

Optimal rate of vortex stretching for axisymmetric flows without swirl

In-Jee Jeong (Seoul National University)
joint work with Deokwoo Lim
arXiv:2409.19497

Singularities in Fluids
National University of Singapore
Dec 16, 2024

Vorticity equation for incompressible inviscid flows

We consider in $\mathbb{R}^3$ the vorticity equation

$$\partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u},$$

$$\mathbf{u} = \nabla \times (-\Delta)^{-1} \boldsymbol{\omega}$$

where

$$\boldsymbol{\omega}(t, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \mathbf{u}(t, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

denote the vorticity and velocity of an incompressible inviscid flow occupying $\mathbb{R}^3$ at time t .

Axisymmetric flows without swirl

A special class of solutions is given by

$$\boldsymbol{\omega} = \omega^\theta(t, r, z) \mathbf{e}^\theta, \quad \mathbf{u} = u^r(t, r, z) \mathbf{e}^r + u^z(t, r, z) \mathbf{e}^z$$

in the cylindrical coordinates (r, θ, z) .

Axisymmetric flows without swirl

A special class of solutions is given by

$$\boldsymbol{\omega} = \omega^\theta(t, r, z) \mathbf{e}^\theta, \quad \mathbf{u} = u^r(t, r, z) \mathbf{e}^r + u^z(t, r, z) \mathbf{e}^z$$

in the cylindrical coordinates (r, θ, z) . The vorticity equation reduces to

$$\partial_t \omega^\theta + [u^r \partial_r + u^z \partial_z] \omega^\theta = \frac{u^r}{r} \omega^\theta.$$

Axisymmetric flows without swirl

A special class of solutions is given by

$$\boldsymbol{\omega} = \omega^\theta(t, r, z) \mathbf{e}^\theta, \quad \mathbf{u} = u^r(t, r, z) \mathbf{e}^r + u^z(t, r, z) \mathbf{e}^z$$

in the cylindrical coordinates (r, θ, z) . The vorticity equation reduces to

$$\partial_t \omega^\theta + [u^r \partial_r + u^z \partial_z] \omega^\theta = \frac{u^r}{r} \omega^\theta.$$

Smooth initial data have unique **global in time** solutions,

Axisymmetric flows without swirl

A special class of solutions is given by

$$\boldsymbol{\omega} = \omega^\theta(t, r, z) \mathbf{e}^\theta, \quad \mathbf{u} = u^r(t, r, z) \mathbf{e}^r + u^z(t, r, z) \mathbf{e}^z$$

in the cylindrical coordinates (r, θ, z) . The vorticity equation reduces to

$$\partial_t \omega^\theta + [u^r \partial_r + u^z \partial_z] \omega^\theta = \frac{u^r}{r} \omega^\theta.$$

Smooth initial data have unique **global in time** solutions, but the vorticity may grow with time, unlike the 2D case.

Axisymmetric flows without swirl

A special class of solutions is given by

$$\boldsymbol{\omega} = \omega^\theta(t, r, z) \mathbf{e}^\theta, \quad \mathbf{u} = u^r(t, r, z) \mathbf{e}^r + u^z(t, r, z) \mathbf{e}^z$$

in the cylindrical coordinates (r, θ, z) . The vorticity equation reduces to

$$\partial_t \omega^\theta + [u^r \partial_r + u^z \partial_z] \omega^\theta = \frac{u^r}{r} \omega^\theta.$$

Smooth initial data have unique **global in time** solutions, but the vorticity may grow with time, unlike the 2D case.

Question: optimal rate of vortex stretching

How fast can $\|\omega^\theta(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}$ grow with time?

Vortex confinement to vortex stretching

The *relative vorticity* satisfies

$$\partial_t \frac{\omega^\theta}{r} + u \cdot \nabla \frac{\omega^\theta}{r} = 0$$

Vortex confinement to vortex stretching

The *relative vorticity* satisfies

$$\partial_t \frac{\omega^\theta}{r} + u \cdot \nabla \frac{\omega^\theta}{r} = 0$$

and gives

$$\left\| \frac{\omega^\theta(t, \cdot)}{r} \right\|_{L^\infty(\mathbb{R}^3)} = \left\| \frac{\omega_0^\theta}{r} \right\|_{L^\infty(\mathbb{R}^3)}$$

where the RHS is bounded for smooth $\omega = \omega^\theta \mathbf{e}^\theta$, say $C^1(\mathbb{R}^3)$.

Vortex confinement to vortex stretching

The *relative vorticity* satisfies

$$\partial_t \frac{\omega^\theta}{r} + u \cdot \nabla \frac{\omega^\theta}{r} = 0$$

and gives

$$\left\| \frac{\omega^\theta(t, \cdot)}{r} \right\|_{L^\infty(\mathbb{R}^3)} = \left\| \frac{\omega_0^\theta}{r} \right\|_{L^\infty(\mathbb{R}^3)}$$

where the RHS is bounded for smooth $\omega = \omega^\theta \mathbf{e}^\theta$, say $C^1(\mathbb{R}^3)$.
For compactly supported vorticity, we have that

$$\|\omega^\theta(t, \cdot)\|_{L^\infty} \leq R(t) \left\| \frac{\omega_0^\theta}{r} \right\|_{L^\infty(\mathbb{R}^3)}$$

where $R(t) := \text{diam}_r \text{supp}(\omega^\theta(t, \cdot))$.

Upper bounds on vorticity growth

- ▶ Classical works of Majda ('94), Danchin ('00) gives

$$\|\omega^\theta(t, \cdot)\|_{L^\infty} \leq C \exp(Ct).$$

Upper bounds on vorticity growth

- ▶ Classical works of Majda ('94), Danchin ('00) gives

$$\|\omega^\theta(t, \cdot)\|_{L^\infty} \leq C \exp(Ct).$$

- ▶ Childress ('07), Feng–Sverak ('15) gives

$$\|\omega^\theta(t, \cdot)\|_{L^\infty} \leq C(1 + t^2)$$

and Childress ('08) conjectures that the optimal rate is $t^{4/3}$.

Upper bounds on vorticity growth

- ▶ Classical works of Majda ('94), Danchin ('00) gives

$$\|\omega^\theta(t, \cdot)\|_{L^\infty} \leq C \exp(Ct).$$

- ▶ Childress ('07), Feng–Sverak ('15) gives

$$\|\omega^\theta(t, \cdot)\|_{L^\infty} \leq C(1 + t^2)$$

and Childress ('08) conjectures that the optimal rate is $t^{4/3}$.

- ▶ Several numerical works verify the rate $t^{4/3}$.

Upper bounds on vorticity growth

- ▶ Classical works of Majda ('94), Danchin ('00) gives

$$\|\omega^\theta(t, \cdot)\|_{L^\infty} \leq C \exp(Ct).$$

- ▶ Childress ('07), Feng–Sverak ('15) gives

$$\|\omega^\theta(t, \cdot)\|_{L^\infty} \leq C(1 + t^2)$$

and Childress ('08) conjectures that the optimal rate is $t^{4/3}$.

- ▶ Several numerical works verify the rate $t^{4/3}$.
- ▶ Lim–J. (preprint): proves upper bound of $t^{4/3}$.

Outline of the talk

- ▶ Experimental/Numerical works on vortex stretching

Outline of the talk

- ▶ Experimental/Numerical works on vortex stretching
- ▶ Classical upper bounds on vortex stretching: $\exp(Ct)$

Outline of the talk

- ▶ Experimental/Numerical works on vortex stretching
- ▶ Classical upper bounds on vortex stretching: $\exp(Ct)$
- ▶ Childress' circulation conserving dipole model: t^2 bound

Outline of the talk

- ▶ Experimental/Numerical works on vortex stretching
- ▶ Classical upper bounds on vortex stretching: $\exp(Ct)$
- ▶ Childress' circulation conserving dipole model: t^2 bound
- ▶ Childress' eroding dipole model: $t^{4/3}$ conjecture

Outline of the talk

- ▶ Experimental/Numerical works on vortex stretching
- ▶ Classical upper bounds on vortex stretching: $\exp(Ct)$
- ▶ Childress' circulation conserving dipole model: t^2 bound
- ▶ Childress' eroding dipole model: $t^{4/3}$ conjecture
- ▶ Feng–Sverak inequality: t^2 bound

Outline of the talk

- ▶ Experimental/Numerical works on vortex stretching
- ▶ Classical upper bounds on vortex stretching: $\exp(Ct)$
- ▶ Childress' circulation conserving dipole model: t^2 bound
- ▶ Childress' eroding dipole model: $t^{4/3}$ conjecture
- ▶ Feng–Sverak inequality: t^2 bound
- ▶ A first variation of Feng–Sverak inequality: $t^{3/2}$ bound

Outline of the talk

- ▶ Experimental/Numerical works on vortex stretching
- ▶ Classical upper bounds on vortex stretching: $\exp(Ct)$
- ▶ Childress' circulation conserving dipole model: t^2 bound
- ▶ Childress' eroding dipole model: $t^{4/3}$ conjecture
- ▶ Feng–Sverak inequality: t^2 bound
- ▶ A first variation of Feng–Sverak inequality: $t^{3/2}$ bound
- ▶ Another variation of Feng–Sverak inequality: $t^{4/3}$ bound

Colliding vortex rings

The simplest setup for vortex stretching:

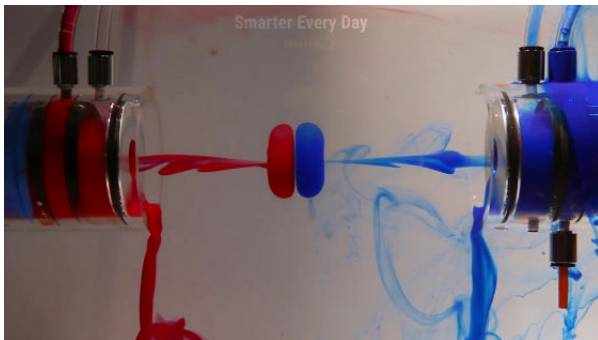
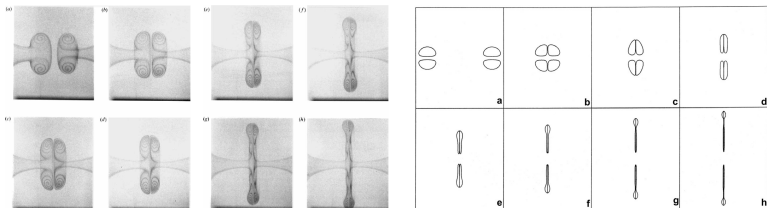


Figure: Two symmetric rings with opposite signs “**anti-parallel**”

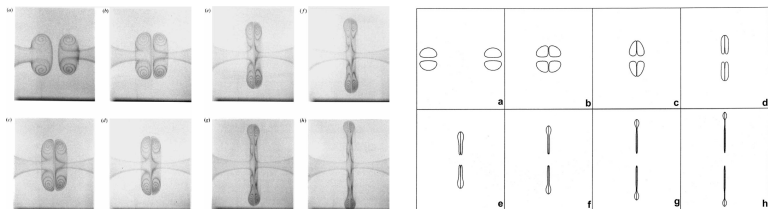
Head-on collision of vortex rings

Extensive experimental works: Oshima (78), Kambe–Minota (83),
Peace–Riley (83), Lim–Nickels (92), Chu–Wang–Chang–Chang–Chang (95), ...



Head-on collision of vortex rings

Extensive experimental works: Oshima (78), Kambe–Minota (83),
Peace–Riley (83), Lim–Nickels (92), Chu–Wang–Chang–Chang–Chang (95), ...

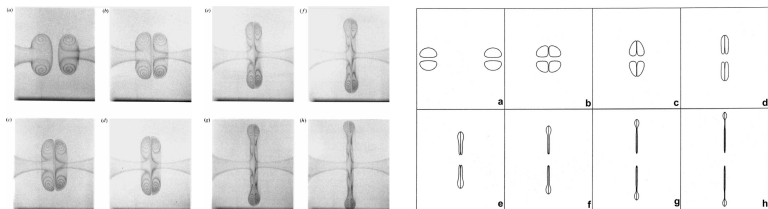


Modeled by odd symmetry and sign assumptions

$$\omega^\theta(r, z) = -\omega^\theta(r, -z), \quad \omega^\theta \leq 0 \quad \text{on} \quad \mathbb{R}_+^3 = \{z > 0\}.$$

Head-on collision of vortex rings

Extensive experimental works: Oshima (78), Kambe–Minota (83), Peace–Riley (83), Lim–Nickels (92), Chu–Wang–Chang–Chang–Chang (95), ...



Modeled by odd symmetry and sign assumptions

$$\omega^\theta(r, z) = -\omega^\theta(r, -z), \quad \omega^\theta \leq 0 \quad \text{on} \quad \mathbb{R}_+^3 = \{z > 0\}.$$

In this geometry, one has monotonicity:

$$\frac{d}{dt} \|r\omega^\theta(t, \cdot)\|_{L^1(\mathbb{R}^3)} = \frac{d}{dt} \iint |r^2 \omega^\theta(t, r, z)| dr dz > 0.$$

Exponential upper bounds

- ▶ Various experiments and numerical computations clearly show large vortex stretching, and the question is the rate in time.

Exponential upper bounds

- ▶ Various experiments and numerical computations clearly show large vortex stretching, and the question is the rate in time.
- ▶ From the axisymmetric Euler equations, we get

$$\frac{d}{dt} \|\omega^\theta\|_{L^\infty} \leq \left\| \frac{u^r \omega^\theta}{r} \right\|_{L^\infty},$$

and how should we bound the RHS in time?

Exponential upper bounds

- Majda's estimate '94: use $|\text{supp}_{\mathbb{R}^3}(\omega^\theta(t, \cdot))| \sim L^3$ and

$$\|u\|_{L^\infty} \lesssim \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x - y|^2} |\omega^\theta(y)| dy \lesssim L \|\omega^\theta\|_{L^\infty},$$

Exponential upper bounds

- Majda's estimate '94: use $|\text{supp}_{\mathbb{R}^3}(\omega^\theta(t, \cdot))| \sim L^3$ and

$$\|u\|_{L^\infty} \lesssim \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x - y|^2} |\omega^\theta(y)| dy \lesssim L \|\omega^\theta\|_{L^\infty},$$

and bound

$$\frac{d}{dt} \|\omega^\theta\|_{L^\infty} \leq \|u\|_{L^\infty} \|\omega^\theta / r\|_{L^\infty} \lesssim \|\omega^\theta\|_{L^\infty}.$$

Exponential upper bounds

- ▶ Majda's estimate '94: use $|\text{supp}_{\mathbb{R}^3}(\omega^\theta(t, \cdot))| \sim L^3$ and

$$\|u\|_{L^\infty} \lesssim \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x - y|^2} |\omega^\theta(y)| dy \lesssim L \|\omega^\theta\|_{L^\infty},$$

and bound

$$\frac{d}{dt} \|\omega^\theta\|_{L^\infty} \leq \|u\|_{L^\infty} \|\omega^\theta/r\|_{L^\infty} \lesssim \|\omega^\theta\|_{L^\infty}.$$

- ▶ Danchin's estimate '00: use weighted Lebesgue inequality

$$\|u/r\|_{L^\infty} \lesssim \|\omega^\theta/r\|_{L^{3,1}}$$

Exponential upper bounds

- ▶ Majda's estimate '94: use $|\text{supp}_{\mathbb{R}^3}(\omega^\theta(t, \cdot))| \sim L^3$ and

$$\|u\|_{L^\infty} \lesssim \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x - y|^2} |\omega^\theta(y)| dy \lesssim L \|\omega^\theta\|_{L^\infty},$$

and bound

$$\frac{d}{dt} \|\omega^\theta\|_{L^\infty} \leq \|u\|_{L^\infty} \|\omega^\theta/r\|_{L^\infty} \lesssim \|\omega^\theta\|_{L^\infty}.$$

- ▶ Danchin's estimate '00: use weighted Lebesgue inequality

$$\|u/r\|_{L^\infty} \lesssim \|\omega^\theta/r\|_{L^{3,1}}$$

to bound

$$\frac{d}{dt} \|\omega^\theta\|_{L^\infty} \leq \|u/r\|_{L^\infty} \|\omega^\theta\|_{L^\infty} \lesssim \|\omega^\theta\|_{L^\infty}.$$

Exponential upper bounds

- Majda's estimate '94: use $|\text{supp}_{\mathbb{R}^3}(\omega^\theta(t, \cdot))| \sim L^3$ and

$$\|u\|_{L^\infty} \lesssim \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x - y|^2} |\omega^\theta(y)| dy \lesssim L \|\omega^\theta\|_{L^\infty},$$

and bound

$$\frac{d}{dt} \|\omega^\theta\|_{L^\infty} \leq \|u\|_{L^\infty} \|\omega^\theta/r\|_{L^\infty} \lesssim \|\omega^\theta\|_{L^\infty}.$$

- Danchin's estimate '00: use weighted Lebesgue inequality

$$\|u/r\|_{L^\infty} \lesssim \|\omega^\theta/r\|_{L^{3,1}}$$

to bound

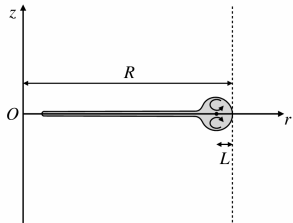
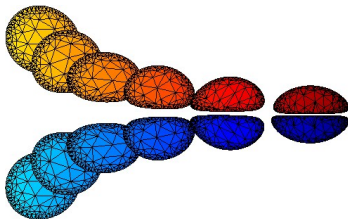
$$\frac{d}{dt} \|\omega^\theta\|_{L^\infty} \leq \|u/r\|_{L^\infty} \|\omega^\theta\|_{L^\infty} \lesssim \|\omega^\theta\|_{L^\infty}.$$

- Both estimates give exponential upper bound.

Childress' t^2 argument using circulation '07

- Assume: $\omega^\theta(t, \cdot)$ is a “blob” centered at $(r, z) = (R(t), 0)$ with radius $L(t)$, where $R(t) \gg L(t)$. Then:

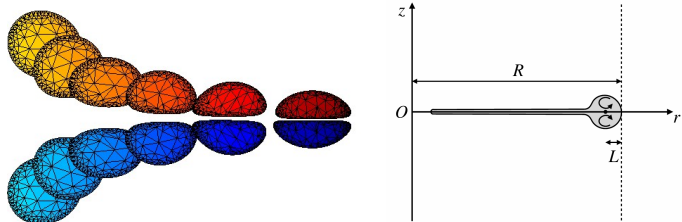
$$\|\omega^\theta/r\|_{L^1(\mathbb{R}^3)} = 2\pi \iint |\omega^\theta(t, r, z)| dr dz \gtrsim R(t)L^2(t).$$



Childress' t^2 argument using circulation '07

- Assume: $\omega^\theta(t, \cdot)$ is a “blob” centered at $(r, z) = (R(t), 0)$ with radius $L(t)$, where $R(t) \gg L(t)$. Then:

$$\|\omega^\theta/r\|_{L^1(\mathbb{R}^3)} = 2\pi \iint |\omega^\theta(t, r, z)| dr dz \gtrsim R(t)L^2(t).$$



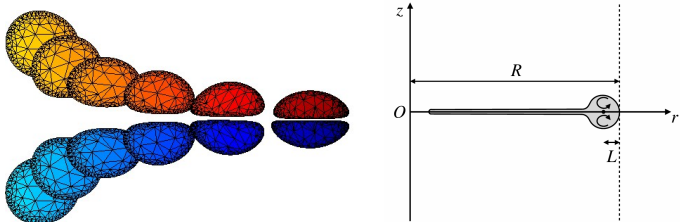
- This dictates $L(t) \lesssim R^{-\frac{1}{2}}(t)$ and then

$$|u(R(t), 0)| \sim \|\omega^\theta\|_{L^\infty} L(t) \sim R(t)L(t) \lesssim R^{\frac{1}{2}}(t).$$

Childress' t^2 argument using circulation '07

- Assume: $\omega^\theta(t, \cdot)$ is a “blob” centered at $(r, z) = (R(t), 0)$ with radius $L(t)$, where $R(t) \gg L(t)$. Then:

$$\|\omega^\theta/r\|_{L^1(\mathbb{R}^3)} = 2\pi \iint |\omega^\theta(t, r, z)| dr dz \gtrsim R(t)L^2(t).$$



- This dictates $L(t) \lesssim R^{-\frac{1}{2}}(t)$ and then

$$|u(R(t), 0)| \sim \|\omega^\theta\|_{L^\infty} L(t) \sim R(t)L(t) \lesssim R^{\frac{1}{2}}(t).$$

- This gives $\frac{d}{dt}R(t) \lesssim R^{\frac{1}{2}}(t)$, i.e. $R(t) \lesssim 1 + t^2$.

Childress' $t^{4/3}$ conjecture '08

- ▶ Childress observes that t^2 growth scenario **contradicts** kinetic energy conservation, namely $\|u\|_{L^2}^2$.

Childress' $t^{4/3}$ conjecture '08

- ▶ Childress observes that t^2 growth scenario **contradicts** kinetic energy conservation, namely $\|u\|_{L^2}^2$.
- ▶ Indeed, recalling that $\|u\|_{L^\infty} \sim \|\omega^\theta\|_{L^\infty} L \sim RL \sim R^{1/2}$ and $|\text{supp}(\omega^\theta)| \sim RL^2 \sim 1$, we have a lower bound

$$\|u\|_{L^2}^2 \gtrsim \|u\|_{L^\infty}^2 RL^2 \sim (R^{\frac{1}{2}})^2 \rightarrow \infty \quad \text{as } R \rightarrow \infty.$$

Childress' $t^{4/3}$ conjecture '08

- ▶ Childress observes that t^2 growth scenario **contradicts** kinetic energy conservation, namely $\|u\|_{L^2}^2$.
- ▶ Indeed, recalling that $\|u\|_{L^\infty} \sim \|\omega^\theta\|_{L^\infty} L \sim RL \sim R^{1/2}$ and $|\text{supp}(\omega^\theta)| \sim RL^2 \sim 1$, we have a lower bound

$$\|u\|_{L^2}^2 \gtrsim \|u\|_{L^\infty}^2 RL^2 \sim (R^{\frac{1}{2}})^2 \rightarrow \infty \quad \text{as } R \rightarrow \infty.$$

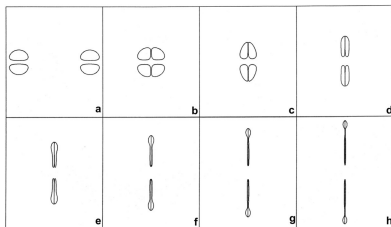
- ▶ But imposing energy conservation contradicts circulation conservation.

Childress' $t^{4/3}$ conjecture '08

- ▶ Childress observes that t^2 growth scenario **contradicts** kinetic energy conservation, namely $\|u\|_{L^2}^2$.
- ▶ Indeed, recalling that $\|u\|_{L^\infty} \sim \|\omega^\theta\|_{L^\infty} L \sim RL \sim R^{1/2}$ and $|\text{supp}(\omega^\theta)| \sim RL^2 \sim 1$, we have a lower bound

$$\|u\|_{L^2}^2 \gtrsim \|u\|_{L^\infty}^2 RL^2 \sim (R^{\frac{1}{2}})^2 \rightarrow \infty \quad \text{as } R \rightarrow \infty.$$

- ▶ But imposing energy conservation contradicts circulation conservation. This is due to strong **filamentation**!



Childress' $t^{4/3}$ conjecture '08

- Now, energy conservation (with filamentation) gives

$$1 \sim \|u\|_{L^2}^2 \sim \|u\|_{L^\infty}^2 RL^2 \sim \|\omega^\theta\|_{L^\infty}^2 RL^4 \sim R^3 L^4$$

Childress' $t^{4/3}$ conjecture '08

- Now, energy conservation (with filamentation) gives

$$1 \sim \|u\|_{L^2}^2 \sim \|u\|_{L^\infty}^2 RL^2 \sim \|\omega^\theta\|_{L^\infty}^2 RL^4 \sim R^3 L^4$$

- This gives $L \sim R^{-3/4}$ and $\|u\|_{L^\infty} \sim RL \sim R^{1/4}$, suggesting

$$R(t) \sim 1 + t^{4/3}.$$

Childress' $t^{4/3}$ conjecture '08

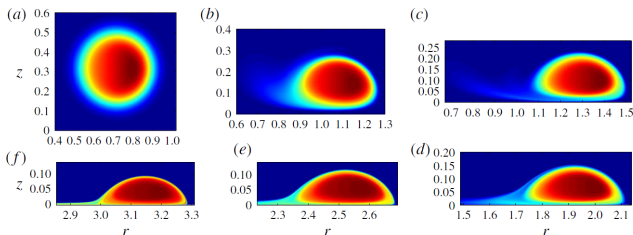
- Now, energy conservation (with filamentation) gives

$$1 \sim \|u\|_{L^2}^2 \sim \|u\|_{L^\infty}^2 RL^2 \sim \|\omega^\theta\|_{L^\infty}^2 RL^4 \sim R^3 L^4$$

- This gives $L \sim R^{-3/4}$ and $\|u\|_{L^\infty} \sim RL \sim R^{1/4}$, suggesting

$$R(t) \sim 1 + t^{4/3}.$$

- Childress–Gilbert–Valiant '16: numerics showing $t^{4/3}$ and convergence to Sadovskii vortex after dynamical rescaling!



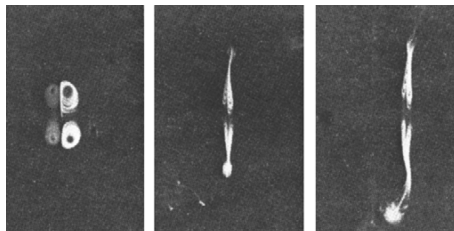
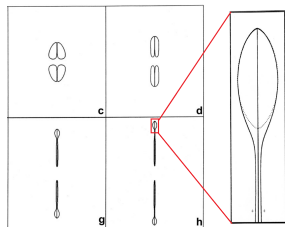
Childress' $t^{4/3}$ conjecture '08

- Namely, numerics suggests that

$$\omega^\theta(t, r, z) \simeq R(t)\Omega(L^{-1}(t)(r - R(t), z)), \quad R(t) \sim t^{4/3},$$

where Ω is a special traveling wave of the 2D Euler equations, called the Sadvorskii vortex (Huang–Tong, Choi–J.–Sim).

- This was already observed in Shariff–Leonard–Ferziger '08 in the case of patches, who also compared it with experiments by Oshima '78. This also shows remarkable convergence to the Sadvorskii vortex patch.



Main result

Theorem (Lim-J., preprint)

For compactly supported data with $\|r^{-1}\omega_0^\theta\|_{L^\infty} < \infty$, we have

$$\|\omega^\theta(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C(1 + t^{4/3})$$

for the solution of axisymmetric Euler without swirl.

Main result

Theorem (Lim–J., preprint)

For compactly supported data with $\|r^{-1}\omega_0^\theta\|_{L^\infty} < \infty$, we have

$$\|\omega^\theta(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C(1 + t^{4/3})$$

for the solution of axisymmetric Euler without swirl.

Remarks

- ▶ No extra conditions are assumed on the data.
- ▶ Applies to **high dimension** axisymmetric Euler equations without swirl, giving global regularity without sign assumption up to dimension **6** (Previously known: 4).
- ▶ For **signed** data, Maffei–Marchioro ('02) gives upper bound of $t^{1/4} \log(e + t)$ based on the conservation of $\|r\omega^\theta\|_{L^1}$.

Feng–Sverak Lemma '15

Lemma (Feng–Sverak '15)

For axisymmetric velocity $u = (u^r, u^z)$,

$$\|u\|_{L^\infty} \lesssim \|r\omega^\theta\|_{L^1}^{\frac{1}{4}} \|\omega^\theta/r\|_{L^1}^{\frac{1}{4}} \|\omega^\theta/r\|_{L^\infty}^{\frac{1}{2}}. \quad (1)$$

Feng–Sverak Lemma '15

Lemma (Feng–Sverak '15)

For axisymmetric velocity $u = (u^r, u^z)$,

$$\|u\|_{L^\infty} \lesssim \|r\omega^\theta\|_{L^1}^{\frac{1}{4}} \|\omega^\theta/r\|_{L^1}^{\frac{1}{4}} \|\omega^\theta/r\|_{L^\infty}^{\frac{1}{2}}. \quad (1)$$

► The t^2 bound is immediate from (1): Rewriting

$$\|r\omega^\theta\|_{L^1}^{\frac{1}{4}} = \|r^2 \cdot \omega^\theta/r\|_{L^1}^{\frac{1}{4}} \leq R^{\frac{1}{2}}(t) \|\omega^\theta/r\|_{L^1}^{\frac{1}{4}},$$

where $R(t)$ is the maximal support radius in r , we get

$$\|u\|_{L^\infty} \lesssim R^{\frac{1}{2}}(t) \|\omega^\theta/r\|_{L^1}^{\frac{1}{2}} \|\omega^\theta/r\|_{L^\infty}^{\frac{1}{2}} \implies R(t) \lesssim 1 + t^2.$$

Feng–Sverak Lemma '15

Lemma (Feng–Sverak '15)

For axisymmetric velocity $u = (u^r, u^z)$,

$$\|u\|_{L^\infty} \lesssim \|r\omega^\theta\|_{L^1}^{\frac{1}{4}} \|\omega^\theta/r\|_{L^1}^{\frac{1}{4}} \|\omega^\theta/r\|_{L^\infty}^{\frac{1}{2}}. \quad (1)$$

- The t^2 bound is immediate from (1): Rewriting

$$\|r\omega^\theta\|_{L^1}^{\frac{1}{4}} = \|r^2 \cdot \omega^\theta/r\|_{L^1}^{\frac{1}{4}} \leq R^{\frac{1}{2}}(t) \|\omega^\theta/r\|_{L^1}^{\frac{1}{4}},$$

where $R(t)$ is the maximal support radius in r , we get

$$\|u\|_{L^\infty} \lesssim R^{\frac{1}{2}}(t) \|\omega^\theta/r\|_{L^1}^{\frac{1}{2}} \|\omega^\theta/r\|_{L^\infty}^{\frac{1}{2}} \implies R(t) \lesssim 1 + t^2.$$

- Q. How are the exponents determined in this inequality?

Understanding Feng–Sverak inequality

- To have

$$\|u\|_{L^\infty} \lesssim \|r\omega^\theta\|_{L^1}^\alpha \|\omega^\theta/r\|_{L^1}^\beta \|\omega^\theta/r\|_{L^\infty}^\gamma,$$

we need

Understanding Feng–Sverak inequality

- To have

$$\|u\|_{L^\infty} \lesssim \|r\omega^\theta\|_{L^1}^\alpha \|\omega^\theta/r\|_{L^1}^\beta \|\omega^\theta/r\|_{L^\infty}^\gamma,$$

we need

- $\alpha + \beta + \gamma = 1$ (homogeneity)
- $-3\alpha - \beta + 2\gamma = 0$ (3D scaling)

Understanding Feng–Sverak inequality

- To have

$$\|u\|_{L^\infty} \lesssim \|r\omega^\theta\|_{L^1}^\alpha \|\omega^\theta/r\|_{L^1}^\beta \|\omega^\theta/r\|_{L^\infty}^\gamma,$$

we need

- $\alpha + \beta + \gamma = 1$ (homogeneity)
- $-3\alpha - \beta + 2\gamma = 0$ (3D scaling)
- *Two-dimensionalization*: for ω^θ localized near $(1, 0)$, the velocity kernel is well approximated by 2D Biot–Savart law.

$$\|u\|_{L^\infty} \lesssim \|\omega^\theta\|_{L^1}^{\alpha+\beta} \|\omega^\theta\|_{L^\infty}^\gamma$$

and this forces $\alpha + \beta = \frac{1}{2} = \gamma$ (2D scaling).

Two-dimensional velocity estimate

Lemma

For $\omega_{2D} : \mathbb{R}^2 \rightarrow \mathbb{R}$, $u_{2D} = \nabla^\perp \Delta^{-1} \omega_{2D}$ satisfies

$$\|u_{2D}\|_{L^\infty} \leq C \|\omega_{2D}\|_{L^1}^{1/2} \|\omega_{2D}\|_{L^\infty}^{1/2}.$$

Two-dimensional velocity estimate

Lemma

For $\omega_{2D} : \mathbb{R}^2 \rightarrow \mathbb{R}$, $u_{2D} = \nabla^\perp \Delta^{-1} \omega_{2D}$ satisfies

$$\|u_{2D}\|_{L^\infty} \leq C \|\omega_{2D}\|_{L^1}^{1/2} \|\omega_{2D}\|_{L^\infty}^{1/2}.$$

► We recall that

$$u_{2D}(0) = \int_{\mathbb{R}^2} \frac{y^\perp}{2\pi|y|^2} \omega_{2D}(y) dy \implies |u_{2D}(0)| \lesssim \int_{\mathbb{R}^2} \frac{|\omega_{2D}(y)|}{|y|} dy.$$

Two-dimensional velocity estimate

Lemma

For $\omega_{2D} : \mathbb{R}^2 \rightarrow \mathbb{R}$, $u_{2D} = \nabla^\perp \Delta^{-1} \omega_{2D}$ satisfies

$$\|u_{2D}\|_{L^\infty} \leq C \|\omega_{2D}\|_{L^1}^{1/2} \|\omega_{2D}\|_{L^\infty}^{1/2}.$$

- We recall that

$$u_{2D}(0) = \int_{\mathbb{R}^2} \frac{y^\perp}{2\pi|y|^2} \omega_{2D}(y) dy \implies |u_{2D}(0)| \lesssim \int_{\mathbb{R}^2} \frac{|\omega_{2D}(y)|}{|y|} dy.$$

- We divide the integral using a cutoff length K

$$\int_{|y| < K} \frac{|\omega_{2D}(y)|}{|y|} dy \lesssim K \|\omega_{2D}\|_{L^\infty}, \quad \int_{|y| \geq K} \frac{|\omega_{2D}(y)|}{|y|} dy \lesssim \frac{\|\omega_{2D}\|_{L^1}}{K}.$$

Two-dimensional velocity estimate

Lemma

For $\omega_{2D} : \mathbb{R}^2 \rightarrow \mathbb{R}$, $u_{2D} = \nabla^\perp \Delta^{-1} \omega_{2D}$ satisfies

$$\|u_{2D}\|_{L^\infty} \leq C \|\omega_{2D}\|_{L^1}^{1/2} \|\omega_{2D}\|_{L^\infty}^{1/2}.$$

- We recall that

$$u_{2D}(0) = \int_{\mathbb{R}^2} \frac{y^\perp}{2\pi|y|^2} \omega_{2D}(y) dy \implies |u_{2D}(0)| \lesssim \int_{\mathbb{R}^2} \frac{|\omega_{2D}(y)|}{|y|} dy.$$

- We divide the integral using a cutoff length K

$$\int_{|y| < K} \frac{|\omega_{2D}(y)|}{|y|} dy \lesssim K \|\omega_{2D}\|_{L^\infty}, \quad \int_{|y| \geq K} \frac{|\omega_{2D}(y)|}{|y|} dy \lesssim \frac{\|\omega_{2D}\|_{L^1}}{K}.$$

- Optimizing in K gives the bound.

Understanding Feng–Sverak inequality further

- Need a pointwise estimate for

$$u(\bar{r}, \bar{z}) = \iint_{\{r>0, z\in\mathbb{R}\}} \mathcal{F}[\bar{r}, r, z - \bar{z}] \omega^\theta(r, z) dr dz,$$

where $\mathcal{F}$ is the axisymmetric velocity kernel.

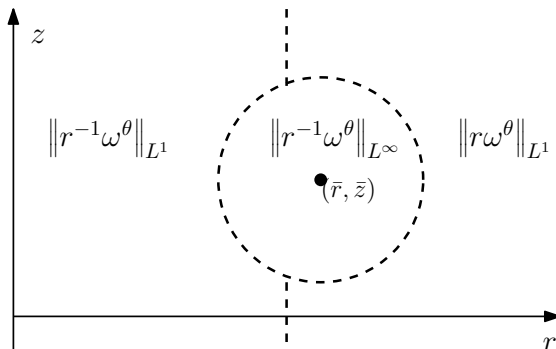
Understanding Feng–Sverak inequality further

- Need a pointwise estimate for

$$u(\bar{r}, \bar{z}) = \iint_{\{r>0, z \in \mathbb{R}\}} \mathcal{F}[\bar{r}, r, z - \bar{z}] \omega^\theta(r, z) dr dz,$$

where $\mathcal{F}$ is the axisymmetric velocity kernel.

- Split the domain and optimize :



First variant of Feng–Sverak inequality and $t^{3/2}$ bound

- Observe: Childress' t^2 proof and Feng–Sverak inequality both uses $\|\omega^\theta/r\|_{L^1}$: “circulation”.

First variant of Feng–Sverak inequality and $t^{3/2}$ bound

- Observe: Childress' t^2 proof and Feng–Sverak inequality both uses $\|\omega^\theta/r\|_{L^1}$: “circulation”. Then, Childress' $t^{4/3}$ conjecture involves the kinetic energy.

First variant of Feng–Sverak inequality and $t^{3/2}$ bound

- ▶ Observe: Childress' t^2 proof and Feng–Sverak inequality both uses $\|\omega^\theta/r\|_{L^1}$: “circulation”. Then, Childress' $t^{4/3}$ conjecture involves the kinetic energy.
- ▶ Goal: find the inequality involving conserved quantities including $\|u\|_{L^2}$, having the smallest exponent on $\|r\omega^\theta\|_{L^1}$.

First variant of Feng–Sverak inequality and $t^{3/2}$ bound

- ▶ Observe: Childress' t^2 proof and Feng–Sverak inequality both uses $\|\omega^\theta/r\|_{L^1}$: “circulation”. Then, Childress' $t^{4/3}$ conjecture involves the kinetic energy.
- ▶ Goal: find the inequality involving conserved quantities including $\|u\|_{L^2}$, having the smallest exponent on $\|r\omega^\theta\|_{L^1}$.
- ▶ We could get in the end, by replacing circulation with energy,

$$\|u\|_{L^\infty} \lesssim \|r\omega^\theta\|_{L^1}^{1/6} \|\omega^\theta/r\|_{L^\infty}^{1/2} \|u\|_{L^2}^{1/3}.$$

First variant of Feng–Sverak inequality and $t^{3/2}$ bound

- ▶ Observe: Childress' t^2 proof and Feng–Sverak inequality both uses $\|\omega^\theta/r\|_{L^1}$: “circulation”. Then, Childress' $t^{4/3}$ conjecture involves the kinetic energy.
- ▶ Goal: find the inequality involving conserved quantities including $\|u\|_{L^2}$, having the smallest exponent on $\|r\omega^\theta\|_{L^1}$.
- ▶ We could get in the end, by replacing circulation with energy,

$$\|u\|_{L^\infty} \lesssim \|r\omega^\theta\|_{L^1}^{1/6} \|\omega^\theta/r\|_{L^\infty}^{1/2} \|u\|_{L^2}^{1/3}.$$

- ▶ This only gives the upper bound of $t^{3/2}$, and did not seem possible to lower the exponent $1/6$.

First variant of Feng–Sverak inequality and $t^{3/2}$ bound

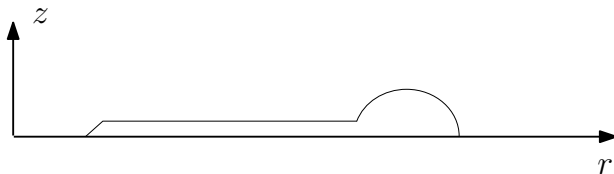
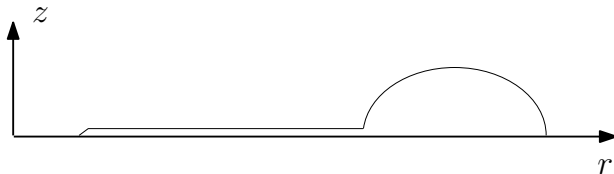
- ▶ Observe: Childress' t^2 proof and Feng–Sverak inequality both uses $\|\omega^\theta/r\|_{L^1}$: “circulation”. Then, Childress' $t^{4/3}$ conjecture involves the kinetic energy.
- ▶ Goal: find the inequality involving conserved quantities including $\|u\|_{L^2}$, having the smallest exponent on $\|r\omega^\theta\|_{L^1}$.
- ▶ We could get in the end, by replacing circulation with energy,

$$\|u\|_{L^\infty} \lesssim \|r\omega^\theta\|_{L^1}^{1/6} \|\omega^\theta/r\|_{L^\infty}^{1/2} \|u\|_{L^2}^{1/3}.$$

- ▶ This only gives the upper bound of $t^{3/2}$, and did not seem possible to lower the exponent $1/6$.
- ▶ This is wasteful since we just need the velocity estimate **on the maximal support** (Marchioro '94). However, is there any reason to believe that we get a better estimate there?

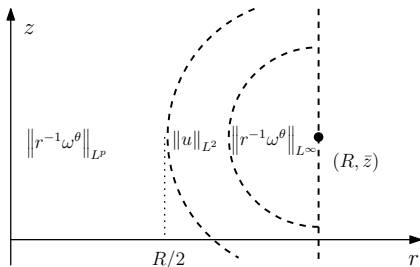
Strong versus weak filamentation

- ▶ Due to **strong filamentation**, the velocity maximum may not be achieved around the maximum support.
- ▶ This is genuinely different from 2D under odd symmetry.



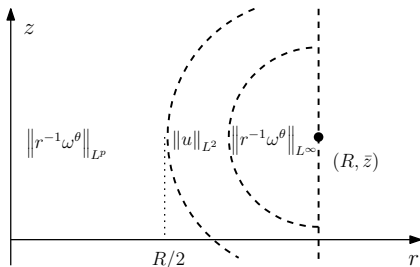
Splitting the regions: axial and two-dimensional

- New idea: estimate the velocity on $(R, \bar{z})$ and decompose the integral into “axial” and “two-dimensional” regions.



Splitting the regions: axial and two-dimensional

- New idea: estimate the velocity on $(R, \bar{z})$ and decompose the integral into “axial” and “two-dimensional” regions.

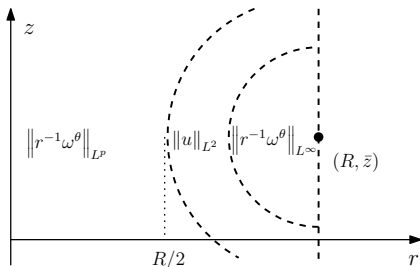


- Contribution from the axial region $(\mathbb{R}_+ \times \mathbb{R}) \setminus B_{(R, \bar{z})}(R/2)$:

$$\|\omega^\theta / r\|_{L^\infty}^{1/3} \|\omega^\theta / r\|_{L^1}^{2/3}.$$

Splitting the regions: axial and two-dimensional

- ▶ New idea: estimate the velocity on $(R, \bar{z})$ and decompose the integral into “axial” and “two-dimensional” regions.



- ▶ Contribution from the axial region $(\mathbb{R}_+ \times \mathbb{R}) \setminus B_{(R, \bar{z})}(R/2)$:

$$\|\omega^\theta / r\|_{L^\infty}^{1/3} \|\omega^\theta / r\|_{L^1}^{2/3}.$$

- ▶ Contribution from the 2D region $B_{(R, \bar{z})}(R/2)$:

$$R^{1/4} \|\omega^\theta / r\|_{L^\infty}^{1/2} \|u\|_{L^2}^{1/2}.$$

Understanding the final estimate

Lemma

In the “2D region” $B = B_{(R,\bar{z})}(R/2)$, we have

$$\left| \iint_B \mathcal{F}[R, r, z - \bar{z}] \omega^\theta(r, z) dr dz \right| \lesssim R^{1/4} \|\omega^\theta / r\|_{L^\infty}^{1/2} \|u\|_{L^2}^{1/2}.$$

Understanding the final estimate

Lemma

In the “2D region” $B = B_{(R,\bar{z})}(R/2)$, we have

$$\left| \iint_B \mathcal{F}[R, r, z - \bar{z}] \omega^\theta(r, z) dr dz \right| \lesssim R^{1/4} \|\omega^\theta / r\|_{L^\infty}^{1/2} \|u\|_{L^2}^{1/2}.$$

- Key observation: in 2D vorticity equation,

$$\|u_{2D}\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\omega_{2D}\|_{L^\infty(\mathbb{R}^2)}^{1/2} \|u_{2D}\|_{L^2(\mathbb{R}^2)}^{1/2}.$$

Understanding the final estimate

Lemma

In the “2D region” $B = B_{(R,\bar{z})}(R/2)$, we have

$$\left| \iint_B \mathcal{F}[R, r, z - \bar{z}] \omega^\theta(r, z) dr dz \right| \lesssim R^{1/4} \|\omega^\theta / r\|_{L^\infty}^{1/2} \|u\|_{L^2}^{1/2}.$$

- Key observation: in 2D vorticity equation,

$$\|u_{2D}\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\omega_{2D}\|_{L^\infty(\mathbb{R}^2)}^{1/2} \|u_{2D}\|_{L^2(\mathbb{R}^2)}^{1/2}.$$

- Incorporating the metric factor $2D \mapsto 3D$:

$$\|u_{2D}\|_{L^2(\mathbb{R}^2)} \mapsto R^{-1/2} \|u\|_{L^2(B)}, \quad \|\omega_{2D}\|_{L^\infty(\mathbb{R}^2)} \mapsto R \|\omega^\theta / r\|_{L^\infty(B)},$$

$$\|\omega_{2D}\|_{L^\infty(\mathbb{R}^2)}^{1/2} \|u_{2D}\|_{L^2(\mathbb{R}^2)}^{1/2} \mapsto R^{1/4} \|\omega^\theta / r\|_{L^\infty}^{1/2} \|u\|_{L^2}^{1/2}.$$

Second inequality in the 2D case

We want a proof of $\|u_{2D}\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\omega_{2D}\|_{L^\infty(\mathbb{R}^2)}^{1/2} \|u_{2D}\|_{L^2(\mathbb{R}^2)}^{1/2}$ which generalizes to the 3D axisymmetric case.

Second inequality in the 2D case

We want a proof of $\|u_{2D}\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\omega_{2D}\|_{L^\infty(\mathbb{R}^2)}^{1/2} \|u_{2D}\|_{L^2(\mathbb{R}^2)}^{1/2}$ which generalizes to the 3D axisymmetric case.

- Split the 2D kernel as before and recall

$$\left| \int_{|y| < K} \frac{y^\perp}{|y|^2} \omega_{2D}(y) dy \right| \lesssim K \|\omega_{2D}\|_{L^\infty}.$$

Second inequality in the 2D case

We want a proof of $\|u_{2D}\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\omega_{2D}\|_{L^\infty(\mathbb{R}^2)}^{1/2} \|u_{2D}\|_{L^2(\mathbb{R}^2)}^{1/2}$ which generalizes to the 3D axisymmetric case.

- Split the 2D kernel as before and recall

$$\left| \int_{|y| < K} \frac{y^\perp}{|y|^2} \omega_{2D}(y) dy \right| \lesssim K \|\omega_{2D}\|_{L^\infty}.$$

- In the far field region, we now integrate by parts:

$$\left| \int_{|y| \geq K} \frac{y^\perp}{|y|^2} \omega_{2D}(y) dy \right| = \left| \int_{|y| \geq K} \frac{y^\perp}{|y|^2} \nabla_y \times u_{2D}(y) dy \right|$$

Second inequality in the 2D case

We want a proof of $\|u_{2D}\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\omega_{2D}\|_{L^\infty(\mathbb{R}^2)}^{1/2} \|u_{2D}\|_{L^2(\mathbb{R}^2)}^{1/2}$ which generalizes to the 3D axisymmetric case.

- Split the 2D kernel as before and recall

$$\left| \int_{|y| < K} \frac{y^\perp}{|y|^2} \omega_{2D}(y) dy \right| \lesssim K \|\omega_{2D}\|_{L^\infty}.$$

- In the far field region, we now integrate by parts:

$$\begin{aligned} \left| \int_{|y| \geq K} \frac{y^\perp}{|y|^2} \omega_{2D}(y) dy \right| &= \left| \int_{|y| \geq K} \frac{y^\perp}{|y|^2} \nabla_y \times u_{2D}(y) dy \right| \\ &\simeq \left| \int_{|y| \geq K} \nabla_y \times \left(\frac{y^\perp}{|y|^2} \right) u_{2D}(y) dy \right| \lesssim \frac{1}{K} \|u_{2D}\|_{L^2}. \end{aligned}$$

Summary and Final Remarks

- ▶ We proved the **upper** bound $t^{4/3}$, and there is numerical evidence for its sharpness (with rescaled Sadvskii vortex).
- ▶ Without compact support assumption, the growth rate might be faster, possibly $t^{3/2}$.
- ▶ The best **lower** bound so far is *sublinear*: $t^{3/8-\varepsilon}$ (Choi–J., Gustafson–Miller–Tsai preprint). This actually shows the growth of the “average” of the vortex.
- ▶ With **signed** vorticity, the best upper bound is $t^{1/4} \log(e + t)$ but no evidence for **any** infinite growth in infinite time.

Thank you for your attention!