Opportunities for SQG

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APDE Seminar at Berkeley

The generalized SQG equations

α-SQG equations

Consider the "active scalar" equation for ω:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = \nabla^{\perp} (-\Delta)^{-1+\frac{\alpha}{2}} \omega, \\ \omega(t=0) = \omega_0 \end{cases} \tag{α-SQG}$$

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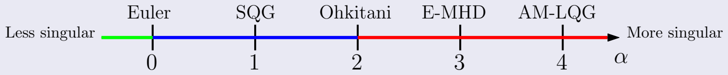
- Domain: $\mathbb{R}^2$ or $\mathbb{R}^2_+$.
- Strongest conservation law: $\|\omega\|_{L^\infty} = \|\omega_0\|_{L^\infty}$.
- Scaling symmetry $\omega(x) \mapsto \lambda^{-\alpha}\omega(\lambda x)$ leaves the $\dot{C}^\alpha$ invariant, which is supercritical if and only if $\alpha > 0$.

gSQG equations



- 2D Euler: u is one order regular than ω.
- Criticality of SQG: $u = R^{\perp}\omega \sim \omega$.
- Local wellposedness for smooth data when $\alpha \le 2$.

- $\alpha \le 0$: global wellposedness
- $\alpha > 2$: strong illposedness in any high Sobolev (Chae–J.–Oh)

Purpose of this talk

Global regularity vs. Finite-time singularity problem

For a given $\omega_0 \in C_c^\infty$, is the corresponding local in time C_c^∞ solution to the α-SQG equation global?

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Our goal

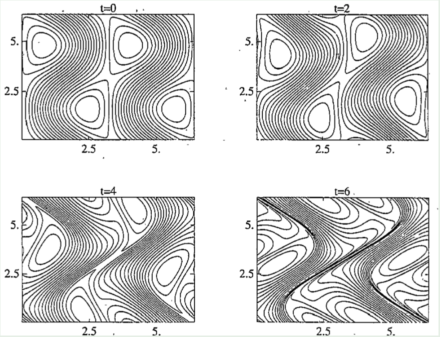
Review classical results and various attempts for finite time singularity formation, and discuss remaining opportunities.

Organization

1. Remarks on smooth solutions
2. Critical regularity problem
3. Patch regularity problem
4. Half-plane regularity problem

Constantin–Majda–Tabak ('94)

Sharp gradient formation for SQG ($\alpha = 1$) is numerically observed by **closing of the hyperbolic saddle**.



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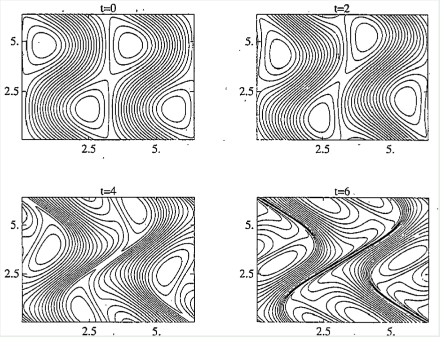

- Cordoba ('98): double exponential upper bound
- Open problem: lower bound on gradient growth

Preliminary Remarks

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The situation is different for limited regularity solutions...

Preliminary Remarks

Recall: α-SQG is C^α critical

- Blow-up criteria for smooth solutions
- Uniqueness threshold $L^1([0,T]; C^\alpha)$
- Nonexistence in $L^\infty([0,T]; C^\alpha)$ (J.–Kim '24)

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Anisotropic feature in $D_t(\nabla^\perp\omega) = \nabla u(\nabla^\perp\omega)$

Concretely, $D_t(\partial_1\omega) = \partial_2 u_2 \partial_1\omega - \partial_1 u_2 \partial_2\omega$.

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Concretely, $D_t(\partial_1\omega) = \partial_2 u_2 \partial_1\omega - \partial_1 u_2 \partial_2\omega$. A consequence is the directional blow-up criteria (cf. Yamazaki): for $0 < \alpha \le 1$,

$$\int_0^T \|\partial_1\omega(t)\|_{L^\infty}\,dt \qquad \text{or} \qquad \int_0^T \|\partial_2\omega(t)\|_{L^\infty}\,dt$$

controls singularity formation.

Critical dynamics

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$$\omega(t,x) = |x|^{\alpha} g(t, x/|x|)$$

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Issues

- Local wellposedness: OK under a *symmetry* assumption on g
- Infinite energy criticism: propagation of local structure by cutoff. Consequence: 1D PDE blowup gives blowup for compactly supported C^{α} solutions to α-SQG.

What is this 1D PDE?

Scale-invariant 1D dynamics (Elgindi–J. '21)

The scale-invariant profile g satisfies

$$\partial_t g + 2G\partial_\theta g = \alpha \partial_\theta G g, \tag{1}$$

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Bad news: long time existence (Castro–Cordoba–Zhang '21)

In the SQG case, there is long time existence for smooth data for (1): data of size ϵ in $H^{16}(\mathbb{T})$ can survive at least for $T = O(\epsilon^{-4})$. Furthermore, there exist nontrivial global rotating solutions to (1).

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When $\alpha = 1$ (SQG), $\partial_\theta G \approx H[g]$.

Bad news: regularity of de Gregorio

Generalized de Gregorio equations (Okamoto–Sakajo–Wunsch '08)

On $\mathbb{T}$, consider the equation ($a \in \mathbb{R}$ is a parameter)

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Chen '21 obtained similar results for $a > 1$, for certain initial data. Singularity formation for $a < 1$ is known (Constantin–Lax–Majda '85, Castro–Cordoba–Fontelos '05, Elgindi–J. '20, Chen '21, ...).

Slightly subcritical dynamics

C^α blowup (Elgindi '21, Cordoba–Martinez-Zoroa–Zheng)

Self-similar singularity formation for 3D axisymmetric no swirl Euler with C^α vorticity, $\omega \approx |x|^\alpha F(|\theta|^\alpha)$. Key observation:

$$\partial_t \omega + u \cdot \nabla \omega \approx \omega R_{12} \omega.$$

Taking $\alpha \to 0$ increases the relative strength of RHS. The model $\partial_t \omega = \omega R_{12} \omega$ admits self-similar singularity.

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Bad news for SQG

While we still have a similar structure for $\nabla^\perp \omega$, the corresponding model is $\partial_t(\nabla^\perp \omega) \approx \nabla u(\nabla^\perp \omega)$ and now there is a parity difference from the above growth scenario.

Patch problem

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Local regularity for patches in $\partial\Omega \in H^s$ with s large

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Question

Global regularity for smooth Euler patch is known. Is there finite-time singularity formation for α-SQG patches with $\alpha > 0$?

Remarks on the patch problem

Serious issues for α-SQG patch solutions

- Velocity is singular (BMO for $\alpha = 1$)
- Issue of defining the CDE (Uniqueness, especially when $\alpha > 1$)
- Convergence from smooth solutions?
- Failure of regularity propagation for piecewise smooth data

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Remark

None of the above issues exist for Euler patches.

Numerical simulations for SQG patches

- Cordoba–Fontelos–Mancho–Rodrigo '05, Mancho '05, Scott–Dritschel '14, '19
- Reports self-similar curvature growth up to order 10^{10}

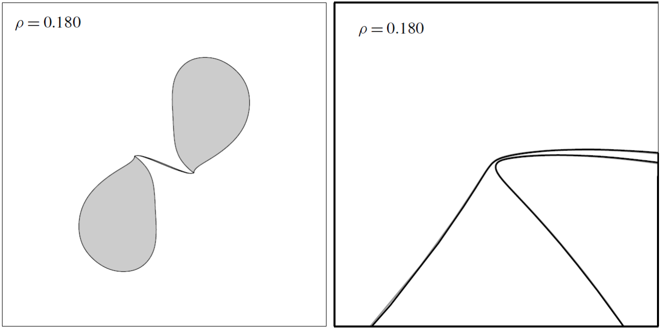
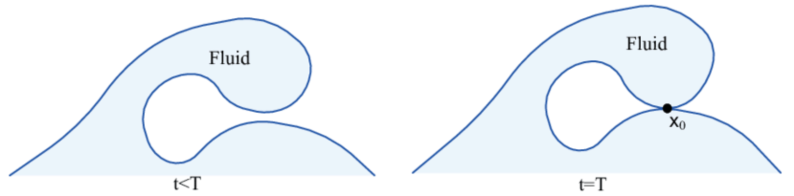


Figure: Scott–Dritschel '19

Bad news: Regularity criterion for patches

No Splash Theorem (Gancedo–Strain, Kiselev–Luo, Jeon–Zlatos)

No splash singularities exist for smooth α-SQG patches for $0 < \alpha < 2$. Actually, the $C^{1,\alpha/(1-\alpha)}$ norm of the boundary **must** blow up for a patch singularity to occur, for $0 < \alpha \leq 1/2$.



Comparison with Euler patches

- Chemin '93: Smooth patches for Euler are global
- Baker '13: Curvature growth up to order 10^3
- Kiselev–Luo '23: C^2 illposedness for Euler patch $\partial_t \kappa \approx H[\kappa]$

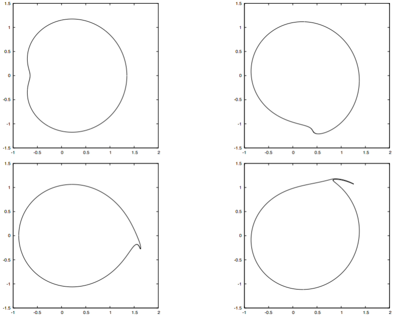


Figure: Baker '13

Theorem (Denisov '15)

Bahouri–Chemin patch $\bar{\omega} = \mathrm{sgn}(x_1)\mathrm{sgn}(x_2)$ is formally a singular steady solution to 2D Euler in $\mathbb{R}^2$.

- With the help of an external strain, there is an infinite smooth Euler patch converging to $\bar{\omega}$ in infinite time.
- There is a continuous family of "steady" patches to 2D Euler which converges to $\bar{\omega}$.

Infinite time corner formation for Euler patch

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A potential blow-up proof for α-SQG

Repeat this for the α-SQG case.

The front problem for α-SQG

A **front** is a patch solution $\omega = \chi_{\Omega(t)}$ of the form where $\Omega(t) = \{(x, y) \in \mathbb{R}^2 : y < h(t, x)\}$ for a height function h. The case $h \equiv 0$ corresponds to the half-plane steady state.

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Global wellposedness and stability for small and localized perturbations for the half-plane state for $1 < \alpha < 2$. This is thanks to a linear dispersive effect of the half-plane state.

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Case of the perturbed circular patch?

More global solutions for α-SQG

- Castro–Cordoba–Gomez-Serrano '16, Gomez-Serrano '19: Analytic steady patches
- Castro–Cordoba–Gomez-Serrano '21: smooth rotating SQG solutions
- Cao–Qin–Zhan–Zou '22: smooth rotating α-SQG
- Hmidi–Xue–Xue, Cao–Lai–Qin, ... : relative patch equilibria
- Gomez-Serrano–Ionescu–Park: **Quasiperiodic patches**

Patch problem on the half-plane

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How about the case of α-SQG **patches**?

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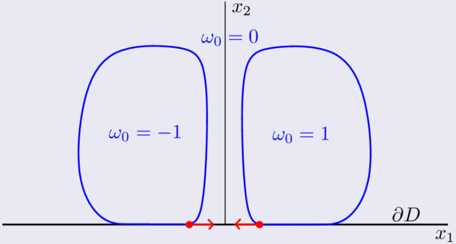
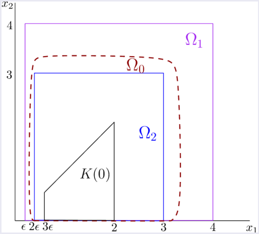
Theorem (Kiselev–Ryzhik–Yao–Zlatos '16, Gancedo–Patel '21)

There is local wellposedness and finite time singularity formation for α-SQG patches in $\mathbb{R}^2_+$ for $0 < \alpha < 1/3$. On the other hand, there is global regularity for such patches in the Euler case.





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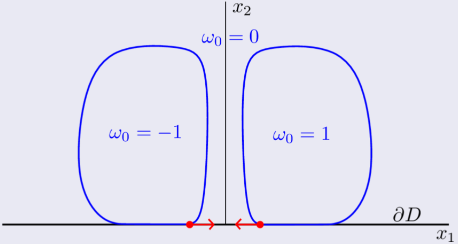

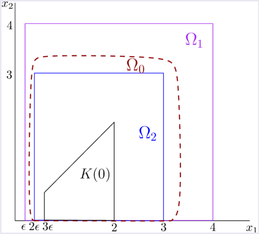

Not known what actually happens at the time of singularity.

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Theorem (Zlatos, J.–Kim–Yao)

Consider the α-SQG on $\mathbb{R}^2_+$ with $1/2 < \alpha \leq 1$. Then, for **any** C^α initial data **not** identically vanishing on $\partial\mathbb{R}^2_+$, there is **NO** $L^\infty([0,\delta]; C^\alpha(\mathbb{R}^2_+))$ solution to α-SQG for any $\delta > 0$.

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This settles illposedness since C^α is the LWP critical class. Still, it raises several questions:

- Potential existence of C^β solution with $\beta < \alpha$
- Wellposedness theory with DBC data.

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Theorem (Zlatos, J.–Kim–Yao)

Consider the α-SQG on $\mathbb{R}^2_+$ with $0 < \alpha \le 1/2$. Then, there is local wellposedness in the following anisotropic class $X^\alpha(\mathbb{R}^2_+) \subset C^\alpha(\mathbb{R}^2_+)$

$$\|\omega\|_{X^\alpha} := \|\omega\|_{C^\alpha} + \|\partial_1\omega\|_{L^\infty} + \|x_2^{1-\alpha}\partial_2\omega\|_{L^\infty}$$

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Theorem (Zlatos)

For all $0 < \alpha \le 1/2$, there is finite time singularity in the class X^α.

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For SQG ($\alpha = 1$), there is local wellposedness in $W^{2,p} \cap H^1_0(\mathbb{R}^2_+)$ with any $2 < p < \infty$.

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Theorem (J.–Kim–Miura)

Consider SQG with Dirichlet boundary conditions on $\mathbb{R}^2_+$. Then:

- Higher regularity: local wellposedness in $W^{3,p} \cap H^1_0$ for any $1 < p < \infty$. There is illposedness from C^∞_0 to C^3.
- Low regularity: local wellposedness in an anisotropic subspace of $C^{0,1}$, similarly defined as the X^α-space.

Smooth solutions on the half-plane

The previous result, which can be easily generalized to all α-SQG with $0 < \alpha < 2$, provides natural classes of functions for which one can still search for finite time singularities. The low regularity result is particularly interesting since the SQG equation is strongly illposed in $\mathbb{R}^2$ in $C^{0,1}$ and C^1 (Cordoba–Martinez-Zoroa '22, J.–Kim '24). Illposedness occurs even for data in $C_c^1 \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$. That is, presence of the boundary and tangential regularity "stabilizes" the lack of normal regularity.

Final remarks

Many attempts towards singularity formation for α-SQG equations have failed. This can be attributed to:

- Anisotropic regularity propagation
- Cancellation structure in nonlinearity
- Dispersive regularizing mechanisms

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Thank you for listening!