Logarithmic vortex spirals

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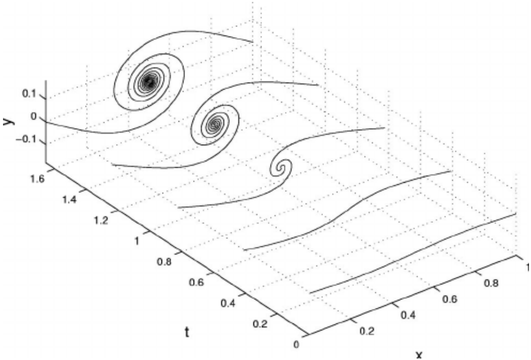
Analysis and computation of vortical flows
TOKYO ICIAM
August 21, 2023

Vortex spirals in incompressible flows



Vortex spirals

Figure: Kelvin–Helmholtz instability

Vortex spirals: waterspout

Figure: Florida, 1969

Vortex spirals past a corner

Vortex spirals past a corner

On the formation of vortex rings

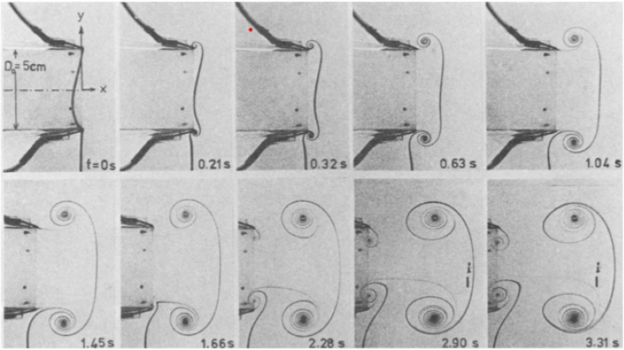
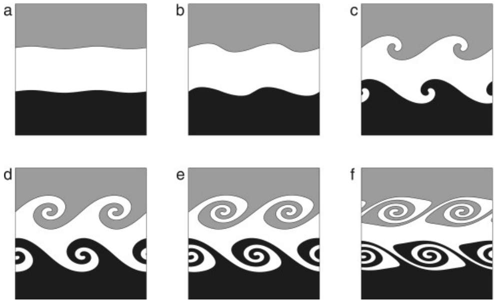


Figure: Didden '79

Kelvin–Helmholtz instability



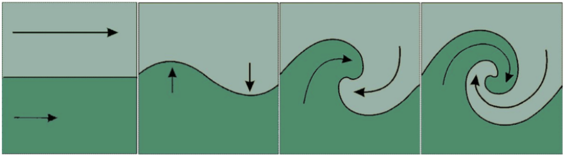


Figure: Mechanism for spiral formation

Kelvin–Helmholtz instability

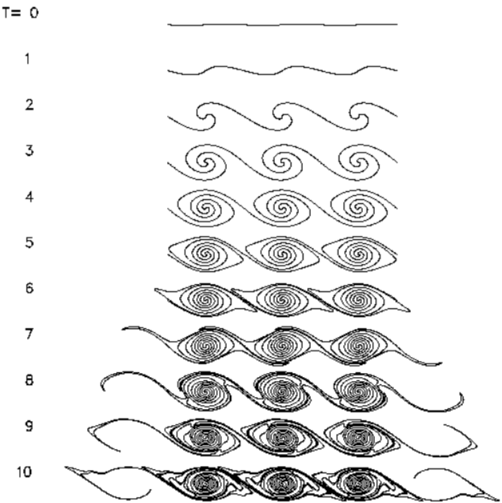


Figure: Krasny '86

Mathematical problems

Questions:

- construction of fluid flows with spiral behavior
- dynamics of spiral flows
- spontaneous creation of spiral flows

Difficulty: Fluid flows containing spirals are not in the standard well-posedness class of the PDE. Even justifying them as weak solutions gives rise to interesting challenges.

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This talk: dynamics of logarithmic vortex spirals

Intro. to Spirals

Spirals are classified according to the rate that "turns" are made. Typical models of the spirals are of the form

$$\theta \simeq f(r), \qquad r \to 0^+, \qquad |f(r)| \to \infty.$$

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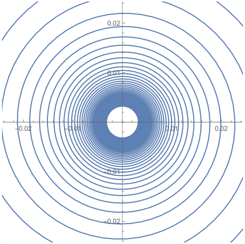
Logarithmic spiral: $f(r) \sim \ln r$. The turns are made along a geometric progression.

Algebraic spiral: $f(r) \sim r^{-a}$ for some $a > 0$. The turns are becoming more and more dense near the origin.



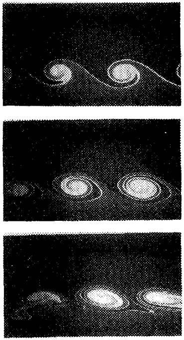
Logarithmic Spirals: common in nature

- Characteristic property: self-similarity
- Invariance under rotation + scaling.

Algebraic vs. Logarithmic for some fluid flows

Question

Spirals in fluid flows are algebraic or logarithmic?
Everson–Sreenivasan '92 has investigated this in various cases.



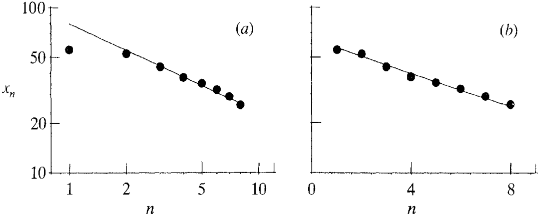
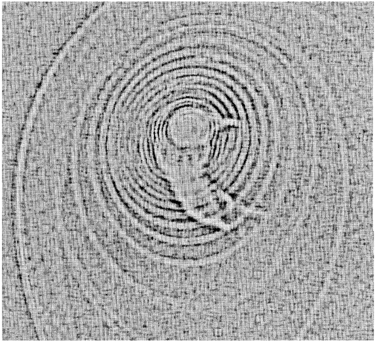


Figure: On direct experiments for shear flows by Sreenivasan et al '89

Algebraic vs. Logarithmic for some fluid flows



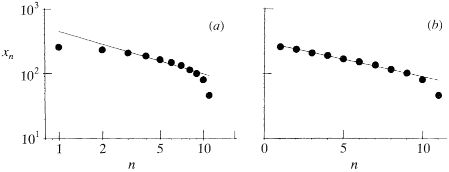


Figure: On smoke rings Magarvey–MacLatchy ('64)

Algebraic vs. Logarithmic for some fluid flows

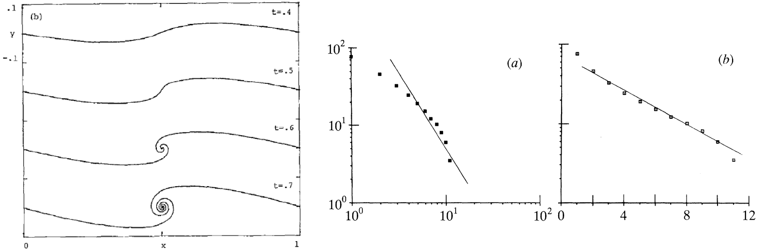


Figure: Numerical computations of Krasny '86

cf. Algebraic spirals provide a better fit in some cases (e.g. Krasny '87 simulations of the roll-up of tip-vortex of an elliptically loaded wing).

Logarithmic spirals

Definition. $\Omega : \mathbb{R}^2 \to \mathbb{R}$ satisfies logarithmic spiral symmetry if there exists some $c > 0$ such that Ω is invariant under the transformation

$$(r, \theta) \mapsto (\lambda r, \theta + c \ln \lambda), \qquad \forall \lambda > 0.$$

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In other words, there exists a function h defined on $\mathbb{S}^1$ such that

$$\Omega(r, \theta) = h(\theta - c \ln r).$$

Logarithmic spirals

In this framework, the Prandtl and Alexander spirals are given by

$$h(t,\theta) = \frac{a}{t - t_0}\delta(\theta - \theta(t)), \qquad \sum_{j=0}^{m-1} \frac{a^{(m)}}{t - t_0}\delta(\theta - \theta^{(m)}(t) - 2\pi j/m).$$

These are self-similar solutions.

Elling–Gnann, Cieslak–Kokocki–Ozanski: non symmetric self-similar solutions

Logarithmic spirals for Navier–Stokes flows

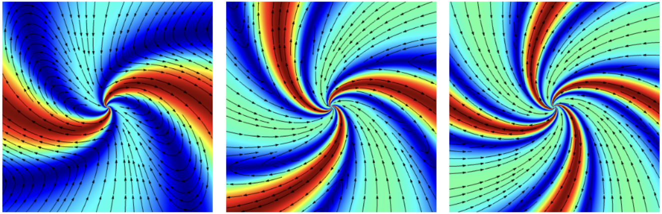
Steady Navier–Stokes flows on $\mathbb{R}^2 \backslash \{0\}$:

$$\Delta\omega + \mathbf{u} \cdot \nabla\omega = 0, \qquad \mathbf{u} = \nabla^{\perp}\Delta^{-1}\omega.$$

In this case, $r^2\omega$ satisfies the spiral symmetry: $\omega = r^{-2}h$.

- Hamel (1917)
- Wang (1991)
- Sverak (2011)
- Guillod–Wittwer (2015)

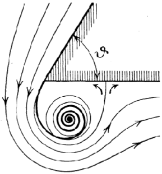
Point: abundance of steady NS flows on $\mathbb{R}^2 \backslash \{0\}$



Logarithmic spirals for Euler flows

Assume ω satisfies logarithmic spiral symmetry, is a vortex sheet, and satisfies time self similarity $\omega(t,x) = t^{-1}\mathring{\omega}(t^{-\mu}x)$.

- Prandtl (1922)
- Alexander (1971)
- Kambe (1989)
- Elling–Gnann (2019)
- Cieslak–Kokocki–Ozanski (21, 22 preprint)



Auch hier ist wieder die logarithmische Spirale eine mögliche Gestalt der Trennungsfläche. Für diese gilt offenbar neben der dynamischen Bedingung der Druckgleichheit noch die kinematische Beziehung: $\psi_1 = \psi_2$ auf beiden Seiten der Trennungslinie. Mit Rücksicht auf die Helmholtzschen Sätze ist weiter noch zu verlangen, daß die Geschwindigkeit, mit der sich der geometrische Ort $\psi_1 = \psi_2$ normal zur Trennungsfläche verschiebt, mit der Normalgeschwindigkeit der Flüssigkeit übereinstimmt. Hierdurch wird jedem Wert von m eindeutig eine bestimmte Steigung der Spirale zugeordnet[2]). Dem obigen Fall entspricht eine solche von nicht ganz 2^0.

Hieraus wird mit (10) für die Trennungsschicht die Gleichung gefunden:

$$r = a e^{\frac{\lambda_1}{\beta}} e^{-\frac{\alpha}{\beta}\vartheta} \left(\frac{t}{T}\right)^m \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

Logarithmic spirals for Euler flows

Previous works: under **time self-similarity and vortex sheet** assumptions, write down the ansatz for the vorticity (Prandtl, Alexander), and prove that it is indeed a weak solution to 2D Euler (Elling–Gnann, Cieslak–Kokocki–Ozanski)

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Our work (J.–Said, 23 preprint): provide a well-posedness class for logarithmic spiral vorticities of 2D Euler, within which

- time self-similar solutions are just a special sub-class,
- vortex sheet solutions are just another sub-class, and realized as well-defined limits of smooth solutions in $\mathbb{R}^2\setminus\{0\}$,
- and furthermore, we obtain a monotonicity formula which gives long time dynamics for general initial data.

Logarithmic spirals for Euler flows

Some further context for our work:

- Elgindi–J., 20, 23: scale-invariant wellposedness theory for Euler, corresponds to the case $c = 0$
- Guillod: log spiral solutions can be studied under the same framework (cf. Work on the 2D Navier–Stokes)
- Elling–Gnann 19: Justification of Alexander spirals
- Cieslak–Kokocki–Ozanski 21, 22 preprint: Justification of Prandtl spirals
- Elgindi–Murray–Said, 22 preprint: long-time dynamics for scale-invariant (different behavior with $c \neq 0$)

Intro. to 2D incompressible inviscid fluid

2D incompressible Euler equations in vorticity form:

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0, \qquad \mathbf{u} = \nabla \times (-\Delta)^{-1} \omega.$$

ω: vorticity, $\mathbf{u}$: velocity.

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Well-posedness of the initial value problem:

- Global with uniqueness $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$ (Yudovich '63)
- Global $\omega_0 \in L^1 \cap L^p$ $(p > 1)$ (DiPerna–Lions)
- Global $\omega_0 \in \mathcal{M}_+ \cap \dot{H}^{-1}$ (Delort),
  $\mathcal{M}_+ \cap \dot{H}^{-1} + L^1$ (Vecchi–Wu)
- Local for analytic vortex sheets (Birkhoff–Rott equation)

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Problem with vortex sheets on logarithmic spirals: no decay at infinity and non-signed in general.

Logarithmic vortex spiral

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Our ansatz:

$$\omega(t, r, \theta) = h(t, \theta - c \ln r), \qquad \mathbf{u} = \nabla^{\perp} \left(r^2 H(t, \theta - c \ln r) \right).$$

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Reduced equations: transport of h

$$\partial_t h + 2H \partial_\theta h = 0, \qquad (1 + c^2) H'' - 4cH' + 4H = h.$$

Surprising cancellation: advecting velocity is just $2H$, which is **order two** smoother than h. This gives:

$$\|2H\|_{Lip} \leq C \|h\|_{\mathcal{M}}.$$

Logarithmic vortex spiral

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$$\partial_t h + 2H\partial_\theta h = 0, \qquad (1+c^2)H'' - 4cH' + 4H = h.$$

Surprising **monotonicity formula**:

$$\frac{d}{dt}\int h d\theta = -4c\int (\partial_\theta H)^2 d\theta.$$

This means decay (for $c > 0$) of the **local circulation**;

$$\int_{B(0,R)} \omega(t,x)dx = \frac{R^2}{2}\int h(t,\theta)d\theta.$$

Logarithmic vortex spiral

Theorem on well-posedness (J.-Said, preprint)

The system

$$\partial_t h + 2H\partial_\theta h = 0, \qquad (1+c^2)H'' - 4cH' + 4H = h.$$

is globally well-posed for any $h_0 \in C^k$ with $k \geq 0$ and locally well-posed for L^p with $p \geq 1$.

Logarithmic vortex spiral

Theorem on vortex sheet spiral limit (J.-Said, preprint)

Let $h_0(\theta) = \sum_{i=1}^{N} \Gamma_i \delta(\theta - \theta_i)$. Then, the sequence of global solutions $\{h^\varepsilon(t, \cdot)\}$ corresponding to regularized initial data h_0^ε converges in distribution to $\sum_{i=1}^{N} \Gamma_i(t)\delta(\theta - \theta_i(t))$.

The functions $\{\Gamma_i, \theta_i\}_{i=1}^{N}$ satisfy a system of ODE, and $\sum_{i=1}^{N} \Gamma_i(t)\delta(\theta - \theta_i(t))$ defines a weak solution to 2D Euler.

- Problem with defining

$$\int 2H\partial_\theta h\varphi d\theta := -\int h\partial_\theta(2H\varphi)d\theta$$

- Prandtl and Alexander spirals are simply special solutions to the ODE system.

Logarithmic vortex spiral

Theorem on long time dynamics (J.-Said, preprint)

(1) In L^∞ case: the solution converges to a constant as $t \to \infty$.

(2) In L^p case with $p < \infty$: either the solution converges to a constant or blows up in finite or infinite time.

(3) In the Dirac measure case: the solution decays to 0 or blows up in finite time.

- Prandtl and Alexander spirals satisfy $|\Gamma(t)| \sim |t - t_0|^{-1}$.

Basic questions

- Characterization of blow up
- Non-symmetric blow up for vortex sheets
- Bifurcation of non-symmetric blow up
- Stability of blow up

cf. Linear instability of Alexander spirals (Cieslak et al, 2023).

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- Formation of logarithmic spirals

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Thank you for listening!