Rotational symmetries and incompressible Euler in high dimensions

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Mathematical Analysis of Viscous Incompressible Fluid
RIMS Workshop
December 12, 2023

Small scale creation in fluid dynamics

Goal: small scale creation for **incompressible inviscid** fluids.

Key points:

- Stability of instability, monotone quantities.
- Differences in two- and three-dimensional cases.
- Bounded and unbounded domains.

2D case

Two-dimensional incompressible Euler equations:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ u = \nabla^\perp \Delta^{-1} \omega. \end{cases}$$

All the L^p norms of ω are conserved.

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Not easy to prove even for carefully designed initial data!
(Review paper of Drivas–Elgindi '22).

Proof of small scale creation in 2D

Breakthroughs by stability of instability: Denissov ('09, '13), Kiselev–Nazarov ('12), Kiselev–Sverak ('14), Zlatos ('15), $\cdots$ (survey paper by Kiselev '18).

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Basic idea: *vortex thinning equation*

$$\partial_t(\nabla^\perp \omega) + u \cdot \nabla(\nabla^\perp \omega) = \nabla u(\nabla^\perp \omega).$$

Want $\nabla u \approx \nabla \bar{u}$ a hyperbolic matrix for all times.

Concrete example: odd-odd symmetry

Choices of Denissov ('09) and Kiselev–Sverak ('14) in $\mathbb{T}^2$ are

$$\sin(x)\sin(y), \qquad \mathrm{sgn}(x)\mathrm{sgn}(y).$$

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Stability: maximizer of kinetic energy $\|\omega\|_{\dot{H}^{-1}}$.

Instability: ∇u is hyperbolic near the origin.

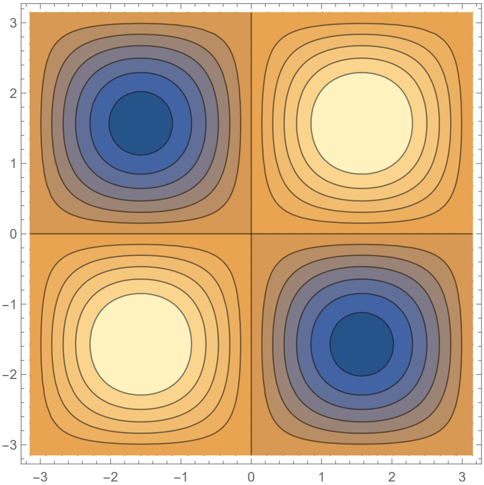

Figure: stream function $\bar{\psi} = \Delta^{-1}\bar{\omega}$

Gradient growth in $\mathbb{R}^2$

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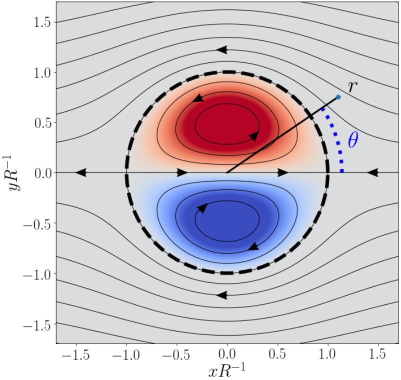
Lamb dipole: explicit *traveling wave* of Euler in $\mathbb{R}^2$:

$$\bar{\omega}(t,x) = \bar{\omega}_0(x - W_{Lamb}t).$$

Nonlinear stability of Lamb dipole: Abe–Choi ('21)



Small scale creation near Lamb dipole

Theorem (Choi–J. '22)

Oblate perturbations of the Lamb dipole go through linear filamentation in time: $\|\nabla\omega(t,\cdot)\|_{L^p} \gtrsim t$ *for all* $1 \le p \le \infty$.

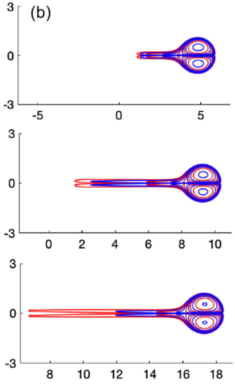


Figure: Filamentation near Lamb dipole in Krasny–Xu ('21)

Ideas of proof

Nonlinear stability: the Lamb dipole $\bar{\omega}$ is the **unique** maximizer of

$$\|\omega\|_{\dot{H}^{-1}} - C\|\omega\|_{L^2}$$

under some constraints up to a shift. This gives:

$$\|\bar{\omega} - \omega_0\|_{L^2} < \delta \implies \|\bar{\omega}(\cdot - \tau(t)\mathbf{e}_1) - \omega(t,\cdot)\|_{L^2} < \varepsilon.$$

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But similar shape implies similar speed:

$$\frac{d}{dt}\int_{\mathbb{R}^2} x_1 \omega dx = \int_{\mathbb{R}^2} u_1 \omega dx$$

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This gives $\tau(t) \simeq W_{Lamb} t$, while the "tail" of the perturbation is always strictly slower than W_{Lamb}.

Vortex stretching in three dimensions

The 3D vorticity equation:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \\ u = \nabla \times (-\Delta)^{-1} \omega. \end{cases}$$

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Our Goal: **Infinite time infinite vortex stretching**.
Namely, can we have $\|\omega(t, \cdot)\|_{L^p} \to \infty$ for global-in-time smooth vorticity as $t \to \infty$?

Axisymmetric Euler equations without swirl

Axisymmetric and no-swirl ansatz: "simplest" 3D flow

$$\boldsymbol{\omega} = \omega^{\theta}(r,z)\mathbf{e}^{\theta}, \qquad \mathbf{u} = u^{r}(r,z)\mathbf{e}^{r} + u^{z}(r,z)\mathbf{e}^{z}.$$

Vorticity equation simplifies to

$$\partial_t \omega^{\theta} + u \cdot \nabla \omega^{\theta} = \frac{u^r}{r}\omega^{\theta}$$

or equivalently

$$\partial_t \frac{\omega^{\theta}}{r} + u \cdot \nabla \frac{\omega^{\theta}}{r} = 0.$$

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Smooth solutions are global, but allows for vortex stretching!

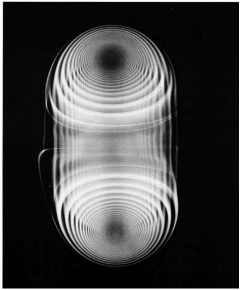
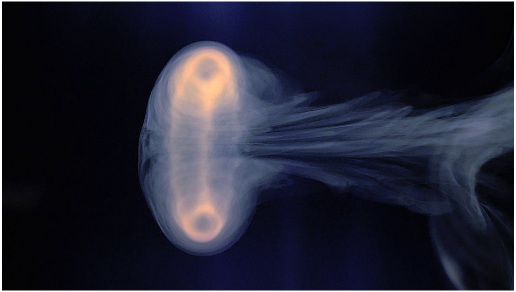
Dynamics of a single vortex ring

Region of concentrated vorticity: travels along the symmetry axis

Explicit traveling wave: **Hill's vortex ring** $\omega^\theta = r\mathbf{1}_B(r,z)$.

Variational nonlinear stability: Choi ('22).

Filamentation near Hill's vortex: all time Hessian growth in $\mathbb{R}^3$
$\|\nabla^2\omega(t,\cdot)\|_{L^\infty} \gtrsim t^{1/2}$ (Choi–J., '22).

Dynamics of multiple vortex rings

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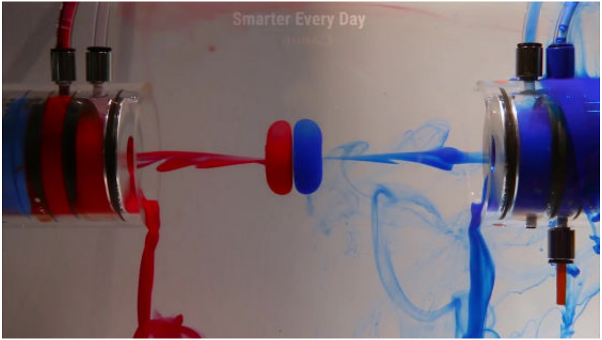
The simplest setup allowing for infinite vortex stretching: two symmetric rings with opposite signs ("anti-parallel').



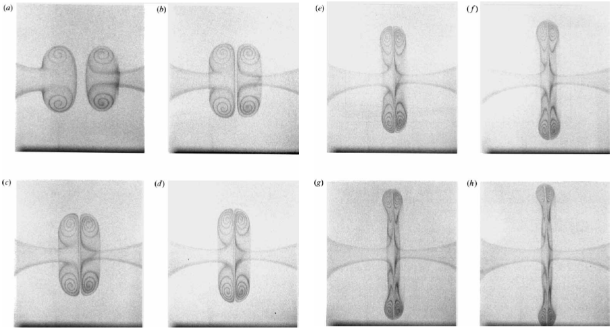
Head-on collision of vortex rings

By **anti-parallel**, we simply mean that

$$\omega^\theta(r,z) = -\omega^\theta(r,-z), \qquad \omega^\theta \le 0 \text{ on } \mathbb{R}^3_+ = \{z > 0\}.$$

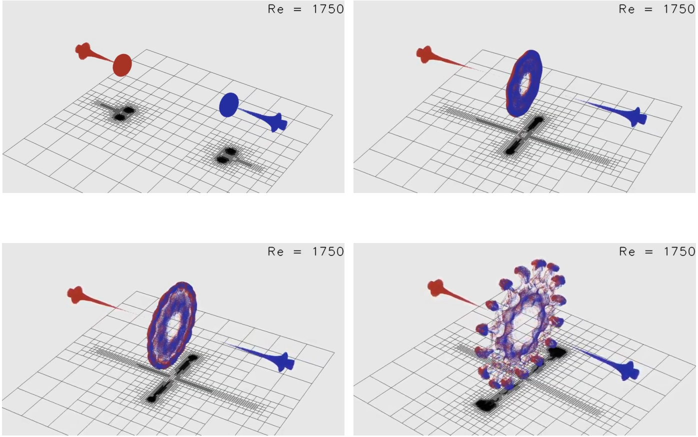
(Equivalent with a single negative vortex ring on $\mathbb{R}^3_+$.)

Extensive numerical works: Oshima (78), Kambe–Minota (83), Peace–Riley (83), Lim–Nickels (92), Chu–Wang–Chang–Chang–Chang (95), ...



Experiments/Numerics show three stages of motion:

- 1. weak interaction and free traveling
- 2. squeezing and vortex stretching
- 3. breakup and reconnection



Head-on collision of vortex rings

Stages 1 and 2 are essentially **inviscid** phenomena. Maximal radius of rings before breakup increases with the Reynolds number, suggesting infinite vortex stretching for Euler.

Cauchy formula for axisymmetric flows without swirl:

$$\omega(t, \Phi(t, r, z)) = \frac{\Phi^r(t, r, z)}{r} \omega_0(r, z),$$

where $\Phi = (\Phi^r, \Phi^z)$ is the flow map.

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Rate of vortex stretching $\approx$ expansion of rings in r.

Difficulty: lack of stability, instead we rely on monotonicity.

Monotonicity

Theorem (Choi–J., preprint)

For anti-parallel flows, the quantities

$$P = \iint_{[0,\infty)^2} -r^2\omega(t,r,z)\,\mathrm{d}r\mathrm{d}z, \quad Z = \iint_{[0,\infty)^2} -z\omega(t,r,z)\,\mathrm{d}r\mathrm{d}z$$

are monotone in time: P increases and Z decreases.

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Recall: circulation conservation

$$\iint_{[0,\infty)^2} -\omega(t,r,z)\,\mathrm{d}r\mathrm{d}z = \|r^{-1}\omega\|_{L^1(\mathbb{R}^3_+)}.$$

Comparison: in the case of single-signed vortex ring, Z is decreasing and P is conserved.

Application I: infinite growth of gradient

Theorem (Choi–J., preprint)

There exists a $C_c^\infty(\mathbb{R}^3)$ *datum* ω_0 *with the unique global solution* $\omega(t,\cdot)$ *satisfying, for all* $\alpha > 0$,

$$\sup_{t\in[0,\infty]} \|\omega(t,\cdot)\|_{C^\alpha(\mathbb{R}^3)} = \infty.$$

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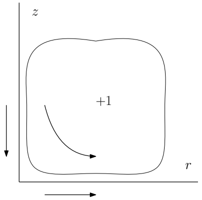
The proof is based on 2D hyperbolic growth scenario, with **lack of stability and compactness** handled by monotonicity.

Application I: growth of gradient

Version of Kiselev–Sverak **Key Lemma** for axisymmetric flows:

$$\begin{pmatrix} u^r(t,r,z) \\ u^z(t,r,z) \end{pmatrix} \simeq \begin{pmatrix} r \\ -2z \end{pmatrix} \mathcal{I}(t,|x|) \qquad \text{cf. Elgindi '21}$$

In our case: mass could escape to infinity, in which case $\mathcal{I} \to 0$.



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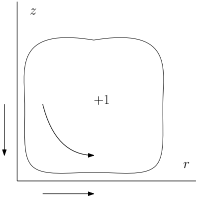

Mass cannot escape to the z-direction due to monotonicity.
Escaping to the r-direction indeed occurs but this gives vortex stretching.

Application II: universal growth rates

Theorem (Choi–J., preprint)

*For **any** compactly supported anti-parallel vorticity, at least $t^{\frac{2}{15}-}$ growth of the vortex impulse occurs:*

$$P(t) := \iint_{[0,\infty)^2} -r^2\omega(t,r,z)\,\mathrm{d}r\mathrm{d}z \gtrsim t^{\frac{2}{15}-}, \qquad \forall t \geq 0.$$

This implies infinite growth of the support diameter.

Roughly, rate of vortex stretching $\sim [r] \gtrsim t^{\frac{1}{15}-}$

cf. variational proof of vortex stretching

Key inequality

- Strategy inspired by Iftimie–Sideris–Gamblin ('99): lower bound on $\dot{P}$ purely based on monotone and conserved quantities.
- Our key inequality: for $1 < q < 15/13$

$$E \lesssim_q X P^{\frac{1}{q}} (\dot{P})^{1-\frac{1}{q}},$$

and $X \leq X_0$ with

$$X = \|\xi\|_{L^1}^{\frac{4}{q}-\frac{10}{3}} \|\xi\|_{L^\infty}^{\frac{1}{3}} Z^{\frac{4(q-1)}{q}} P^{\frac{1-q}{q}} + \|\xi\|_{L^1}^{\frac{3}{q}-\frac{7}{3}} \|\xi\|_{L^\infty}^{\frac{1}{3}} Z^{\frac{2(q-1)}{q}}.$$

- Here, E is the kinetic energy of the fluid and $\xi = \omega/r$.
- Integrating in time gives $P \to \infty$, implying infinite growth of support and maximum, thanks to the Cauchy formula.

Application III: infinite vortex stretching

Corollary

Let $0 \leq \delta < 1/15$. Assume that for some $p \in [2-\delta, \infty]$, the initial data is of compact support and satisfies

$$\left\| r|\omega_0|^{-1} \mathbf{1}_{\{|\omega_0|>0\}} \right\|_{L^{\frac{1-\delta}{1-((2-\delta)/p)}}(\mathbb{R}^3)} < \infty \tag{1}$$

Then, for each $\varepsilon > 0$, we have

$$\|\omega(t,\cdot)\|_{L^p(\mathbb{R}^3)} \geq C_\varepsilon (1+t)^{\frac{1}{2-\delta}(\frac{2}{15}-2\delta)-\varepsilon} \quad \text{for all} \quad t \geq 0.$$

Condition (1) is satisfied for

- Smooth vortex patches
- Vorticity with $C^{1,1/15-}$ regularity.

An additional contradiction argument gives $\limsup_{t\to\infty} \frac{\|\omega(t,\cdot)\|_{L^\infty}}{(1+t)^{0.13+}} = \infty$.

Discussion

Question: actual rate of vortex stretching?

- Dyson model for vortex rings: predicts t as in 2D (false).
- Majda ('94), Danchin ('00): $\exp(Ct)$.
- Childress ('07–08): upper bound of t^2 and then $t^{4/3}$.
- Feng–Sverak ('15): upper bound of t^2.
- Childress–Gilbert–Valiant ('16): $t^{4/3}$ is indeed achievable, by modulated Sadovskii vortex