}, \qquad \mathrm{O}_3^- = \{A \in \mathrm{O}_3 \mid \det A = -1\}$$

$$\mathrm{id}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}_3, \qquad -\mathrm{id}_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{O}_3^-$$

$$\dim \mathrm{O}_3 = \dim \mathrm{SO}_3 = \dim \mathrm{O}_3^- = 3$$

$$\mathrm{SO}_3 = \{\text{rotations}\}, \qquad \mathrm{O}_3^- = \{\text{roto-reflections}\}$$

A **roto-reflection** is a rotation followed by a reflection about a plane which is perpendicular to the axis of rotation.

Spectrum

The **spectrum** of an $n \times n$ real matrix A is the collection of (real) eigenvalues of A:

$$\operatorname{Spec}(A) := \{\lambda \in \mathbb{R} \mid \det(A - \lambda\, \mathrm{id}_n) = 0\}$$

Suppose $R \in \mathrm{SO}_3$.

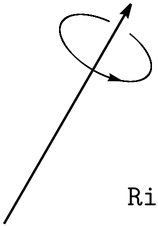
Then 1 is an eigenvalue of R, i.e., $1 \in \operatorname{Spec}(R)$:

$\because$ $\det(R - \mathrm{id}_3) = \det(R - R^{\mathrm{t}} R) = \det((\mathrm{id}_3 - R^{\mathrm{t}})R) = \det(\mathrm{id}_3 - R^{\mathrm{t}}) \cdot \det R = \det(\mathrm{id}_3 - R) = -\det(R - \mathrm{id}_3)$ and hence $\det(R - \mathrm{id}_3) = 0$.

Thus there exists a unit vector $\mathbf{u}$ such that $R\mathbf{u} = \mathbf{u}$.
This $\mathbf{u}$ determines the axis of rotation, unless $R = \mathrm{id}_3$.

We will see soon that it is very easy to find the eigenvector.

Angle of Rotation



Right Hand System

For any $R \in \mathrm{SO}_3$, their exists $S \in \mathrm{SO}_3$ such that

$$SRS^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad R \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in [0, \pi]$. Thus the **trace** of R is

$$\mathrm{tr}(R) = 1 + 2\cos\theta, \qquad -1 \le \mathrm{tr}(R) \le 3.$$

Remark: If $\theta \in \mathbb{R}/2\pi$ is the angle of rotation along the $\mathbf{u}$-direction, then $-\theta = 2\pi - \theta \in \mathbb{R}/2\pi$ is the angle of rotation along the $-\mathbf{u}$-direction.

Left and Right handed helices

Left Handed Helix

Right Handed Helix

cf. 갈등(葛藤)

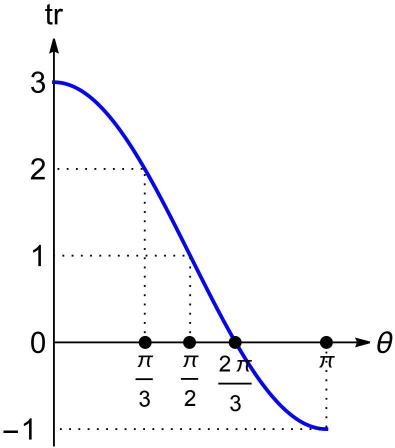
Trace and the angle of rotation

Let $R \in \mathrm{SO}_3$.

$$R \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

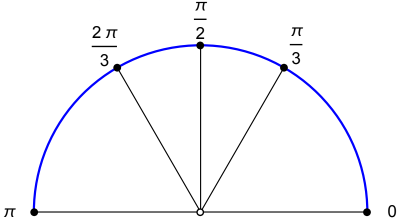
$$\operatorname{tr}(R) = 1 + 2\cos\theta$$

$$\theta \in [0, \pi]$$



We use the notation $A \sim B$ for square matrices A and B, if there exists an invertible matrix T such that $A = TBT^{-1}$.

$R \in \mathrm{SO}_3$



$$\mathrm{tr}(R) = 1 + 2\cos\theta \quad \in [-1, 3]$$

trace	$\cos\theta$	θ	Remarks
3	1	0	identity matrix
2	$\frac{1}{2}$	$\frac{\pi}{3}$	60°-rotations
1	0	$\frac{\pi}{2}$	90°-rotations
0	$-\frac{1}{2}$	$\frac{2\pi}{3}$	120°-rotations
-1	-1	π	half-turns

$$1 < \mathrm{tr} < 3 \;\Rightarrow\; 0 < \theta < \tfrac{\pi}{2}$$
$$-1 < \mathrm{tr} < 1 \;\Rightarrow\; \tfrac{\pi}{2} < \theta < \pi$$

Remark: Trace vs Determinant

For any sqaure matrix M,

$$\det(e^M) = e^{\mathrm{tr}(M)}, \quad \left.\frac{d}{dt}\right|_0 \det(e^{tM}) = \mathrm{tr}(M), \quad d(\det) = \mathrm{tr}$$

$$\begin{tikzcd}
\mathfrak{gl}_n \arrow[r, "\mathrm{tr}"] \arrow[d, "\exp"'] & \mathbb{R} \arrow[d, "\exp"] \\
\mathrm{GL}_n \arrow[r, "\det"'] & \mathbb{R}^\times
\end{tikzcd}$$

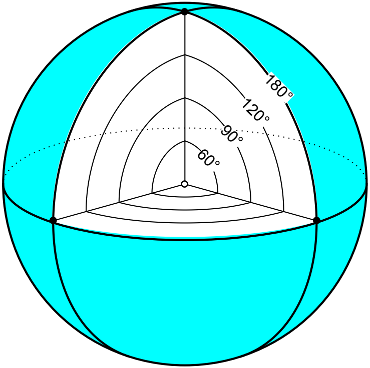
Determinant of a linear transformation detects the volume change.

The **flow** of the vector field $\mathbf{F}_M$ on $\mathbb{R}^n$, with $\mathbf{F}_M(X) := MX$, is

$$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n, \qquad X \mapsto e^{tM}X$$

and $\mathrm{div}(\mathbf{F}_M) = \mathrm{tr}(M)$.

Inside SO_3

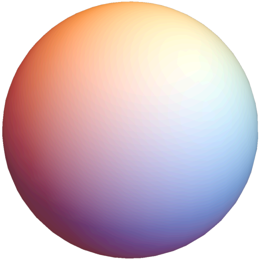


$$\mathrm{SO}_3 = \mathbb{B}^3(\pi)/\sim$$

where two points on the boundary $\mathbb{S}^2(\pi) = \partial(\mathbb{B}^3(\pi))$ are identified $(\sim)$ if and only if they are antipodal.

SO_3

SO_3 is a (solid) ball ($\mathbb{B}^3$) with some identification on the boudary.



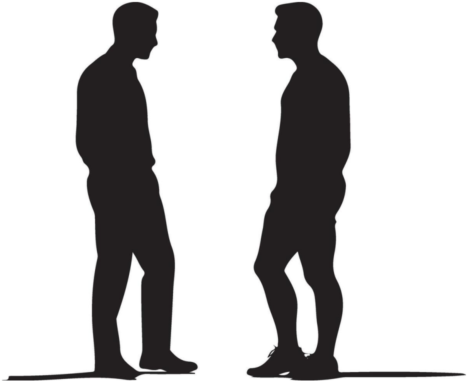
The identity matrix id_3 lies in the **center** of SO_3.

Each point $\mathbf{u}$ on the boundary $\mathbb{S}^2 = \partial\mathbb{B}^3$ represents the **half-turn** along the axis $\mathbf{u}$.

Thus $\mathbf{u}$ and the antipodal point $-\mathbf{u}$ represents the same half-turn.

Thus SO_3 is the **projective 3-space** $\mathbb{P}^3$.
$\mathbb{P}^3$ is $\mathbb{R}^3$ together with the points of plane at infinity.
One usually say that $\mathbb{P}^3$ is the collection of all 1-dimensional linear subspaces in $\mathbb{R}^4$, because $\mathbb{R}^3 \simeq \mathbb{R}^3 \times \{1\} \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$.

Half Turn, $180°$-Rotation



https://www.vecteezy.com/vector-art/60696141-two-men-standing-and-facing-each-other-in-silhouette

Half Turn, an example

Let

$$H := \frac{1}{7}\begin{pmatrix} 2 & 3 & 6 \\ 3 & -6 & 2 \\ 6 & 2 & -3 \end{pmatrix}.$$

Then $H \in \mathrm{SO}_3$ and $\mathrm{tr}(H) = -1$.
Thus H is a half turn, i.e., 180°-rotation. Now

$$\frac{1}{2}(H + \mathrm{id}) = \frac{1}{14}\begin{pmatrix} 9 & 3 & 6 \\ 3 & 1 & 2 \\ 6 & 2 & 4 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}\begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$$

Thus the rotation axis of H is $\mathbf{u} = \pm\frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

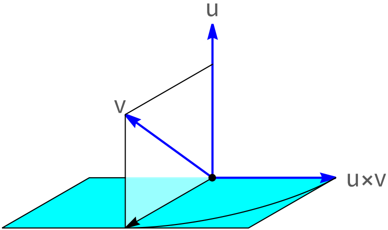
Remark: The transformation $2\mathbf{u}\mathbf{u}^* - \mathrm{id}$ means the half-turn along $\mathbf{u}$-axis.

Cross Product (교차곱) and Orientation

$$\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \qquad (\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \times \mathbf{v}$$

The cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ is characterized by the relation: for all $\mathbf{w} \in \mathbb{R}^3$,

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$$



Remarks: 1. Suppose $R = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \in \mathrm{O}_3$.
Then $R \in \mathrm{SO}_3$ if $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$. $R \in \mathrm{O}_3^-$ if $\mathbf{u}_3 = -\mathbf{u}_1 \times \mathbf{u}_2$.

2. $\mathbf{u} \times \mathbf{v} = \star(\mathbf{u} \wedge \mathbf{v})$

3. $$\begin{aligned} & (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ = \; & (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \end{aligned}$$

Symmetric and Skew-symmetric matrices

$$\boxed{\mathbf{M}_n = \mathbf{M}_n^+ \oplus \mathbf{M}_n^-}$$

Any square matrix M is the sum of symmetric and anti-symmetric (i.e., skew-symmetric) matrices, in a unique way:

$$M = M_+ + M_-.$$

$$M_+ = \tfrac{1}{2}\big(M + M^{\mathrm{t}}\big), \qquad M_- = \tfrac{1}{2}\big(M - M^{\mathrm{t}}\big)$$

Skew-Symmetric Matrices: $\mathfrak{so}_3 \simeq \mathbb{R}^3$

Let $\mathfrak{so}_3$ be the space of all 3×3 skew-symmetric matrices.

$$\mathfrak{so}_3 = \{A \in \mathbf{M}_3 \mid A + A^{\mathrm{t}} = O\}$$

We have the canonical isomorphism:

$$\mathbb{R}^3 \to \mathfrak{so}_3, \qquad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \mathbf{v}^{\times} := \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

Note that

$$\mathbf{v}^{\times}(\mathbf{x}) = \mathbf{v} \times \mathbf{x} \qquad (\mathbf{x} \in \mathbb{R}^3)$$

Exercises: 1. Show that for any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $R \in \mathrm{SO}_3$,

$$(R\mathbf{v})^{\times} = R\,\mathbf{v}^{\times} R^{\mathrm{t}}, \quad (\mathbf{v} \times \mathbf{w})^{\times} = [\mathbf{v}^{\times}, \mathbf{w}^{\times}], \quad R(\mathbf{v} \times \mathbf{w}) = (R\mathbf{v}) \times (R\mathbf{w}).$$

2. If $c : \mathbb{R} \to \mathrm{O}_3$ is a smooth curve with $c(0) = \mathrm{id}_3$, then $c'(0) \in \mathfrak{so}_3$.
Thus $\mathfrak{so}_3$ is the tangent space of O_3 (or SO_3) at the identity element.

To find the axis of a Rotation

$$\mathbb{R}^3 \simeq \mathfrak{so}_3, \qquad \mathbf{v} \mapsto \mathbf{v}^\times$$

Claim: *Suppose $R \in \mathrm{SO}_3$ and $R^2 \neq \mathrm{id}$. Let $t \in (0, \pi)$ be the angle of R, and let R_- be the anti-symmetric part of R so that $R_- = \mathbf{v}^\times$ for some non-trivial vector $\mathbf{v}$. Then*

$$R\mathbf{v} = \mathbf{v}, \quad |\mathbf{v}| = \sin t$$

and, with $\mathbf{u} := \mathbf{v}/|\mathbf{v}|$,

$$R = \exp(t\mathbf{u}^\times) = \mathbf{u}\mathbf{u}^* + (\cos t)(\mathrm{id} - \mathbf{u}\mathbf{u}^*) + (\sin t)\mathbf{u}^\times. \quad (1)$$

Remark: For a unit vector $\mathbf{u}$, $P_{\mathbf{u}} := \mathbf{u}\mathbf{u}^*$ is the projection operator onto the $\mathbf{u}$-axis, and $P_{\mathbf{u}^\perp} := \mathrm{id} - \mathbf{u}\mathbf{u}^*$ is the projection operator onto the orthogonal complement $\mathbf{u}^\perp$ of $\mathbf{u}$.

Proof

$0 = \mathbf{v}^\times(\mathbf{v}) = \frac{1}{2}(R - R^{\mathrm{t}})(\mathbf{v})$ and hence $R\mathbf{v} = R^{\mathrm{t}}(\mathbf{v})$. Thus $R^2\mathbf{v} = R(R^{\mathrm{t}}\mathbf{v}) = \mathrm{id}(\mathbf{v}) = \mathbf{v}$. Now apply this to the (Cayley-Hamilton) characteristic equation

$$R^3 - \tau R^2 + \tau R - \mathrm{id} = 0 \qquad (\tau := \mathrm{tr}(R) > -1)$$

to get $(1+\tau)R\mathbf{v} = (1+\tau)\mathbf{v}$. Thus $R\mathbf{v} = \mathbf{v}$.

Now let $\mathbf{a}$ be a unit vector perpendicular to $\mathbf{u}$, and let $\mathbf{b} := \mathbf{u}^\times(\mathbf{a})$. Then $(\mathbf{u}, \mathbf{a}, \mathbf{b})$ is a positively oriented orthonormal basis of $\mathbb{R}^3$ and

$$R(\mathbf{u}) = \mathbf{u}, \quad R(\mathbf{a}) = \mathbf{a}\cos t + \mathbf{b}\sin t, \quad R(\mathbf{b}) = -\mathbf{a}\sin t + \mathbf{b}\cos t.$$

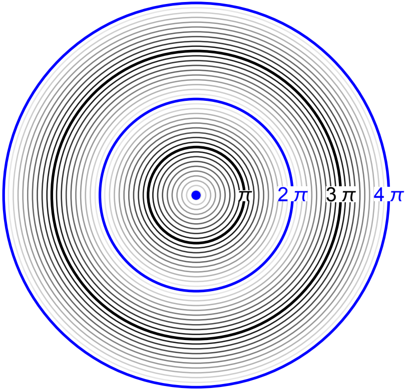
Now $\exp(t\mathbf{u}^\times) = \mathrm{id} + t\mathbf{u}^\times + \frac{t^2}{2!}(\mathbf{u}^\times)^2 + \frac{t^3}{3!}(\mathbf{u}^\times)^3 + \cdots$.
Clearly, $\exp(t\mathbf{u}^\times)(\mathbf{u}) = \mathbf{u}$. Since $\mathbf{u}^\times(\mathbf{b}) = -\mathbf{a}$,

$$\exp(t\mathbf{u}^\times)(\mathbf{a}) = \mathbf{a} + t\mathbf{b} - \frac{t^2}{2!}\mathbf{a} - \frac{t^3}{3!}\mathbf{b} + \cdots = \mathbf{a}\cos t + \mathbf{b}\sin t.$$

Thus $R = \exp(t\mathbf{u}^\times)$, and the equality (1) is easy to see. Finally $R = TDT^{\mathrm{t}}$ for some $T \in \mathrm{SO}_3$, where $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$. Since the anti-symmetric part of D is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin t \\ 0 & \sin t & 0 \end{pmatrix}$, $\mathbf{v} = T\begin{pmatrix} \sin t \\ 0 \\ 0 \end{pmatrix}$ and $|\mathbf{v}| = \sin t$.
This equality also follows from (1), since $R_- = (\sin t)\mathbf{u}^\times$. ☐

$\mathrm{SO}_3 \simeq \mathbb{B}^3(\pi)/\sim$

$$\exp : \mathfrak{so}_3 \to \mathrm{SO}_3, \qquad R_{\mathbf{u},t} = e^{t\mathbf{u}^\times} \quad \text{for } \mathbf{u} \in \mathbb{S}^2 \subset \mathbb{R}^3$$



Spheres of radii $n\pi$ in $\mathbb{R}^3 \simeq \mathfrak{so}_3$.

For any $t \in \mathbb{R}$, and any unit vector $\mathbf{u} \in \mathbb{R}^3$,

$$\exp((t + 2\pi)\mathbf{u}^\times) = \exp(t\mathbf{u}^\times)$$

$$\exp(n\pi\mathbf{u}^\times) = \begin{cases} \mathrm{id} & (n \text{ is even}) \\ H_{\mathbf{u}} & (n \text{ is odd}). \end{cases}$$

Exercises

For a unit vector $\mathbf{u}$ in $\mathbb{R}^3$, let $P = \mathbf{u}\mathbf{u}^*$ and $P_\perp := \mathrm{id} - P$.

1. Show that $\mathbf{u}^\times$ is the projection operator onto the plane $\mathbf{u}^\perp$ (perpendicular to $\mathbf{u}$) followed by the $+90^\circ$-rotation along the $\mathbf{u}$-axis:
$$\mathbf{u}^\times = \exp\left(\frac{\pi}{2}\mathbf{u}^\times\right) \circ P_\perp$$
2. Show that
$$\mathbf{u}^\times \circ P_\perp = \mathbf{u}^\times$$
and, for $n = 1, 2, 3, \ldots,$
$$(\mathbf{u}^\times)^{2n} = (-1)^n P_\perp, \qquad (\mathbf{u}^\times)^{2n+1} = (-1)^n \mathbf{u}^\times.$$
3. $\exp(\pi\mathbf{u}^\times) = H_{\mathbf{u}}$
4. Find the integrals
$$\int_0^{\pi/2} \exp(t\mathbf{u}^\times)\, dt, \quad \int_0^{\pi} \exp(t\mathbf{u}^\times)\, dt, \quad \int_0^{2\pi} \exp(t\mathbf{u}^\times)\, dt.$$

An Example

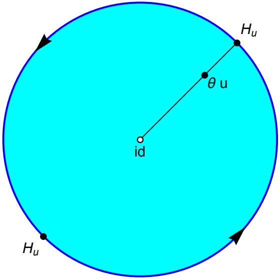
$$R = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix} \in \mathrm{SO}_3, \qquad \tfrac{1}{3} = \operatorname{tr}(R) = 1 + 2\cos\theta$$

$$\cos\theta = -\tfrac{1}{3}$$

$$R_- = \frac{1}{3}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}^{\times}$$

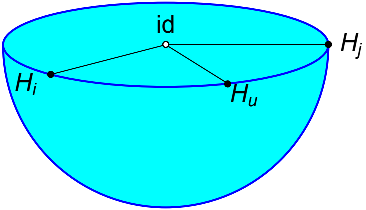
$$\mathbf{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad R\mathbf{u} = \mathbf{u}$$

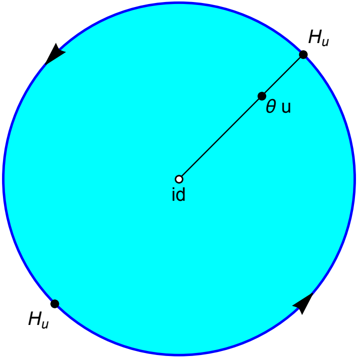
$$R \quad \Leftrightarrow \quad \theta\mathbf{u} \approx 0.6\pi\mathbf{u}$$



A section of SO_3

$$R = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}, \quad \mathbf{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$





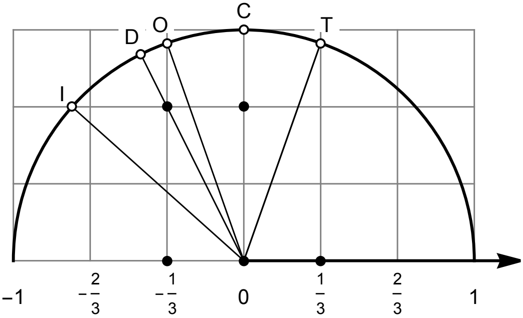
Tetrapod

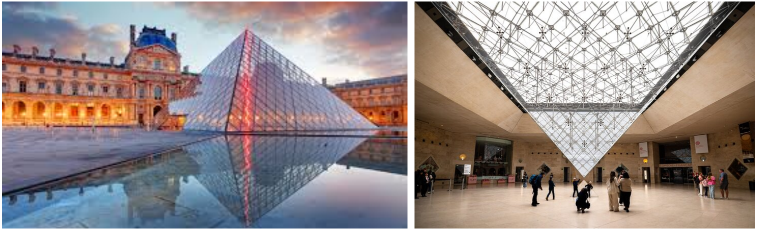
$$\cos\theta = -\frac{1}{3}, \qquad \arccos(-\tfrac{1}{3}) \approx 109.471^\circ \approx 0.6\pi$$

https://uditchbeton.com/material/tetrapod-pantai/

cf. CH_4 and Tetrahedron

Dihedral Angles of Platonic Solids





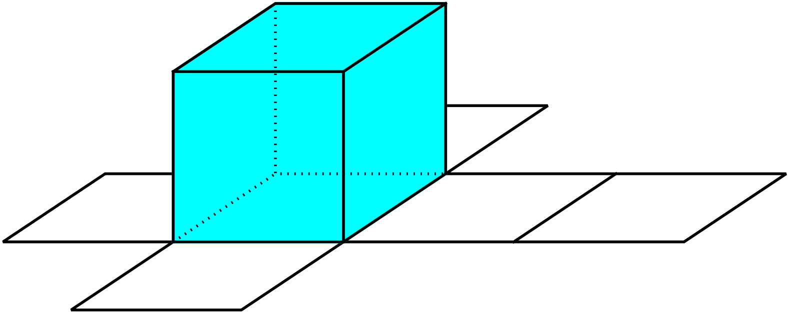
Louvre Museum

[현대수학입문, 경문사]

Exercise

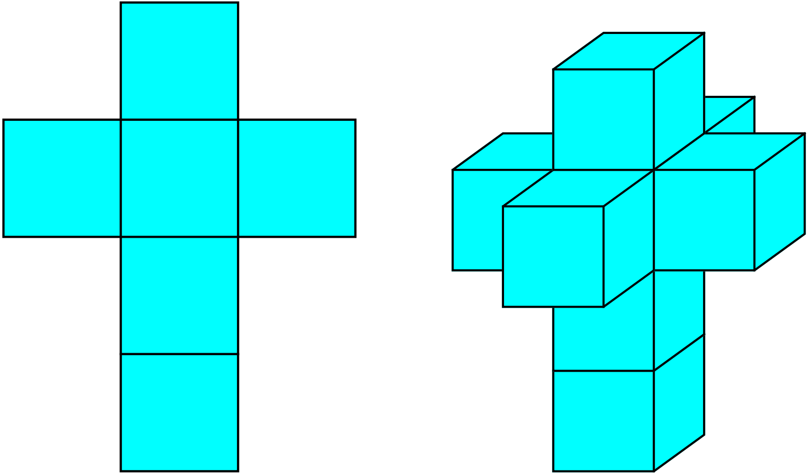
1. Show that $R := \frac{1}{7}\begin{pmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{pmatrix}$ is in SO_3. Find the axis and the angle of R.
2. For $\boldsymbol{\omega} \in \mathbb{R}^3$ and $\mathbf{x}_0 \in \mathbb{R}^3$, let
$$\mathbf{x}(t) := \exp(t\boldsymbol{\omega}^{\times})\mathbf{x}_0.$$
Show that $||\mathbf{x}(t)||$ is constant, $\mathbf{x}(t)$ lies on a circle, and
$$\mathbf{x}'(t) = \boldsymbol{\omega} \times \mathbf{x}(t).$$
3. Find all rational points on $\mathbb{S}^2$.
4. Supppose $A \in \mathrm{O}_3^-$. Show that $\mathrm{tr}(A) \in [-3, 1]$. Show that $\mathrm{tr}(A) = 1$ if and only if A is a reflection. Show that the collection of all reflections in O_3^- form a projective plane.

How to understand high dimensions?



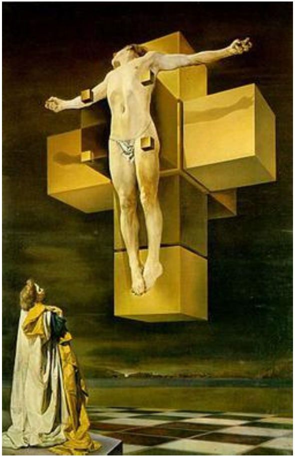
펼친도형 (전개도)

Cube and Hyper-Cube



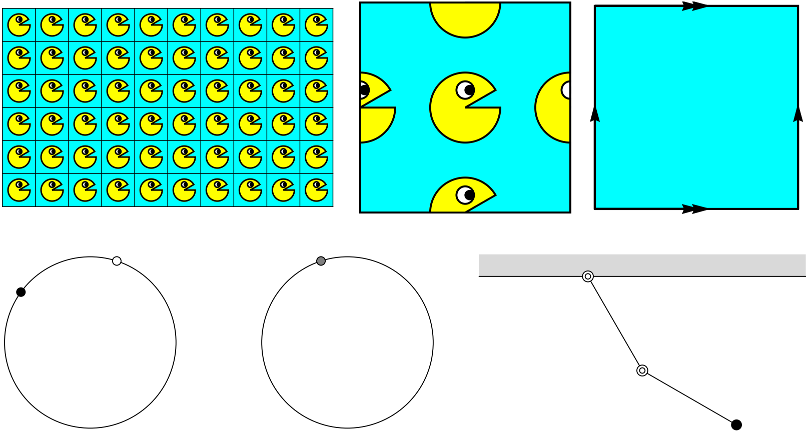
A cube, $[0,1]^3$, has 6 faces. A hypercube, $[0,1]^4$, has 8 faces.

Salvador Dalí, 1904–1989



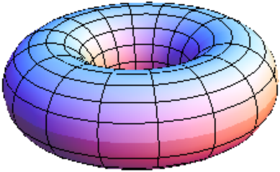
Crucifixion (Corpus Hypercubus), 1954

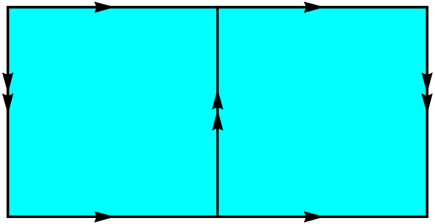
Torus: $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$

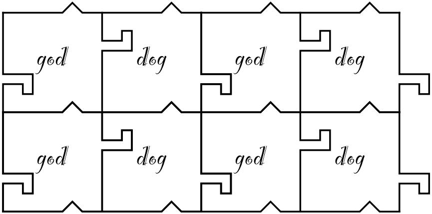


Klein Bottle, $\mathbb{K}$

$$\mathbb{K} = \mathbb{T}/\pm$$







$\mathbb{S}^3$

$$\begin{aligned}\mathbb{S}^3 &= \{t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \mid t^2 + x^2 + y^2 + z^2 = 1\} \\ &= \{\cos\theta + \mathbf{u}\sin\theta \mid \mathbf{u} \in \mathbb{S}^2,\ \theta \in \mathbb{R}\}\end{aligned}$$

$$\mathbb{S}^2 = \{x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \mid x^2 + y^2 + z^2 = 1\} = \{\mathbf{u} \in \mathbb{H} \mid \mathbf{u}^2 = -1\}$$

$$\mathbf{u} \in \mathbb{S}^2 \quad \Rightarrow \quad e^{\theta\mathbf{u}} = 1 + \theta\mathbf{u} + \frac{(\theta\mathbf{u})^2}{2!} + \frac{(\theta\mathbf{u})^3}{3!} + \cdots = \cos\theta + \mathbf{u}\sin\theta$$

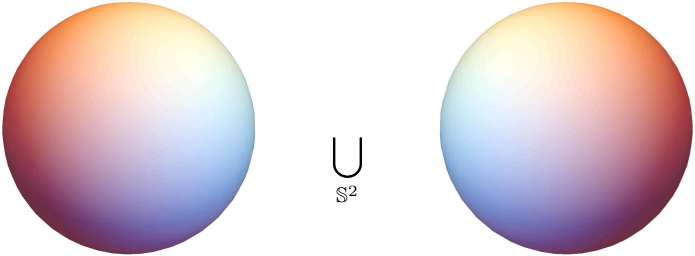
$$\mathbf{u} \in \mathbb{S}^2 \quad \Rightarrow \quad e^{\pi\mathbf{u}} + 1 = 0$$

$$\mathbb{S}^3 = \mathbb{S}^3_+ \cup \mathbb{S}^3_-$$

$$\mathbb{S}^3_+ \cap \mathbb{S}^3_- = \mathbb{S}^2 \qquad \text{(equator)}$$

$$\mathbb{S}^3_+ = \{(t,x,y,z) \mid x^2+y^2+z^2 = 1-t^2,\ t \geq 0\} \simeq \mathbb{B}^3$$

$$\mathbb{S}^3_- = \{(t,x,y,z) \mid x^2+y^2+z^2 = 1-t^2,\ t \leq 0\} \simeq \mathbb{B}^3$$

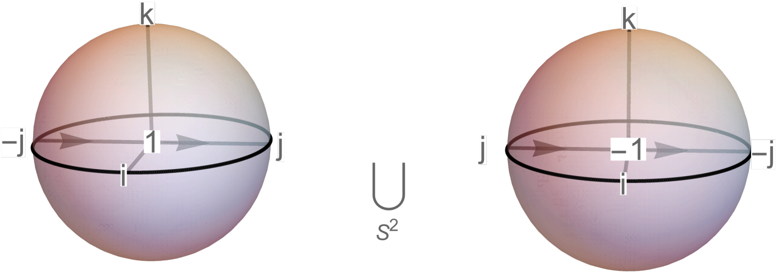


$$\bigcup_{\mathbb{S}^2}$$

$1 \in \mathbb{S}^3_+$

$-1 \in \mathbb{S}^3_-$

$\mathbb{S}^3$



North and south hemispheres

The arrows in the figure shows the circle

$$c(t) := e^{t\mathbf{j}}, \qquad (t \in \mathbb{R}).$$

$$c(0) = 1, \quad c(\pi/2) = \mathbf{j}, \quad c(\pi) = -1, \quad c(3\pi/2) = -\mathbf{j}, \quad c(2\pi) = 1$$

Loop

$$\begin{tikzcd}
\mathbf{u} \in \mathbb{S}^2 \subset \mathbb{R}^3 & \simeq & \mathfrak{so}_3 \arrow[d, "\exp"] & \ni & t\mathbf{u}^\times \arrow[d, mapsto] \\
 & & \mathrm{SO}_3 & \ni & \exp(t\mathbf{u}^\times)
\end{tikzcd}$$

For a unit vector $\mathbf{u} = (a, b, c)$, the path

$$R_\mathbf{u} : [0, 2\pi] \to \mathrm{SO}_3, \qquad t \mapsto \exp\left(t \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \right)$$

represents a continuous 360° rotation around the axis $\mathbf{u}$.

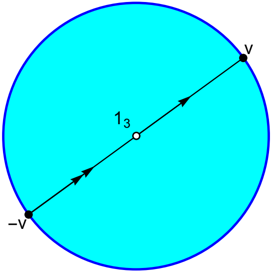
The path $R_\mathbf{u}$ is s loop with $R_\mathbf{u}(0) = R_\mathbf{u}(2\pi) = \mathrm{id}$.
It is NOT homotopic to the constant loop.
But the loop $t \mapsto R_\mathbf{u}(2t)$ is null-homotopic.

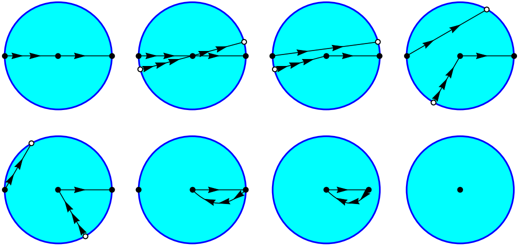
Loops in SO_3

For $\mathbf{v}$ on the boundary of $\mathbb{B}^3$, the loop $p : [0,2] \to \mathrm{SO}_3$

$$p(t) := \begin{cases} t\mathbf{v} & (0 \leq t \leq 1) \\ (t-2)\mathbf{v} & (1 \leq t \leq 2) \end{cases}$$

is NOT null-homotopic, but $p * p$ is null-homotopic.





Every loop in SO_3 is either null-homotopic or homotopic to p.

$\mathbf{u} \in \mathbb{S}^2 \subset \mathbb{S}^3 \cap \operatorname{im}(\mathbb{H})$

$$\begin{tikzcd}
e^{\theta\mathbf{u}} = \cos\theta + \mathbf{u}\sin\theta \arrow[d, mapsto] & \in & \mathbb{S}^3 \arrow[rr, leftrightarrow, "\simeq"] \arrow[dr] & & \mathrm{SU}_2 \arrow[dl] \\
e^{2\theta\mathbf{u}^\times} & \in & & \mathrm{SO}_3 &
\end{tikzcd}$$

$$\mathrm{SO}_3 = \mathbb{S}^3/\pm, \qquad \mathrm{SO}_3 = \mathrm{SU}_2/\pm\,.$$

$$\mathbb{S}^3 = \left\{ a + b\mathbf{j} \,\middle|\, \begin{matrix} a, b \in \mathbb{C} \\ |a|^2 + |b|^2 = 1 \end{matrix} \right\} \simeq \left\{ \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix} \,\middle|\, \begin{matrix} a, b \in \mathbb{C} \\ |a|^2 + |b|^2 = 1 \end{matrix} \right\} = \mathrm{SU}_2$$

The map $\mathrm{SU}_2 \to \mathrm{SO}_3$ is realized through the stereographic projection $\mathbb{S}^2 \to \mathbb{P}^1(\mathbb{C})$, where SU_2 acts on $\mathbb{P}^1(\mathbb{C})$ as Möbius transformations.

Lie Algebra Level

$$\begin{tikzcd}
\operatorname{im}\mathbb{H} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathfrak{su}_2 \arrow[dl, leftrightarrow] \\
& \mathfrak{so}_3 &
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] \& \& \begin{pmatrix} x\mathbf{i} & -y - z\mathbf{i} \\ y - z\mathbf{i} & -x\mathbf{i} \end{pmatrix} \arrow[dl, leftrightarrow] \\
\& 2\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \&
\end{tikzcd}$$

Appendix

Exercises

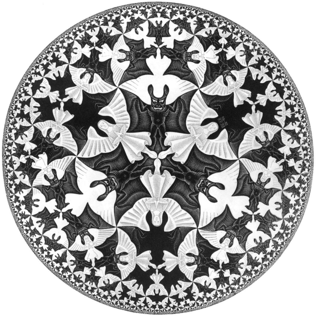
1. Let $A \in \mathrm{SO}_3$ and $A^n = \mathrm{id}_3$ for some positive integer n. Show that A is a rotation by the angle $2\pi/n$.
2. Describe a monomorphism $\mathrm{O}_2 \to \mathrm{SO}_3$.
3. (Maximal Torus Theorem) Show that for any compact connected Lie group G, there exists a *maximal torus* $\mathbb{T}$. Moreover
$$G = \bigcup_{g \in G} g\mathbb{T}g^{-1}.$$

Spaces of Constant Curvature

A *continuous motion* of any rigid body is possible if the underlying space M is of **constant curvature** K.

Then M is locally isometric to

$$\begin{cases} \mathbb{S}^n & \text{if } K = 1 \\ \mathbb{E}^n & \text{if } K = 0 \\ \mathbb{H}^n & \text{if } K = -1\,. \end{cases}$$



M. C. Escher, *Circle Limit, IV*

cf. Space Form Problem, Helmholtz, Killing(1891)-Hopf(1926) Theorem

Motions

Let $(\mathbb{E}, d)$ be the Euclidean space (of any dimension).
It is a **metric space** isometric to $\mathbb{R}^n$ for some positive integer n.

A *transformation* $f : \mathbb{E} \to \mathbb{E}$ is called a (rigid) **motion** if

$$d(p, q) = d(f(p), f(q))$$

for all points p and q in $\mathbb{E}$.

A motion is either **orientation preserving** or **orientation reversing**, but not both.

Two bases $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $\mathbf{b}' = (\mathbf{b}'_1, \ldots, \mathbf{b}'_n)$ of a real vector space V are in the *same orientation* if and only if there exists an $n \times n$ real matrix g of positive determinant such that $\mathbf{b}' = \mathbf{b}g$.
This relation is an equivalence relation.

An **orientation** for a real vector space is a choice of an equivalence class of basis.

Continuous Motion

Let $\mathrm{MO}(\mathbb{E})$ be the group of all motions in $\mathbb{E}$,
and let $\mathrm{MO}^+(\mathbb{E})$ be the group of orientation preserving motions.

$$\dim \mathrm{MO}(\mathbb{E}) = \dim \mathrm{MO}^+(\mathbb{E}) = \tfrac{1}{2}n(n+1) \qquad (n = \dim \mathbb{E})$$

We have the short exact sequences:

$$1 \to \vec{\mathbb{E}} \to \mathrm{MO}(\mathbb{E}) \to \mathrm{O}(\vec{\mathbb{E}}) \to 1, \qquad \mathrm{MO}(\mathbb{E}) = \vec{\mathbb{E}} \rtimes \mathrm{O}(\vec{\mathbb{E}})$$

$$\begin{tikzcd} 1 \arrow[r] & \vec{\mathbb{E}} \arrow[r] \arrow[d, squiggly] & \mathrm{MO}^+(\mathbb{E}) \arrow[r] & \mathrm{SO}(\vec{\mathbb{E}}) \arrow[r] \arrow[d, squiggly] & 1, \\ & \text{linear momentum} & & \text{angular momentum} & \end{tikzcd} \qquad \mathrm{MO}^+(\mathbb{E}) = \vec{\mathbb{E}} \rtimes \mathrm{SO}(\vec{\mathbb{E}})$$

A **continuous motion** of a rigid body B in the space $\mathbb{E}$ is described by a path

$$p : [0,1] \to \mathrm{MO}(\mathbb{E}), \quad p(0) = \mathbf{1}_{\mathbb{E}}.$$

If B_0 is the initial position of the body, then $B_t := p(t)B_0$ is the position and the orientation of the body at time t.

Axis of rotation

Suppose $R \in \mathrm{SO}_3$ is non-trivial and non-half-turn.
Then $R_- = \mathbf{v}^\times$ for some nontrivial vector $\mathbf{v}$. The unit vector $\mathbf{u} := \mathbf{v}/|\mathbf{v}|$ is called the (proper) **axis** (of rotation) for R.

Note that

$$\langle \mathbf{u}, \mathbf{w} \times R\mathbf{w} \rangle \geq 0$$

for any $\mathbf{w} \in \mathbb{R}^3$, so that the rotation angle θ is in the interval $[0, \pi]$. Note that $\langle \mathbf{u}, \mathbf{w} \times R\mathbf{w} \rangle = \det(\mathbf{u}, \mathbf{w}, R\mathbf{w})$ and hence one may assume that $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

Exercise: Let $\mathbf{u}$ be the axis of R, then

$$\langle \mathbf{u}, \mathbf{w} \times R\mathbf{w} \rangle > 0$$

for any $\mathbf{w}$ which is linearly independent of $\mathbf{u}$, the axis of R.
Show that $-\mathbf{u}$ is the axis of R^{t}.

Find the eigenvector

Exercise: For $A \in \mathrm{SO}_3$, let $A^{\mathrm{t}} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$. Suppose $A \neq \mathrm{id}_3$. Show that $\mathrm{rank}(A - \mathrm{id}_3) = 2$ and hence (at least) one of the following is a (nontrivial) eigenvector of A:

$$(\mathbf{b}_1 - \mathbf{e}_1) \times (\mathbf{b}_2 - \mathbf{e}_2), \quad (\mathbf{b}_2 - \mathbf{e}_2) \times (\mathbf{b}_3 - \mathbf{e}_3), \quad (\mathbf{b}_3 - \mathbf{e}_3) \times (\mathbf{b}_1 - \mathbf{e}_1).$$

Orientation

Let M be a connected smooth manifold.

A a **frame** at a point $p \in M$ is a basis $\mathbf{b}$ for the tangent space TM_p at p.

The collection of all frames on M is a fiber bundle $\mathrm{GL}(M)$ over M.

The space M is **not orientable** if for any pair of frames $(p, \mathbf{b})$ and $(p', \mathbf{b}')$ there exists a continuous path $c : [0, 1] \to \mathrm{GL}(M)$ such that $c(0) = (p, \mathbf{b})$ and $c(1) = (p', \mathbf{b}')$.

When M is **orientable**, a choice of a component of $\mathrm{GL}(M)$ is called an **orientation**.

One may discuss orientations for topological manifolds.

Half-turns

Proposition: *For a non trivial $R \in \mathrm{SO}_3$, the following are equivalent:*

(i) R *is a half-turn.*

(ii) $\mathrm{tr}(R) = -1$

(iii) -1 *is an eigenvalue of* R.

(iv) R *is symmetric, i.e.* $R^{\mathrm{t}} = R$.

(v) R *is an involution, i.e.,* $R^{-1} = R$.

(vi) $\frac{1}{2}(R + \mathrm{id})$ *is a projection operator.*

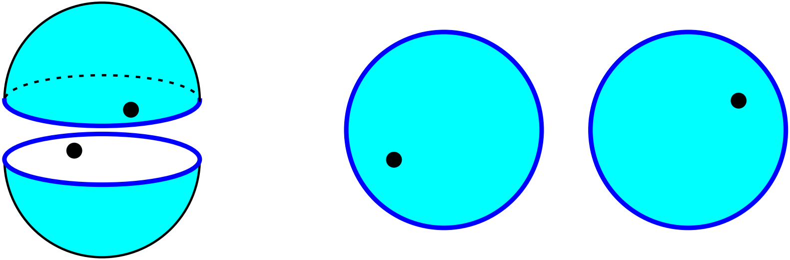
(vii) $\frac{1}{2}(R + \mathrm{id}) = \mathbf{u}\mathbf{u}^*$ *for some unit vector* $\mathbf{u}$.

A linear transformation $P : \mathbb{R}^n \to \mathbb{R}^n$ is called an **orthogonal projection** if $P^2 = P$ and $P^{\mathrm{t}} = P$. Then $P_\perp := \mathrm{id} - P$ is the complementary orthogoonal projection.

For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the map $\mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{x} \mapsto \mathbf{u} \langle \mathbf{v}, \mathbf{x} \rangle$ is denoted by $\mathbf{u} \otimes \mathbf{v}^*$ or $\mathbf{u}\mathbf{v}^*$.

Exercise: Find the matrix form of $\mathbf{u}\mathbf{v}^*$.

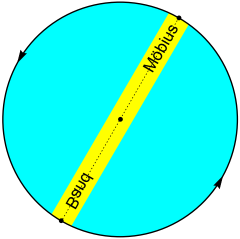
Half-Turns and Projective Plane



Let $\mathcal{H}$ be the collection of all half-turns.
We have the covering map

$$H : \mathbb{S}^2 \to \mathcal{H}, \quad \mathbf{u} \mapsto H_{\mathbf{u}} := 2\mathbf{u}\mathbf{u}^* - \mathrm{id}$$

and hence $\mathcal{H} \simeq \mathbb{P}^2$, the projective plane.



Exercises: 1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be an orthonormal basis for $\mathbb{R}^3$. Show that $H_{\mathbf{u}} \circ H_{\mathbf{v}} = H_{\mathbf{w}}$.

2. Show that the collection of all lines in the projective plane is also a projective plane.

Thus the collection of all lines in the Euclidean plane is a Möbius band.

Half turns

For a unit vector $\mathbf{v}$, let

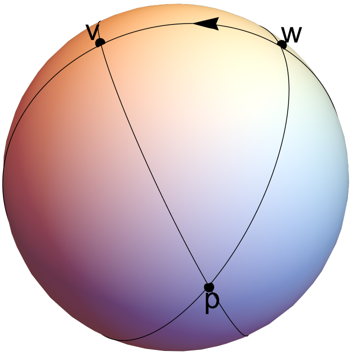
$$H_{\mathbf{v}} : \mathbb{R}^3 \to \mathbb{R}^3$$

be the half-turn operation along the $\mathbf{v}$-axis. Note that $H_{\mathbf{v}} = H_{-\mathbf{v}}$.

Problem: For unit vectors $\mathbf{v}$ and $\mathbf{w}$, is $H_{\mathbf{v}} \circ H_{\mathbf{w}}$ a half-turn?

`Solution:` If $\mathbf{v} = \pm\mathbf{w}$, then $H_{\mathbf{v}} \circ H_{\mathbf{w}} = \mathrm{id}$, the identity transformation.

Thus we will assume that $\mathbf{v} \neq \pm\mathbf{w}$. Let θ be the angular distance between $\mathbf{v}$ and $\mathbf{w}$. Then $0 < \theta < \pi$. Let C be the great circle (in the sphere $\mathbb{S}^2$) which passes through the points $\mathbf{w}$ and $\mathbf{v}$. We consider the orientation on C determined by the ordering $(\mathbf{w}, \mathbf{v})$. Then this orientation determines the **pole** $\mathbf{p} \in \mathbb{S}^2$ of C. Then $H_{\mathbf{v}} \circ H_{\mathbf{w}}$ is the rotation along the $\mathbf{p}$-axis with the angle 2θ.



Let C be the circle which passes through the points $\mathbf{w}$ and $\mathbf{v}$. The orientation of C is given by the ordering $(\mathbf{w}, \mathbf{v})$.

Let $C_{\mathbf{v}}$ be the great circle through $\mathbf{v}$ and perpendicular to C, and let $C_{\mathbf{w}}$ be the great circle through $\mathbf{w}$ and perpendicular to C.

The circles $C_{\mathbf{v}}$ and $C_{\mathbf{w}}$ intersect at $\mathbf{p}$ and $-\mathbf{p}$, where $\mathbf{p}$ is the pole of the oriented circle C.

Note that $(H_{\mathbf{v}} \circ H_{\mathbf{w}})(\mathbf{p}) = H_{\mathbf{v}}(-\mathbf{p}) = \mathbf{p}$ and hence $H_{\mathbf{v}} \circ H_{\mathbf{w}}$ is a rotation along the $\mathbf{p}$-axis. Now $(H_{\mathbf{v}} \circ H_{\mathbf{w}})(\mathbf{w}) = H_{\mathbf{v}}(\mathbf{w})$.

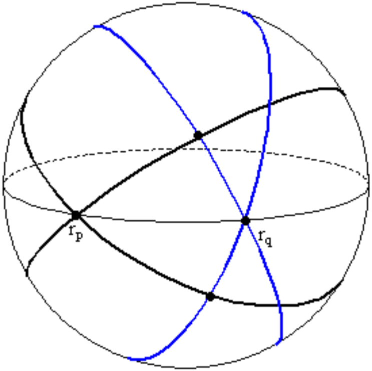
Thus the rotation angle of $H_{\mathbf{v}} \circ H_{\mathbf{w}}$ is 2θ.

In particular, if $\mathbf{v}$ and $\mathbf{w}$ are perpendicular, then $H_{\mathbf{v}} \circ H_{\mathbf{w}}$ is also a half-turn.

Composition of Rotations

Two centers p and q of rotations $r_p = r_{p,\theta_p}$ and $r_q = r_{q,\theta_q}$, respectively, on the sphere $\mathbb{S}^2$ are marked in the figure. (SO_3 is the group of orientation-preserving isometries on the sphere $\mathbb{S}^2$.) We assume that $p \neq q$.

The operation $r_q \circ r_p$ is also a rotation about the point x, which is the intersection of two great circles:



If C is the great circle passing through the points p and q, then C_p is the great circle with $\angle(C, C_p) = \frac{1}{2}\theta_p$ and C_q is the great circle passing through q with $\angle(C, C_q) = \frac{1}{2}\theta_q$.

If we use oriented circles, then the intersection of C_p and C_q is the well-defined antipodal pair. This is pair is the center of rotation.

The rotation angle is $2\angle(C_p, C_q)$.

Quaternion description of rotations maybe easy to find the axis of composition.

$(\cos t + \mathbf{u} \sin t)(\cos s + \mathbf{v} \sin s) = \cos t \cos s + \mathbf{u} \sin t \cos s + \mathbf{v} \cos t \sin t + (\mathbf{u} \times \mathbf{v}) \sin t \sin s.$