Degenerate dispersion and illposedness for generalized SQG equations with singular velocities

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Part I: Introduction to degenerate dispersive PDE

Ref: *Illposedness for dispersive equations: Degenerate dispersion and Takeuchi–Mizohata condition*, arXiv:2402.06278 (with Sung-Jin Oh)

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- That is, for $\operatorname{supp}(a(t,\cdot)) \sim X(t)$, we have $\dot{X}(t) = \partial_\xi p$.

Linear dispersive PDE: variable coefficient case

Consider variable coefficient

$$i\partial_t\phi + p(x, i\partial_x)\phi = 0$$

and take $\phi \simeq a(t,x)e^{i\Xi(t)x+ipt}$ with $|\Xi(t)| \gg 1$ time-dependent.

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- This time, Taylor expansion reads

$$e^{-i\Xi x} \circ p(X, i\partial_x) \circ e^{i\Xi x} \simeq p(X,\Xi) + \partial_x p(X,\Xi)x + \partial_\xi p(X,\Xi)i\partial_x$$

  and the new term gives the *system*

$$\begin{cases} -\Xi'(t) + \partial_x p(X,\Xi) = 0, \\ X'(t) + \partial_\xi p(X,\Xi) = 0. \end{cases} \qquad \text{(Hamiltonian ODE)}$$

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Degenerate case: assume that p vanishes at $x = 0$, with

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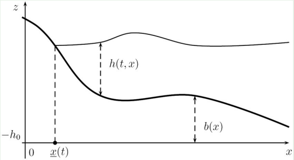
Oscillatory integrals and pseudo-differential calculus: tools for justifying *wave packet heuristics* for actual solutions of in

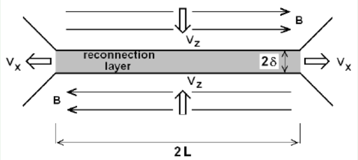
$$i\partial_t \phi + p(x, i\partial_x)\phi = 0.$$

Degenerate dispersion: examples from physics

Severe instability by waves approaching the point of degeneracy

- Water wave models with vanishing height, e.g. shoreline
- MHD models with vanishing magnetic field, e.g. Sweet–Parker model for magnetic reconnection
- Resonant interaction of waves in compressible Euler system (Hunter–Smothers model)
- Surface growth, sedimentation theory, magma dynamics, ...



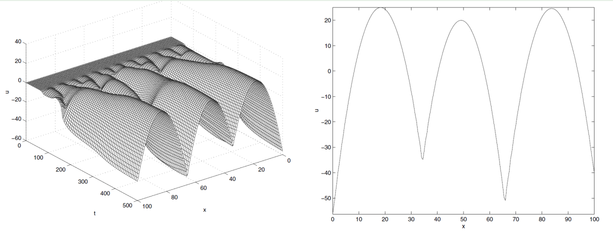


Surface growth model

- Stein–Winkler '05 Ozanski '19 Ozanski–Robinson '18 Burczak–Ozanski–Seregin '21

$$\partial_t u + \partial_{xx}(\partial_x u)^2 = -\nu \partial_{xxxx} u - \kappa \partial_{xx} u$$

- Generic behavior at local extrema: cx^2
- Degenerate dispersion near local minimum ($c > 0$): $\partial_{xx}(\partial_x u)^2 \sim x \partial_{xxx} u$ generates highly unstable behavior



Part II: Illposedness for singular gSQG equations

Refs: *Illposedness via degenerate dispersion for generalized surface quasi-geostrophic equations with singular velocities*, arXiv:2308.02120 (with Dongho Chae and Sung-Jin Oh), 74 pp.

On the Cauchy problem for the Hall and electron magnetohydrodynamic equations without resistivity I: Illposedness near degenerate stationary solutions, (with Sung-Jin Oh) Ann. PDE 8 (2022), no. 2, 106 pp.

Generalized SQG equations

We consider 2D active scalar equations of the following form:

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^{\perp} m(\Lambda)\theta. \end{cases} \tag{gSQG}$$

where $m(\Lambda)$ is a Fourier multiplier, $\Lambda = (-\Delta)^{1/2}$.

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- We take $\Omega = \mathbb{R}^2, \mathbb{R} \times \mathbb{T}, \mathbb{T}^2$ and $\theta : \Omega \to \mathbb{R}$, $u : \Omega \to \mathbb{R}^2$.
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Question: given m, local wellposedness for smooth data?

Assumptions on $m(\Lambda)$

- We take m to be positive, smooth on $\mathbb{R}^2 \backslash \{0\}$ and **elliptic**

$$|\nabla_\xi^n m(|\xi|)| \lesssim_n \langle \xi \rangle^{-n} m(|\xi|).$$

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Proposition (cf. CCCGW '11)

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Theorem (Chae–J.–Oh, preprint)

For **singular** $m(\Lambda)$*, namely if* $m(|\xi|) \to \infty$ *as* $|\xi| \to \infty$*, and if* $|\nabla m|$ *is also elliptic, then* (gSQG) *is* **strongly illposed** *in* $H^s(\Omega)$ *with all* s *sufficiently large.*

Remarks

- For α-SQG equations, $u = \nabla^{\perp}(-\Delta)^{-1+\frac{\alpha}{2}}\theta$, our result gives local wellposedness up to $\alpha \leq 2$ and illposedness for $\alpha > 2$.
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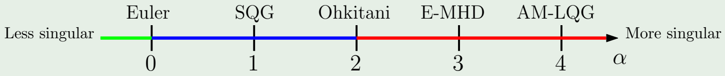
- There are at least three examples of singular multipliers of particular interest:
  - Ohkitani model: $m = \log(\Lambda)$
  - Toy model for electron-MHD: $m = \Lambda$
  - Asymptotic model in LQG: $m = \Delta$.



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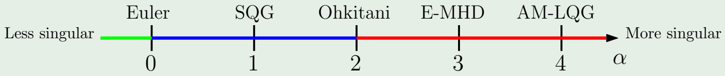

- Interpretation of illposedness (work in progress):
  - Enhanced/Anomalous dissipation
  - Geometry of initial data matters for wellposedness

Trichotomy depending on singularity of multiplier

1. If $m(|\xi|) \gtrsim |\xi|$: illposedness from H^∞ to H^{s_0}
2. If $m(|\xi|) \sim |\xi|^\beta$ for some $0 < \beta < 1$: illposedness from $H^{\gamma s_0}$ to H^{s_0} for some $\gamma > 1$ depending on β.
3. If $1 \ll m(|\xi|) \lesssim \log(|\xi|)$: illposedness from H^{s_0} to itself, and **wellposedness** from $H^{s_0+\epsilon}$ to H^{s_0} for any $\epsilon > 0$.

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Open problem: illposedness in Gevrey/analytic regularity.

Basic properties of gSQG

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- When $m = (-\Delta)^{-1+\frac{\alpha}{2}}$, $\|\theta(t)\|_{\dot{H}^{\frac{\alpha}{2}-1}}$ is conserved: stronger if velocity is more singular.

Theorem (Norm Inflation)

For any $\epsilon, \delta > 0$ *and* $s \geq s_0$, *there is a data* $\theta_0 \in C^{\infty}_{comp}(\Omega)$ *with* $\|\theta_0\|_{H^s} < \epsilon$ *for which any corresponding solution* θ *satisfies*

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Steps

1. **Quantitative illposedness for linearization around shear**
2. Norm inflation for nonlinear equation
3. Patching argument: norm inflation to nonexistence

Linear illposedness: overview of the proof

- Consider solution of the form $\theta = \bar{\theta} + \tilde{\theta}(t, x, y)$.
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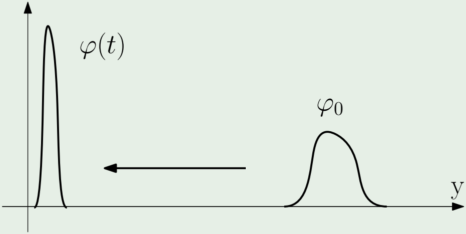

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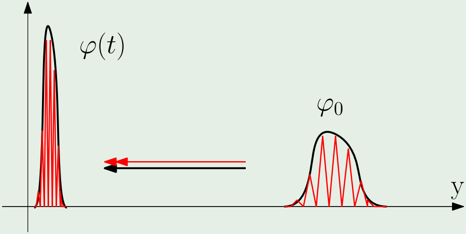

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  namely $[\partial_t - \mathcal{L}^*]\tilde{\varphi} = \epsilon \ll_{L^2} 1$.
- Duality testing:

$$\left| \frac{d}{dt} \langle \varphi, \tilde{\varphi} \rangle \right| = |\langle \varphi, \epsilon \rangle| \ll \|\varphi_0\|_{L^2}^2.$$

  Then, for t small,

$$\|\varphi(t)\|_{L^p} \|\tilde{\varphi}(t)\|_{L^{p^*}} \geq \langle \varphi, \tilde{\varphi} \rangle(t) \geq \frac{1}{2} \|\varphi_0\|_{L^2}^2$$

  which gives L^p illposedness of $\partial_t + \mathcal{L}$ upon decay of $\|\tilde{\varphi}(t)\|_{L^{p^*}}$.

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cf. This is wellposed in H^∞ for nondegenerate case: $\inf |F'| > 0$.

General question: how to construct approximate solutions to

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where p is a pseudo-differential op with

- p is self-adjoint $\implies$ L^2-boundedness
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- dispersive: $\partial_\Lambda p \to \infty$ as $\Lambda \to \infty$.

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- dispersive: $\partial_\Lambda p \to \infty$ as $\Lambda \to \infty$.

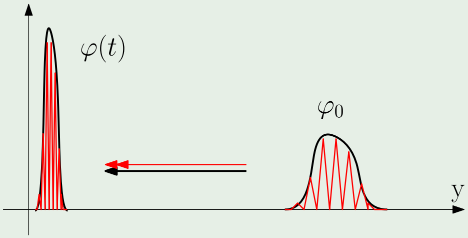
The guideline: *bicharacteristic ODE*

$$\begin{cases} \dot{Y}(t) = -\partial_\Lambda p(Y,\Xi), \\ \dot{\Xi}(t) = \partial_y p(Y,\Xi) \end{cases}$$

and Gaussian wave packet $\tilde{\varphi} = \exp(i\Xi(t)y - |y - Y(t)|^2)$.

Problem of coherence

Recalling that the mechanism of illposedness is accumulation of support near the degeneracy, we see that the naive Gaussian wave packet cannot be a good approximation during the time interval in which frequency growth actually occurs.



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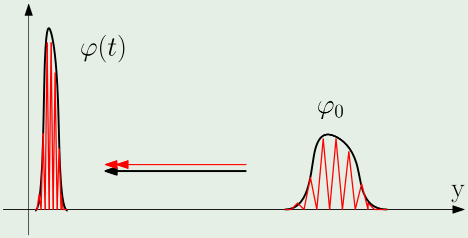

That is, we want $\tilde{\varphi}$ such that the associated error remains small in L^2 in $[0, T]$ with $|\Xi(T)| \gg |\Xi(0)|$.

If $p \sim m(|\Xi|)|Y|^n$, it implies that $|Y(t)| \ll |Y(0)|$.

We consider a general ansatz

$$\tilde{\varphi} = \exp(i\Phi(t, y))A(t, y)$$

and as a general rule, the phase function Φ should satisfy

$$\partial_t \Phi + p(y, \partial_y \Phi) = 0. \qquad \text{(Hamilton–Jacobi)}$$

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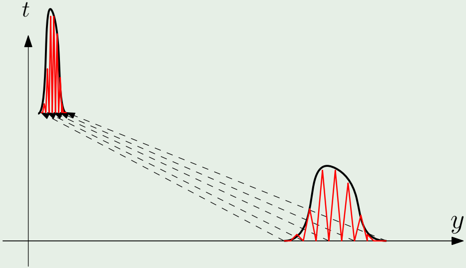
We want: $|\partial_y \Phi| \simeq \Xi(t) \gg 1$ on the support of A, while derivatives of A and *additional* derivatives of $\partial_y \Phi$ cost much less than $\Xi(t)$. However, such an assumption does not propagate in time, in general. (Technical keyword: "scale separation")

When we solve the HJ equation $\partial_t \Phi + p(y, \partial_y \Phi) = 0$, the key idea is to **choose** a "curved" initial data Φ_0 such that the resulting characteristic curves are relatively straight and parallel with each other: **maximal cancellations** in the $\partial_{yy}\Phi$–estimates



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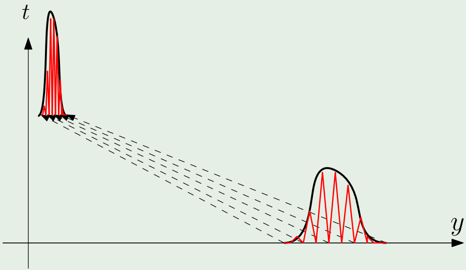

End result: $|\partial_y \Phi(t,\cdot)| \simeq \Xi(t)$ on the support of $A(t)$, while $|\partial_{yy}\Phi(t,\cdot)| \ll \Xi^2(t)$. Gives improved error estimates.

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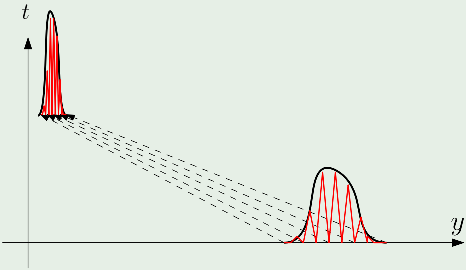

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Rmk: this is "renormalization" in the differential case.

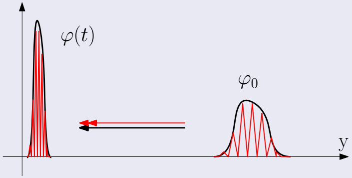
Theorem (Main linear illposedness statement)

For $\lambda_0 \gg 1$, $M \geq 1$, let $0 < \tau_M \ll 1$ be the frequency growth time from λ_0 to $M\lambda_0$ determined by bicharicteristic ODE.
Then any $L^\infty_t L^2$ solution existing on $[0, 1.01\tau_M]$ to the linear equation with data of the form

$$\varphi_0 = e^{i\lambda_0 y} a_0(y)$$

experiences norm growth of the form

$$\sup_{t \in [0, 1.01\tau_M]} \|\varphi(t,\cdot)\|_{H^s} \gtrsim (M\lambda_0)^s \|\varphi_0\|_{L^2}.$$



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- Applications for various degenerate dispersive PDE, arising from water waves, sedimentation, plasma dynamics, ...
- Challenging further directions: **vector-valued case**, **multi-dimensional**, **curved spacetime**, ...

Summary

- We proved strong illposedness for **singular** gSQG, using wave packets for degenerate dispersive symbol. For physical systems, this indicates (1) strong dissipation, (2) necessity for finding geometric conditions for local wellposedness.
- In doing so, we developed a general framework for coherent wave packet construction, with potential applications to various degenerate dispersive PDE.

Thank you very much for your attention!