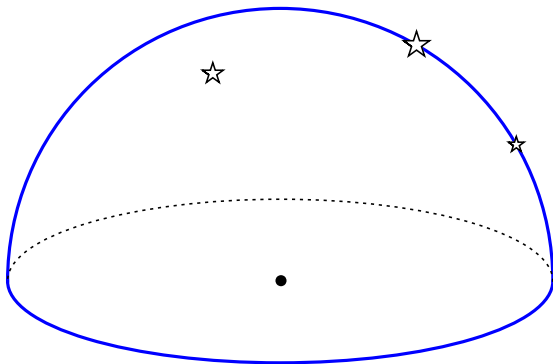


# H e a v e n

(Spheres, 天球)



Kosmos Club, 2026. 4. 11. 김 홍 종

# Ptolemaios (Clavius Ptolemy), c.100 – c.170

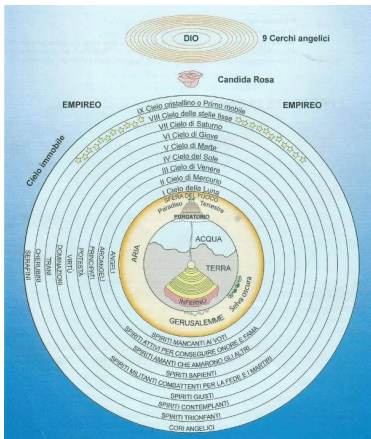


Portrait of Ptolemy (1476)

Schema huius præmissæ diuisionis Sphærarum.



Almagest  
(Mathematical Systematic Treatise)



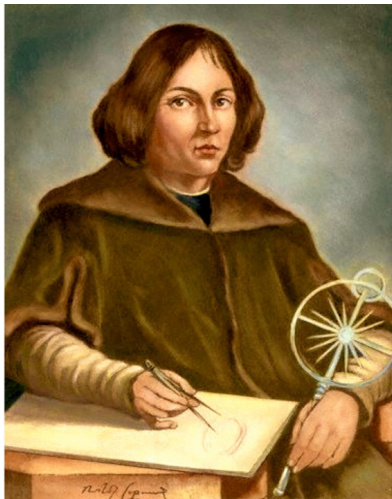
*Paradiso, La Divina Commedia*  
Dante Alighieri (1265–1321)

<https://ko.wikipedia.org/wiki/>



*Decameron*  
Giovanni Boccaccio (1313–1375).

## 1543, *De Revolutionibus*, Copernicus (1473–1543)



### NICOLAI COPERNICI TORINENSIS DE REVOLUTIONIBVS ORBI- um coelestium, Libri VI.

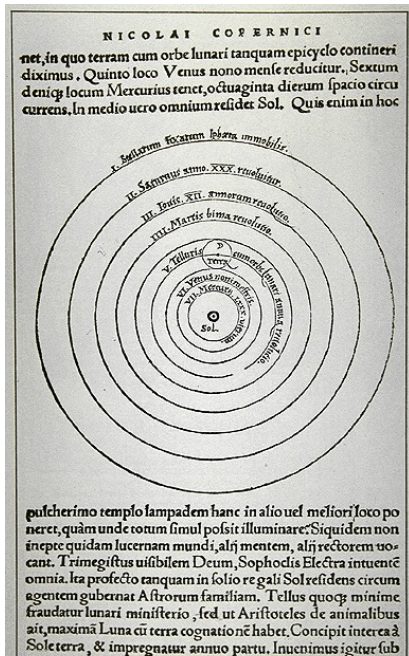
Habes in hoc opere iam recens nato, & ædito, studiose lector, Motus stellarum, tam fixarum, quam erraticarum, cum ex ueteribus, tum etiam ex recentibus obseruationibus restitutos: & nouis insuper ac admirabilibus hypothesibus ornatos. Habes etiam Tabulas expeditissimas, ex quibus cõdem ad quoduis tempus quàm facillime calculare poteris. Igitur eme, lege, fruere.

Ἀγαπίστεο ἄδελφοί μου.

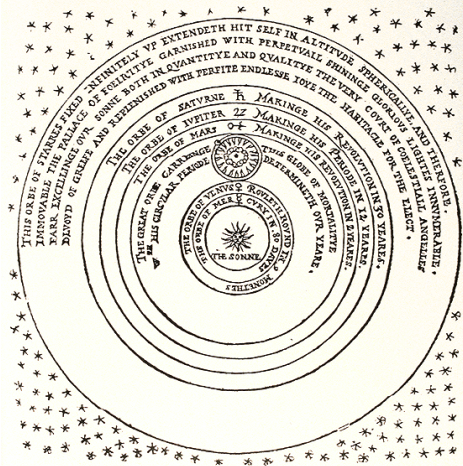
Norimbergæ apud Ioh. Petreium,  
Anno M. D. XLIII.

기하학은 사라지는 일시적인 것이 아닌 영원한 지식을 추구한다. — 플라톤 (국가론 VII)  
플라톤의 아카데미에 적힌 “기하학을 모르는 사람은 들어오지 말라”는 문구는  
“진리(신, 이데아)를 추구하지 않는 사람은 들어오지 말라”는 뜻. 점은 점이 아니요, 자는 자가 아니다.

# De Revolutionibus



A Perfit description of the Cœlestiall Orbes,  
 according to the most auncient doctrine of the  
 Pythagoreans, &c.



1576, *A Perfit Description of the Cœlestiall Orbes*  
 according to the most auncient doctrine of Pythagoreans, etc.  
 Thomas Digges (1546–1595)

# Leonardo da Vinci (1452–1519)

- *Mechanics is the paradise of the mathematical sciences,  
because by means of it one comes to the fruits of mathematics.*

cf. J. Fourier: *The profound study of nature is the most fertile source of mathematical discoveries.*

cf. [Plato, Archimedes, Jacobi]

cf. [Descartes — 이 학문에서 얻을 수 있는 가장 기본적인 이득은 “영혼의 수양”입니다.]

- *Every process in nature will occur in the shortest possible way.*

[E. Zeidler, *Nonlinear Functional Analysis, IV*, 1988.]

- *Let no one who is not a mathematician read my works.*  
– In the opening of *Trattato della Pittura*

- Perspective is the rein(고삐) and rudder(방향타) of painting.

- 화가의 마음은 거울과 같아야 한다.

- 식성과 상관없이 과식하는 것이 건강에 해로운 것처럼 의욕없이 하는 공부는 기억력을 망친다.  
이렇게 해서 공부한 것은 아무것도 남지 않는다.

- 쓸모 없는 사람을 칭찬하는 것은 좋은 사람을 헐뜯는 것만큼 큰 잘못이다.

- 지구는 태양계도나 우주의 중심이 아니다.

- 어떤 것들과 모두 같은 것이 있으면, 그것들은 모두 같다.

⋮

전체는 항상 부분보다 크다. (in the Elements of Euclid)

## Heavenly Sphere (天球)

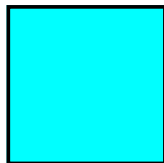
In an affine space  $\mathbb{A}$ , the **sphere**

$$\mathbb{S}(\mathbb{A})$$

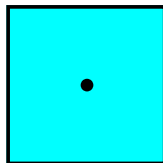
is the collection of all directions.

---

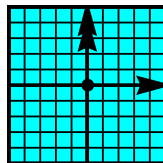
**Remark:** An **affine space** is a vector space without the origin.



affine space



vector space



Cartesian space

[See Appendix: Affine Space]

# Direction

A **direction** in a (real) affine space  $(\mathbb{A}, \vec{\mathbb{A}})$  is a collection

$$\{c\mathbf{v} \mid c \in \mathbb{R}_+\}$$

for some nontrivial vector  $\mathbf{v}$ , where  $\mathbb{R}_+ := \{c \in \mathbb{R} \mid c > 0\}$ .

---

## Remarks:

Any **oriented line** in an affine space determines a direction.

Two parallel oriented lines determine the same direction.

What is a projective space?

For a (real) vector space  $V$ , let  $V_*$  be the set of nontrivial vectors:

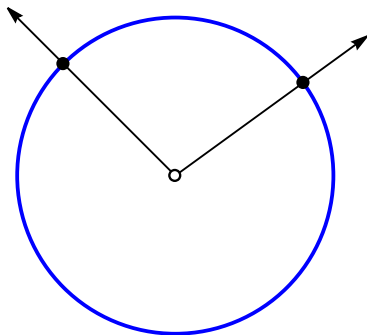
$$V_* := V - \{0\}.$$

Then the collection of all directions in  $V$  is the **sphere**

$$\mathbb{S}(V) = V_*/\mathbb{R}_+ = \{ \text{directions in } V \}.$$

When  $V$  has a **norm**,

$$\mathbb{S}(V) = \{ \mathbf{v} \in V : |\mathbf{v}| = 1 \}.$$



# Standard Spheres

When  $V = \mathbb{R}^{n+1}$ , we have

$$\mathbb{S}^n = \mathbb{S}(\mathbb{R}^{n+1}) = \{\mathbf{r} \in \mathbb{R}^{n+1} : |\mathbf{r}| = 1\}$$

---

**Remark:** The dimension of the ordinary sphere

$$\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is 2, not 3.

The collection of all directions in an  $n + 1$  dimensional (affine, Euclidean, vector, Cartesian) space is the  $n$ -dimensional sphere.

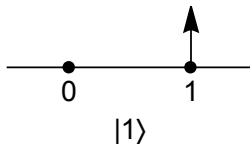
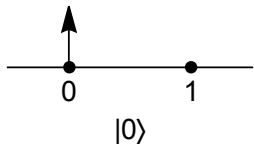
## $S^0$ , 陰陽, parity and bits

$$S^0 = \{\bullet, \circ\} = \{\mathbf{T}, \mathbf{F}\} = \{+, -\}$$

---

A **bit** is an element of  $S^0$ .

A **quantum bit** (i.e., **qubit**) is a nontrivial function  $S^0 \rightarrow \mathbb{C}$  up to a nontrivial multiplicative scalar.



# 《태극도설》(太極圖說), 주돈이(周敦頤, 1017-1073)

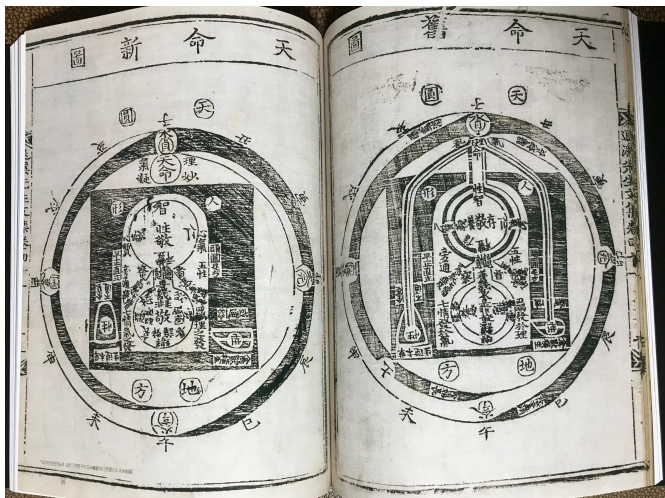
無極而太極 太極動而生陽 動極而靜 靜而生陰 靜極復動  
(무극이태극 태극동이생양 동극이정 정이생음 정극복동)

一動一靜 互爲其根 分陰分陽 兩儀立焉  
(일동일정 호위기근 분양분음 양의입언)

陽變陰合 而生水 火 木 金 土 五氣 順布 四時 行焉  
(양변음합 이생수화목금토 오기순포 사시행언)

⋮

# 퇴계 이항 천명도(天命圖)



天圓地方

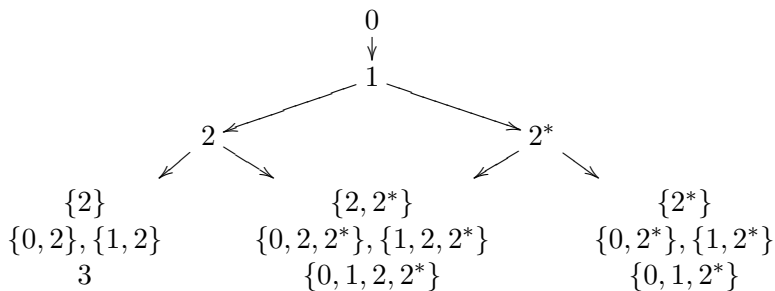
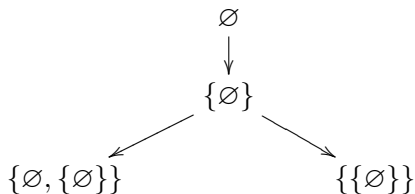
cf. 기대승(奇大升, 1527-1572) & 월봉서원(月峯書院)

cf. 中庸 - 天命之謂性 率性之謂道 修道之謂教

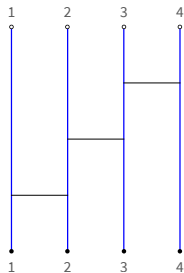
訓民正音, 天地人, 하나



# Set Theory: *Nothing exists*



# Parity



1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

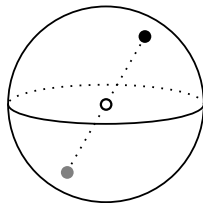
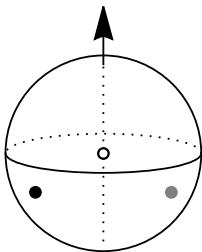
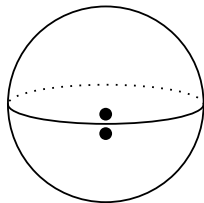
15	14	13	12
11	10	9	8
7	6	5	4
3	2	1	

# Representations of $\mathbb{S}^0 = \{+, -\}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

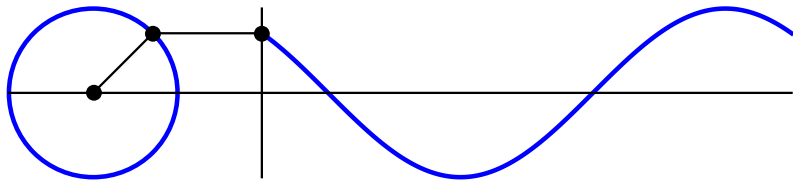
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$



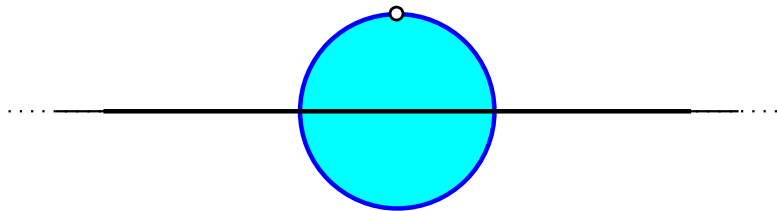
$S^1$ , the circle, angles

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$



# 하나(님)

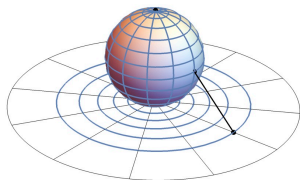
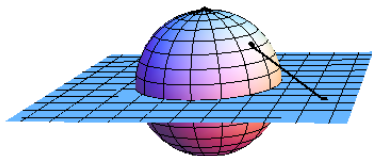
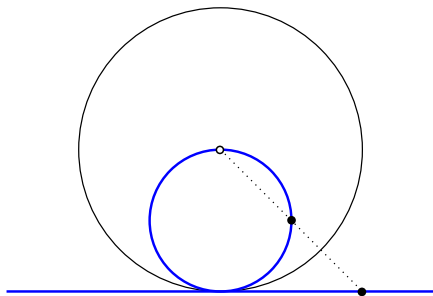
$$\begin{array}{ccc} & S^1 & \\ // & & // \\ \mathbb{R} \cup \{\bullet\} & & \partial\mathbb{B}^2 \end{array}$$

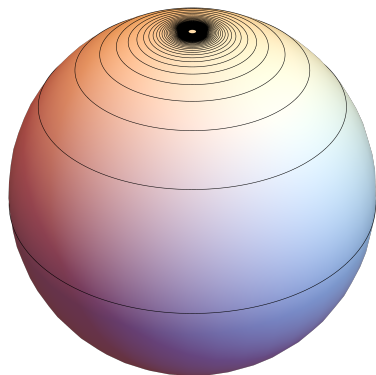
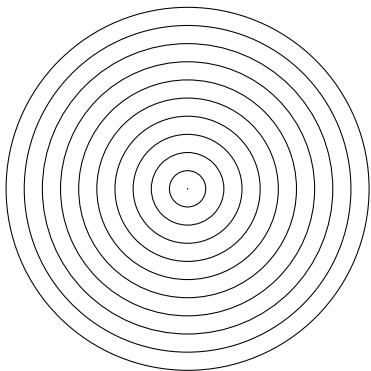


$$\mathbb{R}^n \cup \{\infty\} \simeq S^n = \partial\mathbb{B}^{n+1}$$

無明이란 하나임을 모르는 것을 뜻한다 — 원효 [趙恩秀, 깨달음(覺)의 理論的 解明, 1986]

# Stereographic Projection ( $n = 1, 2, 3, \dots$ )



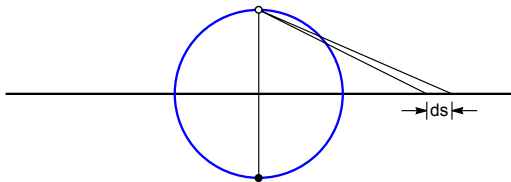


Concentric circles in the plane (Left)  
and its images in the sphere (Right)  
Circle becomes smaller if its radius  $\geq \pi/2$ .

## Metric

In  $\mathbb{R}^2$ , consider the **metric** (or the first fundamental form)

$$ds^2 = \frac{4}{(1+x^2+y^2)^2} (dx^2 + dy^2), \quad ds = \frac{2}{1+r^2} \sqrt{dr^2 + r^2 d\theta^2}$$



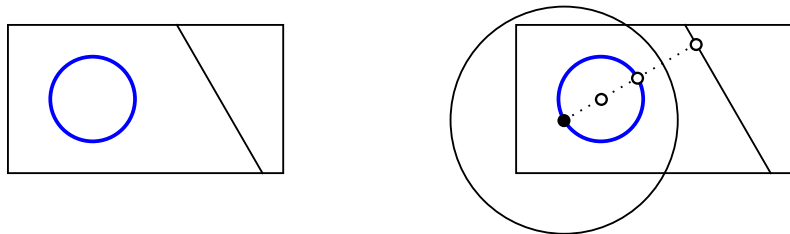
Then the length of the curve  $c(t) := t(a, b)$ ,  $0 \leq t < \infty$ , is

$$\int_0^\infty \|c'(t)\| dt = \int_0^\infty \frac{2\sqrt{a^2 + b^2}}{1 + (a^2 + b^2)t^2} dt = \int_0^\infty \frac{2 d\tau}{1 + \tau^2} = \pi$$

which is the distance between the north pole and the south pole on the standard unit sphere.

**Exercise:** Let  $c_r(t) := r(\cos t, \sin t)$  for  $0 \leq t \leq 2\pi$ . For what value of  $r \in [0, \infty)$ , is the length of  $c_r$  the largest?

## Inversion vs Stereographic Projection



*A circle and a line in a plane are the images of each other through an inversion.*

$$\mathbb{S}_*^n \simeq \mathbb{E}^n, \quad \mathbb{S}^n \simeq \mathbb{E}^n \cup \{\infty\}$$

*A hypersphere and a hyperplane in a Euclidean space are the images of each other through an inversion.*

$$S^{n-1} = \partial B^n$$

$$S^{n-1} \subset B^n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}$$

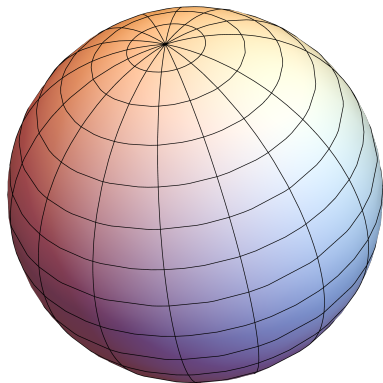
Sphere is the boundary of a ball.

$$S^0 \subset S^1 \subset S^2 \subset S^3 \subset S^4 \subset \dots$$

$$\cap \quad \cap \quad \cap \quad \cap \quad \cap$$

$$B^1 \subset B^2 \subset B^3 \subset B^4 \subset B^5 \subset \dots$$

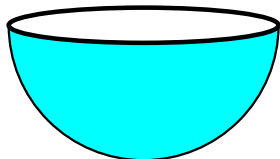
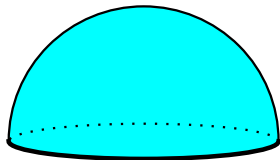
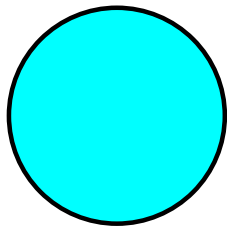
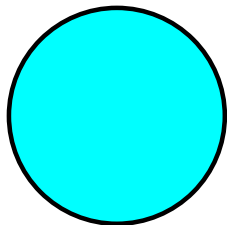
$$\mathbb{S}^4 = \partial\mathbb{B}^5$$



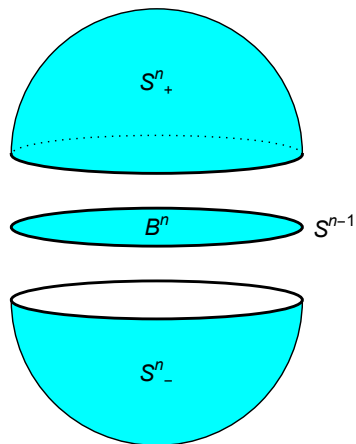
Evolution of  $\mathbb{S}^3$ : From Big Bang to Big Crunch

## Gluing and Surgery

The sphere is the union of two discs along their boundaries.



# Hemispheres and Equator



$$\mathbb{S}_+^n = \{(x_0, \dots, x_n) \in \mathbb{S}^n : x_n \geq 0\}$$

$$\mathbb{S}_-^n = \{(x_0, \dots, x_n) \in \mathbb{S}^n : x_n \leq 0\}$$

$$\mathbb{S}^{n-1} = \{(x_0, \dots, x_n) \in \mathbb{S}^n : x_n = 0\}$$

$$\mathbb{S}^n = \mathbb{S}_+^n \cup \mathbb{S}_-^n$$

$$\mathbb{S}_+^n \cap \mathbb{S}_-^n = \mathbb{S}^{n-1} \quad (\text{equator})$$

## $\mathbb{S}^2$ , the ordinary sphere

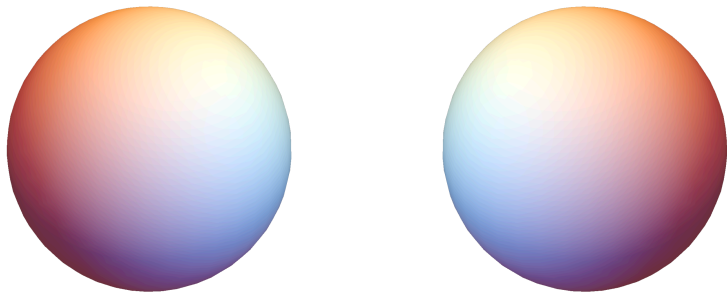
**Theorem:** *Generic vector field on a (2 dimensional) sphere has two singularities.*

There exists no method to identify two vectors in different locations on the sphere.



Q: 어느 시점에 지구 상에서 바람 불지 않는 지점이 있을까?

**Corollary:**  $\mathbb{S}^2$  is not a Lie group.

$S^3$ 

$$S^3 \simeq \mathbb{B}^3 \cup_{S^2} \mathbb{B}^3$$

$$1 \in S^3_+, \quad -1 \in S^3_-$$

## Quaternions and $\mathbb{S}^3$

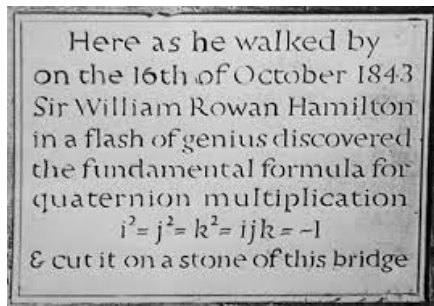
$$\mathbb{H} = \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : x_0, x_1, x_2, x_3 \in \mathbb{R}\} \simeq \mathbb{R}^4$$

$$\mathbb{S}^3 = \{q \in \mathbb{H} : |q| = 1\}$$

## William Hamilton and Quaternions, 1843

The quaternion multiplication of vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  is

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$



Brougham (or Broom) Bridge

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}$$

# William Rowan Hamilton, 1805–1865

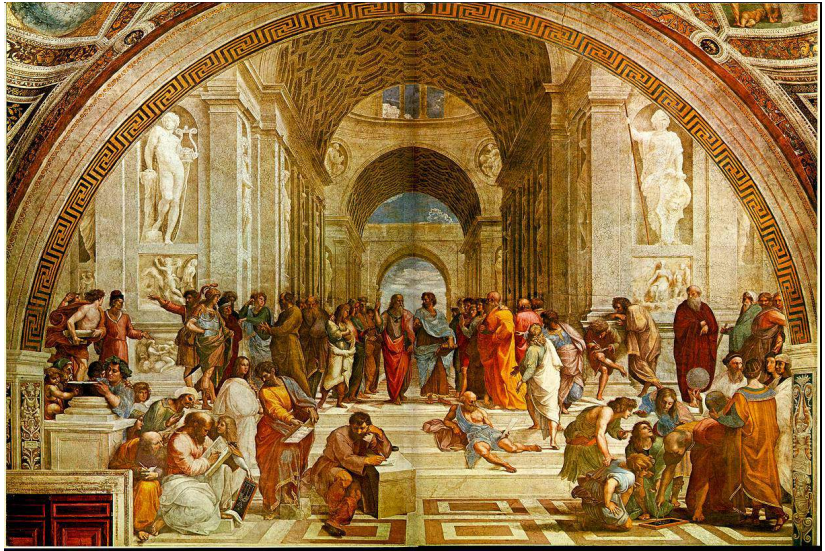
## THE TETRACTYS

Or high Mathesis, with her charm severe,  
Of line and number, was our theme; and we  
Sought to behold her unborn progeny,  
And thrones reserved in Truth's celestial sphere:  
While views, before attained, became more clear;  
And how the One of Time, of Space the Three,  
Might, in the Chain of Symbol, girdled be:  
And when my eager and reverted ear  
Caught some faint echoes of an ancient strain,  
Some shadowy outlines of old thoughts sublime,  
Gently he smiled to see, revived again,  
In later age, and occidental clime,  
A dimly traced Pythagorean lore,  
A westward floating, mystic dream of FOUR.

## Antonio Gaudí (1853–1926) & 四性道



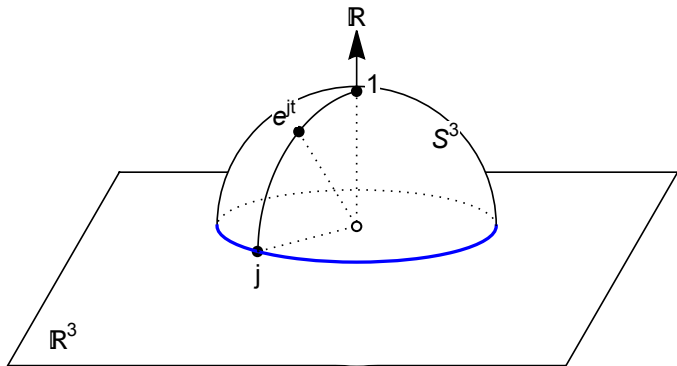
## Scuola di Atene, Raffaello Sanzio (1483–1520)



$$\mathbb{S}^3 = \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$$

The equator:

$$\begin{aligned}\mathbb{S}^2 &= \{x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : x_1^2 + x_2^2 + x_3^2 = 1\} \\ &= \{q \in \mathbb{H} : q^2 = -1\}\end{aligned}$$



For  $-1 \leq t \leq 1$ , let

$$\mathbb{S}_t^2 := \{(t, x, y, z) \mid t^2 + x^2 + y^2 + z^2 = 1\}.$$

Then the radius of  $\mathbb{S}_t^2$  is  $\sqrt{1 - t^2}$  and

$$\mathbb{S}^3 = \bigcup_{t \in [-1, 1]} \mathbb{S}_t^2$$

## The (pure imaginary) space

Consider the (pure imaginary) space

$$\mathbb{R}^3 := \text{Im}(\mathbb{H}) = \{x\mathbf{i} + y\mathbf{j} + z\mathbf{k} : x, y, z \in \mathbb{R}\}$$

in the quaternions  $\mathbb{H}$ .

It is the tangent space of  $\mathbb{S}^3$  at the identity element.

---

For any  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$  and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ , we have

$$\mathbf{u}\mathbf{v} = -\langle \mathbf{u}, \mathbf{v} \rangle + \mathbf{u} \times \mathbf{v}.$$

In particular, if  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular (pure imaginary) vectors, then  $\mathbf{u}\mathbf{v} = \mathbf{u} \times \mathbf{v}$ .

## Adjoint Action

For  $q \in \mathbb{S}^3$ , we have the **adjoint action**

$$\text{Ad}_q : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \mathbf{r} \mapsto \text{Ad}_q(\mathbf{r}) = q\mathbf{r}q^{-1}.$$

It is an orthogonal linear transformation. If

$$q = e^{t\mathbf{u}} = \cos t + \mathbf{u} \sin t$$

for a unit vector  $\mathbf{u} \in \mathbb{S}^2$  and  $t \in \mathbb{R}$ , then

$$q\mathbf{u}q^{-1} = e^{t\mathbf{u}}\mathbf{u}e^{-t\mathbf{u}} = \mathbf{u}.$$

Thus  $\text{Ad}_{e^{t\mathbf{u}}}$  is a rotation operator along the axis  $\mathbf{u}$ .

**Exercise:** Show that  $e^{\pi\mathbf{u}} + 1 = 0$  for any  $\mathbf{u} \in \mathbb{S}^2$ .

$$\mathbb{S}^3 = \text{Spin}(3)$$

Let  $\mathbf{v}$  be any unit vector orthogonal to  $\mathbf{u}$ , and let  $\mathbf{w} := \mathbf{u} \times \mathbf{v} = \mathbf{u}\mathbf{v}$ .  
Then

$$q\mathbf{v}q^{-1} = \mathbf{v} \cos(2t) + \mathbf{w} \sin(2t), \quad q\mathbf{w}q^{-1} = -\mathbf{v} \sin(2t) + \mathbf{w} \cos(2t).$$

Thus  $\text{Ad}_{e^{t\mathbf{u}}} = R_{\mathbf{u}}(2t) \in \text{SO}(3)$ , the rotation along the axis  $\mathbf{u}$  with the angle  $2t$ .

A point  $\mathbf{u}$  on the equator of  $\mathbb{S}^3$  represents the **half-turn** operator

$$\text{Ad}_{\mathbf{u}} : \mathbb{S}^2 \rightarrow \mathbb{S}^2.$$

The circle  $e^{t\mathbf{u}}$ ,  $0 \leq t \leq 2\pi$ , in  $\mathbb{S}^3$  corresponds to the double circle  $R_{\mathbf{u}}(2t)$  in  $\text{SO}(3)$ .

We have the double cover

$$\mathbb{S}^3 \rightarrow \text{SO}(3)$$

$\mathbb{S}^n$  is a Lie group if and only if  $n = 0, 1, 3$ .

$$\mathbb{S}^3 \simeq \mathrm{SU}(2)$$

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**Poincaré Conjecture (1904):**

*A compact simply connected 3-manifold is  $\mathbb{S}^3$ .*

W. Thurston, and G. Perelman

# Appendix

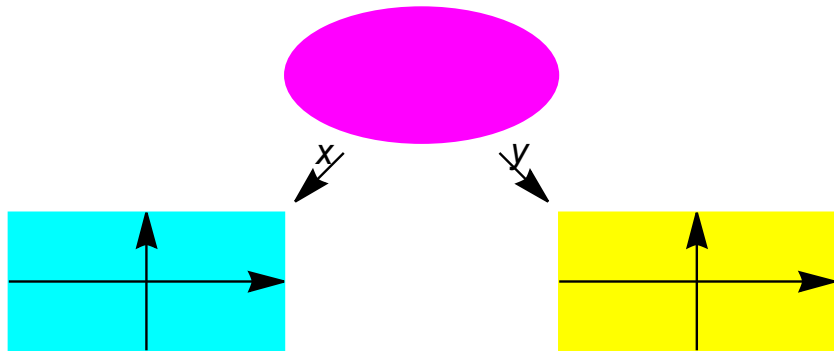
## Affine Space

# Spaces

A **Cartesian space** is

$$\mathbb{R}^n$$

for some  $n$ , or a (finite dimensional vector) space together with a *coordinate system*.



# Vector Space

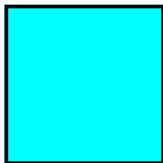
A **vector space** is an *abelian group*  $(V, +)$  together with an action of **scalars**.

$$\mathbb{F} \times V \rightarrow V$$

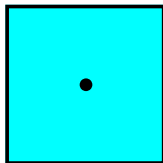
**Remark:** The collection of scalars is called a **field**, which was originated from the German word “Körper” (體, 몸, 근본).

# Affine Space

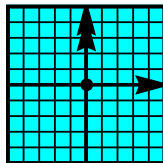
An **affine space** is a vector space without the origin.



affine space



vector space



Cartesian space

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**An example:** For a linear map  $A : X \rightarrow Y$  between vector spaces, the solution space of the equation

$$A\mathbf{x} = \mathbf{y}$$

is an affine space for any  $\mathbf{y} \in Y$ .

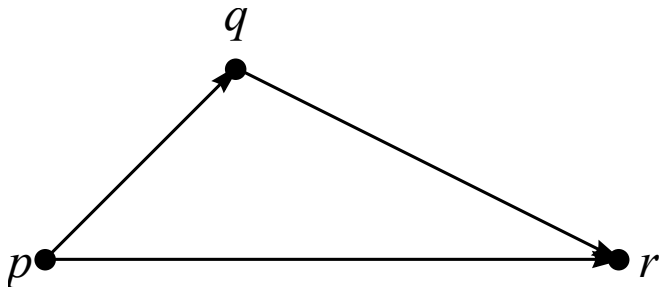
## Affine space

An **affine space** is a pair  $(\mathbb{A}, \vec{\mathbb{A}})$ , where

$\mathbb{A}$  is a collection of points, and

$\vec{\mathbb{A}}$  is a collection of vectors,

satisfying some properties (as stated in the next page).



# Properties

(i) The difference of two points is a vector:

for any pair  $(p, q)$  of points, there corresponds a vector  $\vec{pq}$ .

In other words, a map

$$\text{vec} : \mathbb{A} \times \mathbb{A} \rightarrow \vec{\mathbb{A}}, \quad \text{vec}(p, q) = \vec{pq}$$

is given. (One may write  $\vec{pq} = q - p$  or  $q = p + \vec{pq}$ .)

(Points are of mass 1 and vectors are of mass 0.)

(ii) Once we fix an origin, then the space becomes the vector space. In other words, for each  $p$  in  $\mathbb{A}$ , we have the bijection:

$$\text{vec}_p : \mathbb{A} \rightarrow \vec{\mathbb{A}}, \quad q \mapsto \vec{pq}$$

(iii) for any points  $p, q, r$ ,

$$\vec{pq} + \vec{qr} = \vec{pr}.$$

**Remark:** If  $c(t)$  is a curve in  $\mathbb{A}$ , then  $c'(t)$  and  $c''(t)$  are in  $\vec{\mathbb{A}}$ .

## Examples

- A **Euclidean space** is an affine space with an inner product.
- Any vector space is an affine space.
- The Cartesian space  $\mathbb{R}^n$  is (a vector space and hence is) an affine space.
- $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 1\}$
- for any linear map  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , the solution space

$$\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{y}\}$$

is an affine space for any  $\mathbf{y} \in \mathbb{R}^m$ .

- $\mathbb{A} \simeq \vec{\mathbb{A}} \times \{1\} \subset \bigcup_{t \in \mathbb{R}} \vec{\mathbb{A}} \times \{t\} = \vec{\mathbb{A}} \times \mathbb{R}$