Incompressible Euler equations at critical regularity

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Two-dimensional incompressible fluid

2D Euler in vorticity form

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = \nabla^\perp \Delta^{-1} \omega, \\ \omega(t=0) = \omega_0 \end{cases} \tag{Euler}$$

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- Vorticity ω, Velocity u. Domain $\mathbb{R}^2$ or $\mathbb{T}^2$.
- Strongest conservation law: $\|\omega\|_{L^\infty} = \|\omega_0\|_{L^\infty}$.
- Classical ('30s): global wellposedness for C_c^∞
- Yudovich ('63): wellposedness in L^∞.

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Incompressible Euler is a transport equation, regularity of the solution expected to be preserved in time. Most typical choice: $Y = L^\infty([0,T];X)$ or $C([0,T];X)$.

Sobolev wellposedness and critical regularity

Theorem (Ebin–Marsden, Kato–Ponce, Kato '70s)

The two-dimensional incompressible Euler equation is globally wellposed with $X = W^{s,p}(\mathbb{R}^2)$ if $sp > 2$.

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Scaling criticality: $\omega(t,x) \mapsto \omega(t,\lambda x)$ leaves $\dot{W}^{2/p,p}$ invariant.

Why do we care?

Motivations

- Theoretical desire: wellposedness in the largest possible class
- Coincide with strongest conservation law
- Some physical situations (e.g. logarithmic spirals)
- Slightly sub/super-critical dynamics

Understanding criticality

The H^1 estimate for ω:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 = -\int \nabla u : \nabla\omega\nabla\omega \leq \|\nabla u\|_{L^\infty}\|\nabla\omega\|_{L^2}^2$$

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If we want $\omega_0 \in H^1$ with $\omega(t,\cdot) \notin H^1$, then

- Pick initial data for which the RHS is divergent,
- Propagate a lower bound for the RHS for some time.

Failure of critical $\|\nabla u\|_{L^\infty}$ bound

Note $\nabla u = \nabla\nabla^\perp\Delta^{-1}\omega$.

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$$\Delta(fg) = \Delta(f)g + f\Delta(g) + 2\nabla(f)\cdot\nabla(g).$$

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Generalizing, for $\alpha < 1/2$,

$$\Delta(x_1x_2|\ln|x||^\alpha) = O(|\ln|x||^{\alpha-1}) \in L^\infty\cap H^1.$$

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On the other hand, for $\alpha > 0$

$$\nabla u = \nabla\nabla^\perp(x_1x_2|\ln|x||^\alpha) = O(|\ln|x||^\alpha) \notin L^\infty.$$

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Understanding criticality

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- Log-Lipschitz continuity

$$\sup_{|x-x'|<1/4} \frac{|u(x)-u(x')|}{|x-x'||\ln|x-x'||} \le C\|\omega\|_{L^\infty}$$

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- Log-inequality

$$\|\nabla u\|_{L^\infty} \le C\|\omega\|_{L^\infty} \ln(10+\|\omega\|_X), \quad X = \text{subcritical}$$

  Potential expexp growth (Denisov '09, Kiselev–Sverak '14,...)

Theorem (Bourgain–Li '15 '19, Elgindi–J. '17, J.–Kim '24)

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Precise illposedness statements in 2D:

- (norm inflation) for any $\epsilon, \delta > 0$, there exists $\omega_0 \in C_c^\infty$ s.t.

$$\|\omega_0\|_{W^{2/p,p}} < \epsilon, \qquad \sup_{t\in(0,\delta)} \|\omega(t)\|_{W^{2/p,p}} > \frac{1}{\epsilon}.$$

- (nonexistence) there exists $\omega_0 \in W^{2/p,p}_{comp} \cap L^\infty$ such that the Yudovich solution escapes $W^{2/p,p}$ instantaneously, i.e.

$$\|\omega_0\|_{W^{2/p,p}} < \epsilon, \qquad \|\omega(t)\|_{W^{2/p,p}} = +\infty, \quad t \in (0, \delta]$$

Critical Besov case

Critical Besov: Vishik ('98, '99), Chae ('04), Pak–Park ('04, '13).

The n-dim'l Euler is locally wellposed in the critical Besov spaces $B^{n/p}_{p,1}$ for all $1 \leq p \leq \infty$.

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One of the key points is that

$$\omega \in B^{n/p}_{p,1} \implies \nabla u \in B^{n/p}_{p,1} \implies \nabla u \in L^\infty \implies \text{regularity}$$

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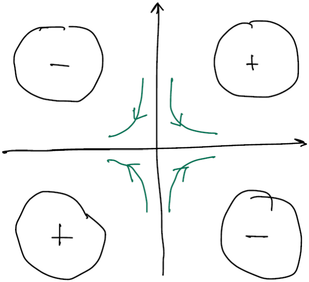
Hence, for illposedness, it is **essential** to take $\omega_0 \in W^{n/q,q} \setminus \cup_p B^{n/p}_{p,1}$.

Geometry of illposedness

- Pioneering works: Denisov ('09), Bourgain–Li ('13), Kiselev–Sverak ('14)
- Vorticity **odd** in both axes and non-negative on $(\mathbb{R}_+)^2$: generates hyperbolic flow near the origin



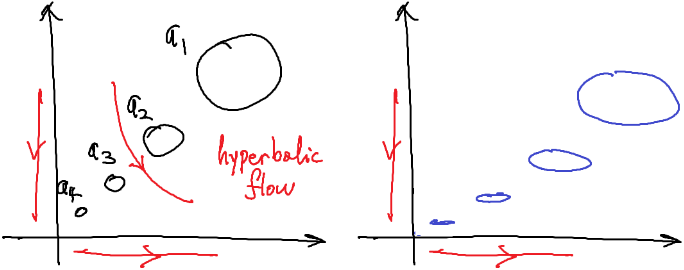
$$u(x) \simeq C \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

$$C = \frac{4}{\pi} \int_{(\mathbb{R}_+)^2} \frac{y_1 y_2}{|y|^4} \omega(y)\, dy$$

Geometry of illposedness

Data in the form of dyadic bubbles:

$$\omega_0(x) = \sum_{j=1}^{\infty} a_j \varphi(2^j x), \qquad \{a_j\}_{j\geq 1} \in \ell^\infty$$



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$$\|\omega_0\|_{L^\infty} \sim \| \{a_j\} \|_{\ell^\infty}, \quad \|\omega_0\|_{\dot{H}^1} \sim \| \{a_j\} \|_{\ell^2}, \quad \|\omega_0\|_{\dot{B}^1_{2,1}} \sim \| \{a_j\} \|_{\ell^1}$$

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This suggests taking $a_j = j^{-\alpha}$, $1 \geq \alpha > 1/2$. Essentially,

$$\omega_0(x) \sim |\ln|x||^{-\alpha} \frac{x_1 x_2}{|x|^2} \chi(|x| \leq 1).$$

Main issues

The goal:

$$\|\omega(t)\|_{H^1}^2 - \|\omega_0\|_{H^1}^2 \sim \int_0^t \int_{\mathbb{R}^2} \nabla u \nabla\omega\nabla\omega dxdt = \infty.$$

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Difficulty: understanding $\omega \mapsto \nabla u$, a **regularization** effect due to vortex thinning (cf. Elgindi–J. '23, Elgindi–Shikh Khalil '22):

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In the end: we need to take $\alpha = 1/2 + \epsilon$ ("barely" H^1).

Localization argument with Hardy inequality

- Decompose $\omega(t) = \sum_{j\geq 1} \omega^{(j)}(t)$ with

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- Orthogonality and Hardy inequality

$$\|\omega(t)\|_{\dot{H}^1}^2 \gtrsim \|x_2^{-1}\omega(t)\|_{L^2}^2 = \sum_{j\geq 1} \|x_2^{-1}\omega^{(j)}(t)\|_{L^2}^2$$

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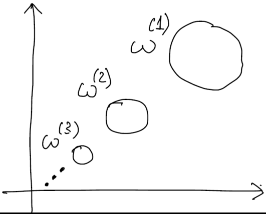
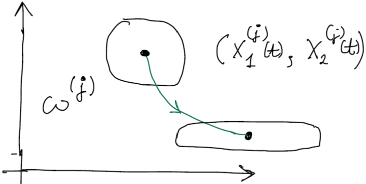
- Localization:

$$\|x_2^{-1}\omega^{(j)}(t)\|_{L^2} \sim \frac{1}{x_2^{(j)}(t)} \|\omega^{(j)}(t)\|_{L^2} = \frac{1}{x_2^{(j)}(t)} \|\omega_0^{(j)}\|_{L^2}$$





Localization argument with Hardy inequality

- Recall:

$$\|\omega(t)\|_{\dot{H}^1}^2 \gtrsim \sum_{j\geq 1} (x_2^{(j)}(t))^{-2} \|\omega_0^{(j)}\|_{L^2}^2.$$

- Dynamics of $x_2^{(j)}$:

$$\frac{d}{dt}(x_2^{(j)})^{-2} = -\frac{2u_2(t, x_2^{(j)})}{x_2^{(j)}}(x_2^{(j)})^{-2}$$

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- Application of Key Lemma:

$$\frac{d}{dt}\ln (x_2^{(j)})^{-2} \sim \sum_{1\leq k<j} \frac{1}{k^\alpha}(x_1^{(k)}(t))^{-4}$$

Localization argument with Hardy inequality

- Observation: *invariant timescale* (localized stability)

$$T^{(j)} \sim \frac{1}{\sum_{1 \le k < j} k^{-\alpha}}$$

and on $[0, T^{(j)}]$, jth bubble remains essentially unchanged.

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- Finally, check $\sum_{j\ge1} (x_2^{(j)}(t))^{-2} \|\omega_0^{(j)}\|_{L^2}^2 = \infty$ for $\alpha = 1/2 + \epsilon$.

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Different approaches:

- Cordoba–Martinez-Zoroa, Cordoba–Martinez-Zoroa–Ozanski
- Elgindi, Elgindi–Khalil

Many questions remain open: general critical data, subcritical data (Cordoba–Martinez-Zoroa–Ozanski), degeneration of Hölder regularity, ...

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Thank you for listening!