Twisting in Hamiltonian Flows with Applications to Fluids

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New Trends in Hyperbolic Conservation Laws and Related Models
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Hamiltonian flows in a 2D domain Ω

- Given smooth $\Psi : [0,\infty) \times \Omega \to \mathbb{R}$, we consider the ODE

$$\begin{cases} \dot{X}(t) = -\partial_y \Psi(t, X(t), Y(t)), \\ \dot{Y}(t) = \partial_x \Psi(t, X(t), Y(t)) \end{cases}$$

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- Alternatively, f solves the transport equation

$$\partial_t f - \partial_y \Psi \partial_x f + \partial_x \Psi \partial_y f = 0.$$

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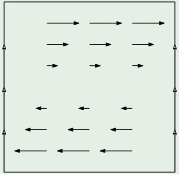
- Observe growth of spatial gradients:

$$\|\nabla_{x,y}\Phi(t,\cdot)\|_{L^1(\Omega)} = C_0(1+Ct), \quad \text{“twisting”}$$
$$\|\nabla_{x,y} f(t,\cdot)\|_{L^1(\Omega)} = c_0(1+ct), \quad \text{“filamentation”}$$

with $c \neq 0$ unless f_0 is independent of x.



Example 2

- Assume that Ψ is only *partially* known: for a small $\delta > 0$,

$$\Psi(t,x,y) = \begin{cases} -y & \text{if} \quad |y - \frac{1}{2}| < \delta, \\ y & \text{if} \quad |y + \frac{1}{2}| < \delta \end{cases}$$

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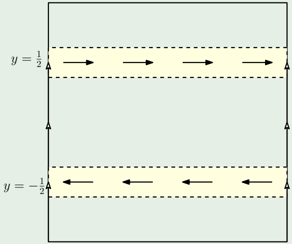

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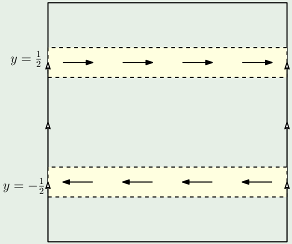


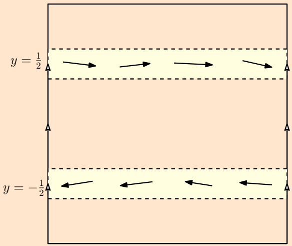
- Still, we have $\|\nabla_{x,y}\Phi(t,\cdot)\|_{L^1(\Omega)} \gtrsim t$, $\|\nabla_{x,y} f(t,\cdot)\|_{L^1(\Omega)} \gtrsim t$.

Question (Example 3)

- Assume that Ψ is partially and *perturbedly* known:

$$\Psi(t,x,y) = \begin{cases} -y + \psi(t,x,y) & \text{if} \quad |y - \frac{1}{2}| < \delta, \\ y + \psi(t,x,y) & \text{if} \quad |y + \frac{1}{2}| < \delta \end{cases}$$

with some ψ satisfying $\|\psi\|_{L^\infty([0,\infty)\times W^{1,1}(\Omega))} \leq \varepsilon_0 \ll 1$.



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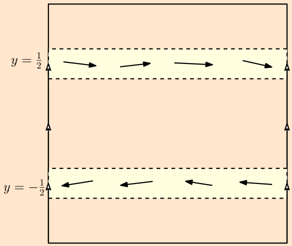

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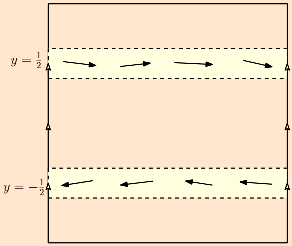

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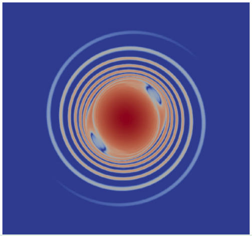
- PDE applications: fluid, kinetic, MHD, ...
- DS applications: Arnold diffusion, complexity growth, ...



Filamentation in fluid flows

Evolution of elliptical vortex in incompressible flows

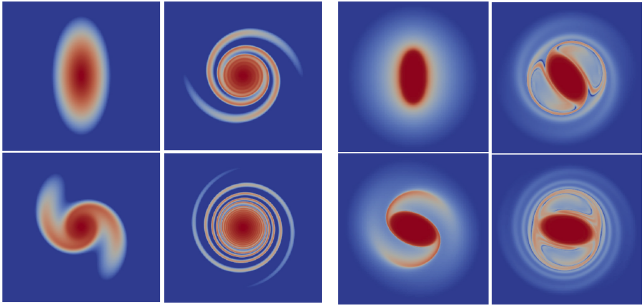


Figure: Krasny–Xu 2023

Filamentation in fluid flows

cf. Growth of $\|\nabla f\|_{L^1}$ versus $\|\nabla f\|_{L^\infty}$.

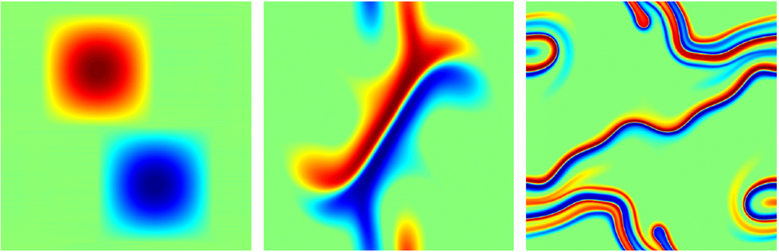


Figure: Iyer–Xu: Optimal mixing velocity field

Filamentation in plasma dynamics

Evolution of $f(t, x, v)$ in Landau damping for 1D Vlasov–Poisson

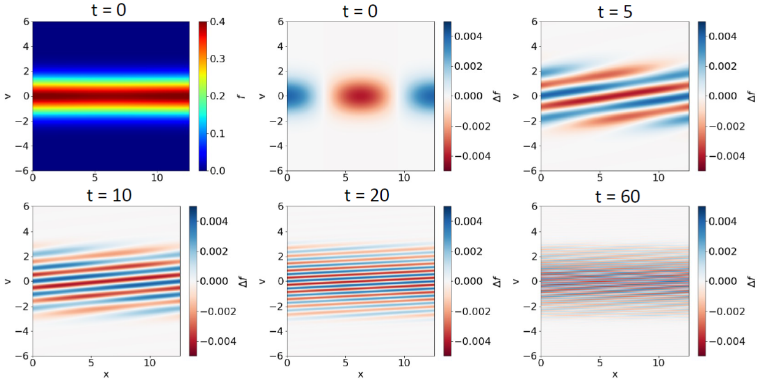


Figure: Krasny–Thomas–Sandberg 2023

Filamentation in plasma dynamics

- Two-stream instability: phase space description
- Filamentation still occurs without Landau damping.

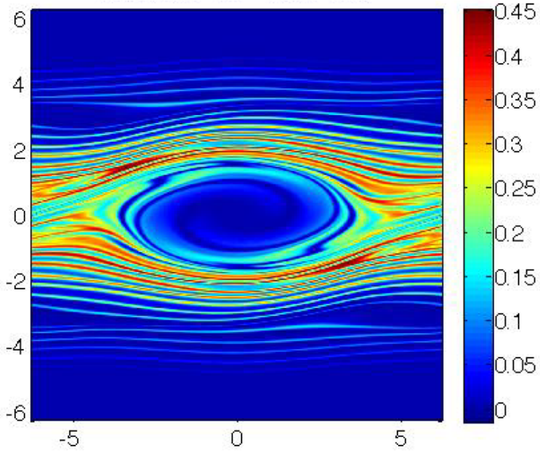
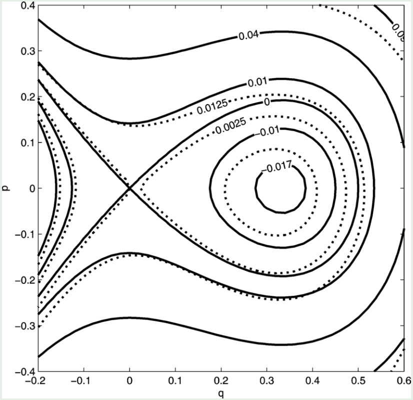


Figure: Liu–Chen–Quan–Zhou 2020

Twisting for steady Hamiltonian flows

Let $\bar{\Psi}$ be a smooth steady Hamiltonian on Ω. (Generic picture: periodic orbits separated by fix points and connecting orbits.) We say that the corresponding flow Φ is **twisting** if there is an annular region $\mathbf{A} \subset \Omega$ foliated with periodic orbits such that the two connected components of $\partial\mathbf{A}$ have **different periods**.



Model Example 1: Shear flows

- Take $\Omega = \mathbb{T} \times [-1, 1]$ and consider $\bar{\Psi}(x, y) = G(y)$.
- Then

$$\dot{X} = -G'(Y), \qquad \dot{Y} = 0.$$

- We have $\Phi = (X, Y)$ with $X(t, x, y) = x - tG'(y) \pmod 2$ and $Y(t, x, y) = y$.
- In this case, Φ is **twisting** if and only if G' is nonconstant:

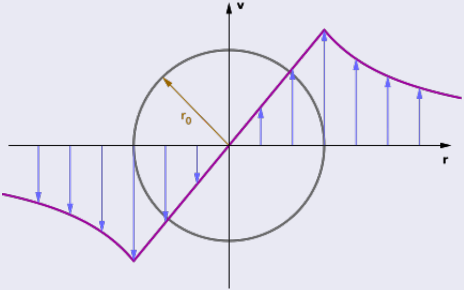
$$\|\nabla\Phi(t, \cdot)\|_{L^1(\Omega)} \geq \|\partial_y X(t, \cdot)\|_{L^1(\Omega)} = Ct.$$

Example 2: Radial flows

Domains $\Omega = \mathbb{R}^2, B_0(1), \cdots$. Consider in polar coordinates

$$\dot{\Theta} = g(R), \qquad \dot{R} = 0.$$

We have $\Theta(t) = \theta + tg(r)$. Twisting occurs if and only if $g' \not\equiv 0$.



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Then, there exists $\varepsilon_0 = \varepsilon_0(\bar{\Psi}) > 0$ *such that:*
if $\Psi(t,\cdot)$ *is a time-dependent Hamiltonian on* Ω *satisfying*

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Difficulties

- Twisting is a local information
- No invariant regions: individual particles are free to travel

Counterexample?

On $\mathbb{T}^2$, we have that $\bar{\Psi}(x, y) = \cos(y)$ is twisting. However, consider its perturbation $\Psi(x, y) = \cos(y) + \varepsilon x$. Then

$$\dot{X} = \sin(Y), \qquad \dot{Y} = \varepsilon.$$

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Note that Ψ is actually not smooth on $\mathbb{T}^2$.

Indeed $\bar{\Psi} - \Psi \notin W^{1,1}(\mathbb{T}^2)$.

Theorem (From twisting to filamentation)

In the same setting, there is filamentation of advected scalars for an L^∞ open set of initial data; that is

$$\|\nabla f(t,\cdot)\|_{L^1(\Omega)} \geq ct, \quad \text{as} \quad t \to \infty.$$

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Applications to PDE

Consider the PDEs of the form

$$\dot{\Phi}(t) = \nabla^\perp \Psi(t), \quad f(t) = f_0 \circ \Phi^{-1}(t)$$

and $f(t) \mapsto \Psi(t)$ by a functional relation. We need a steady solution $(\bar{f}, \bar{\Psi})$ which is **stable** just in the $W^{1,1}$ norm of $\bar{\Psi}$.

Key PDE Examples

- Incompressible 2D Euler equations:

$$\dot{\Phi} = \nabla^{\perp}\Psi,$$
$$\Psi = -(-\Delta)^{-1}\omega,$$
$$\omega \circ \Phi = \omega_0.$$

- Vlasov–Poisson equations:

$$\dot{\Phi} = -\nabla^{\perp}_{x,v}\left(\frac{1}{2}|v|^2 + U(x)\right),$$
$$U = \pm(-\Delta_x)^{-1}\int_{\mathbb{R}} f(t,x,v)dv,$$
$$f \circ \Phi = f_0.$$

- SQG, Vlasov–Riesz, ideal MHD, MRE, ...

Stablility of incompressible 2D Euler

There are many known L^p stable steady vortex $\bar{\omega}$: i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all ω_0 satisfying

$$\|\omega_0 - \bar{\omega}\|_{L^p} < \delta,$$

we have that the solution $\omega(t,\cdot)$ with initial data ω_0 satisfies

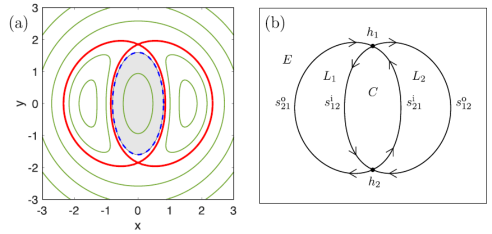
$$\|\omega(t,\cdot) - \bar{\omega}\|_{L^p} < \varepsilon, \quad \text{for all} \quad t \geq 0.$$

This guarantees $\|\Psi(t,\cdot) - \bar{\Psi}\|_{W^{1,1}} \ll 1$.

$$\begin{tikzcd}
 & \Psi(t,\cdot) \arrow[dr, "\text{Hamiltonian flow}"] & \\
\omega(t,\cdot) \arrow[ur, "\Delta^{-1}"] & & \Phi(t,\cdot) \arrow[ll, "\text{Scalar Advection}"']
\end{tikzcd}$$

A collection of stable Euler flows

- Rankine vortex and any monotone radial vortex.
- Kirchhoff Ellipses with aspect ratio < 3.
- First eigenfunctions on $\mathbb{T}^2$ under a symmetry.
- Second eigenfunctions on $\mathbb{T}^2$ under two symmetries.
- Lamb dipole / Hill's vortex / ...

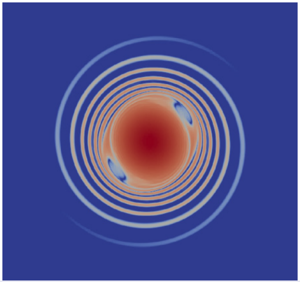


Application to incompressible 2D Euler

Theorem (A sample theorem)

Let $\bar{\omega}$ be an L^p stable 2D Euler solution, whose associated flow map is twisting. Then, for an open set of perturbations ω_0, we have

$$\|\nabla \omega(t)\|_{L^1} \geq c_0 t \qquad \text{for all } t \geq 0.$$



cf. A similar result applies to inviscid SQG in $\mathbb{T}^2$.

Ideas of the proof

- Recall the case 2: it is natural to "unfold" the dynamics to the universal cover $\tilde{\Omega} = \mathbb{R} \times [-1, 1]$ and consider $\tilde{\Phi}$ on $\tilde{\Omega}$

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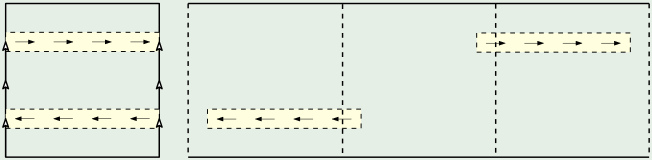

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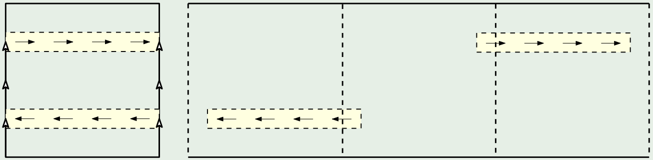

Recall the difficulties

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- No invariant regions: individual particles are free to travel

Introduction of twisting quantity

- Define localized and averaged winding number:

$$\mathcal{I}_i(t) := \iint_{\mathbb{T}\times[-1,1]} \tilde{X}(t,x,y)F_i(Y(t,x,y))\,\mathrm{d}x\mathrm{d}y, \quad i = 1,2$$

where $F_i(y)$ is sharply concentrated at $y_i = (-1)^i/2$.

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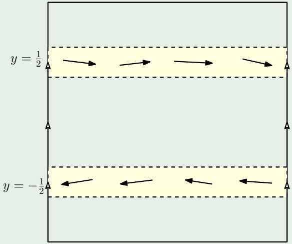
- **Steady case**: $\tilde{X} = x + tV(y) \implies \mathcal{I}_i(t) \simeq \mathcal{I}_i(0) + tV(y)$, which immediately gives $|\mathcal{I}_1(t) - \mathcal{I}_2(t)| \gtrsim |t|$.



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- We compute:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_i(t) = \iint_{\mathbb{T}\times[-1,1]} \left(\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}(t,x,y)\right) F_i(Y(t,x,y))\,\mathrm{d}x\mathrm{d}y$$
$$+ \iint_{\mathbb{T}\times[-1,1]} \tilde{X}(t,x,y)(\frac{\mathrm{d}}{\mathrm{d}t}F_i(Y(t,x,y)))\,\mathrm{d}x\mathrm{d}y.$$

- The first term gives linear growth as in the steady case.

Heart of the matter

- Interpretation of $\mathcal{I}_i(t)$: counting of all "winding numbers" of particles at time t passing through the segment $\{y = y_i\}$.
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Summary

- Filamentation is very common in advection equations
- Twisting for the flow map gives filamentation
- Main result: stability of twisting in the time-dependent case
- Weak requirement $W^{1,1}$ facilitates PDE applications

Thank you for your attention!