Recent progress on the distillability problem

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The talk is based on two papers:


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The distillability problem and entanglement distillation

$M \times N$ NPT states of rank $\max\{M, N\}$

Open problems
The distillability problem and entanglement distillation

$m \times n$ NPT states of rank $\max\{m, n\}$

- The distillability problem and entanglement distillation
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Open problems
Outlines

- The distillability problem and entanglement distillation
- $M \times N$ NPT states of rank $\max\{M, N\}$
- Two-qutrit NPT states of rank four and five
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- The distillability problem and entanglement distillation
- $M \times N$ NPT states of rank $\max\{M, N\}$
- Two-qutrit NPT states of rank four and five
- Open problems
Entanglement distillation

- Pure entangled states are essential resources in quantum information
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- Pure entangled states become mixed entangled states by noise
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**Entanglement distillation.** Bennett et al, 1996.

**Definition**

We transform $N$ copies of an arbitrary entangled state $\rho$ into a pure entangled state $|\psi\rangle$ asymptotically under local operations and classical communications (LOCC).
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**Definition**
If pure entangled states are obtained then $\rho$ is distillable.
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Entanglement distillation

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**Definition**
If pure entangled states are obtained then \( \rho \) is distillable.

- and

**Definition**
If no pure entangled states can be obtained, then \( \rho \) is not distillable, or equivalently \( \rho \) is undistillable.
Distillability problem

- The positive-partial-transpose (PPT) states are not distillable.
The distillability problem and entanglement distillation

$M \times N$ NPT states of rank $\max\{M, N\}$

Distilling two-qutrit NPT states

## Distillability problem

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**Distillability problem.** Is every NPT state distillable?
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- Proof of the existence 2-undistillable NPT Werner states: **Not found yet.**
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- Proof of the existence 2-undistillable NPT Werner states: Not found yet.

- Attempts for the proof: Yes, there is something...
Attempts to solve the distillability problem

- Any NPT state is convertible to an NPT Werner state,
  Divincenzo et al, Dur et al 2000
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Distilling two-qutrit NPT states

PPT and NPT

**Definition**

The partial transpose of a bipartite quantum state $\rho$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ is computed in an orthonormal (o .n.) basis $\{|a_i\rangle\}$ of system A, is defined by

$$\rho^\Gamma := \sum_{ij} |a_i\rangle\langle a_j| \otimes \langle a_j|\rho|a_i\rangle.$$
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**Definition**

$\rho$ is **PPT** if the partial transpose of $\rho$ is positive semidefinite. Otherwise, $\rho$ is **NPT**.

For example, all separable states are PPT. All pure entangled states are NPT.
The distillability problem and entanglement distillation

Distilling two-qutrit NPT states of rank four

Open problems

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The **partial transpose** of a bipartite quantum state \( \rho \) acting on \( \mathcal{H}_A \otimes \mathcal{H}_B \) is computed in an orthonormal (o.n.) basis \( \{|a_i\rangle\} \) of system A, is defined by

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Example. If

$$\rho = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$
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The mathematical formulation of distillability problem

The distillability problem and entanglement distillation

\( M \times N \) NPT states of rank \( \max\{M, N\} \)

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The mathematical formulation of distillability problem


<table>
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The distillability problem and entanglement distillation

$M \times N$ NPT states of rank $\max\{M, N\}$

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**Definition**

$\rho$ is **1-distillable** if there exists a pure bipartite state $|\psi\rangle$ of Schmidt rank two such that $\langle \psi | \rho \Gamma | \psi \rangle < 0$.

Otherwise, $\rho$ is **1-undistillable**.
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(1) \( \rho \) is \( n \)-distillable if the bipartite state \( \rho \otimes^n \) is 1-distillable.
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for a bipartite state \( |\psi\rangle \) of Schmidt rank two.
The distillability problem and entanglement distillation

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The distillability problem and entanglement distillation

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Distilling two-qutrit NPT states

The math/mess of many-copy states

- $\rho^{\otimes n} = \rho_{A_1B_1} \otimes \cdots \otimes \rho_{A_nB_n} \equiv \rho_{A_1\ldots A_n:B_1\ldots B_n}$. 

Example. Consider the “critical” Werner state

$$\rho_{A_1B_1} = \sum_{i,j} \left( |i,j\rangle \langle i,j| - \frac{1}{2} |j,i\rangle \langle j,i| \right)$$

$$\rho_{A_2B_2} = \sum_{m,n} \left( |m,n\rangle \langle m,n| - \frac{1}{2} |n,m\rangle \langle n,m| \right)$$

Then

$$\rho^{\otimes 2} = \rho_{A_1B_1} \otimes \rho_{A_2B_2} = \sum_{i,j,m,n} \left( |im,jn\rangle \langle im,jn| - \frac{1}{2} |im,jn\rangle \langle jn,im| + \frac{1}{4} |im,jn\rangle \langle jn,im| \right)$$
The distillability problem and entanglement distillation

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- \( \rho \otimes^n = \rho_{A_1B_1} \otimes \cdots \otimes \rho_{A_nB_n} := \rho_{A_1\cdots A_n:B_1\cdots B_n} \).

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List of 1-distillable NPT states

- We say a bipartite state $\rho_{AB}$ is $M \times N$ if $\text{rank} \rho_A = M$ and $\text{rank} \rho_B = N$. 
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- $M \times N$ states of rank $\max\{M, N\}$
  - Horodecki et al, 1999
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  - LC and DZ, 2011
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- Two-qutrit states of rank four
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The strategy of entanglement distillation

- Convert the target state $\rho$ or $\rho \otimes^n$ to a distillable state by LOCC.
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- Experience: $n = 2$ is hard!
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- Convert the target state $\rho$ or $\rho^\otimes n$ to a distillable state by LOCC.
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- The normalization factors of quantum states are often ignored in entanglement distillation because it does not affect the distillability of quantum states.
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Example 1. If

$$P = |1\rangle\langle 1| + |2\rangle\langle 2|,$$
$$\rho = (|11\rangle + |22\rangle)(\langle 11| + \langle 22|) + |33\rangle\langle 33|,$$
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then

$$(P \otimes I_B)\rho(P \otimes I_B) = (|11\rangle + |22\rangle)(\langle 11| + \langle 22|)$$

is a Bell state.
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- So \( \rho \) is 1-distillable.
The strategy of entanglement distillation

**Example 2.** If

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then

\[ (P \otimes I_B)\rho(P \otimes I_B) \]

is a two-qubit mixed entangled state. So \( \rho \) is also distillable.
The difficulty of entanglement distillation

- Finding a good $P$ is hard, although $P$ belongs to LOCC.
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- When is $(P \otimes I_B)\rho(P \otimes I_B)$ entangled?
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- Finding a good $P$ is hard, although $P$ belongs to LOCC.
- When is $(P \otimes I_B)\rho(P \otimes I_B)$ entangled?
- What if $(P \otimes I_B)\rho(P \otimes I_B)$ is PPT?
The difficulty of entanglement distillation

- Finding a good $P$ is hard, although $P$ belongs to LOCC.

- When is $(P \otimes I_B)\rho(P \otimes I_B)$ entangled?

- What if $(P \otimes I_B)\rho(P \otimes I_B)$ is PPT?

- A popular trick: let $(P \otimes I_B)\rho(P \otimes I_B)$ be a $2 \times N$ state then it has to be PPT, or some entries have to be zero.
The distillability problem and entanglement distillation

$M \times N$ NPT states of rank $\max\{M, N\}$

- Distilling two-qutrit NPT states of rank four

Open problems
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$


**Lemma**

$M \times N$ NPT states of rank smaller than $\max\{M, N\}$ is 1-distillable.
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$


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Distilling $M \times N$ NPT states of rank $\max\{M, N\}$


**Lemma**

$M \times N$ NPT states of rank **smaller than** $\max\{M, N\}$ is 1-distillable.

- LC and DZ, 2011.

**Lemma**

$M \times N$ NPT states of rank **equal to** $\max\{M, N\}$ is 1-distillable.
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- **LFRP.** Let $\rho_{AB}$ be an $M \times N$ NPT states of rank $\max\{M, N\}$. We say $\rho_{AB}$ has left full-rank property (LFRP) if there is some state $|x\rangle$ such that $\langle x|B\rho_{AB}|x\rangle_B$ is invertible.
Distilling \( M \times N \) NPT states of rank \( \max\{M, N\} \)

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- **Example.** If

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\rho_{AB} = (|11\rangle + |22\rangle)(\langle 11 | + \langle 22 |) + |33\rangle\langle 33 |
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\sigma_{AB} = (|11\rangle + |22\rangle)(\langle 11 | + \langle 22 |) + |22\rangle\langle 22 | + |33\rangle\langle 33 |
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  then

  $$\max_x \left( \text{rank}(\langle x|_B \rho_{AB} |x\rangle_B) \right) = 2 < \text{rank} \rho_A = 3.$$  

  $$\max_x \left( \text{rank}(\langle x|_B \sigma_{AB} |x\rangle_B) \right) = 3 = \text{rank} \rho_A = 3.$$
So $\rho_{AB}$ has no LFRP, and $\sigma_{AB}$ has LFRP.
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- So $\rho_{AB}$ has no LFRP, and $\sigma_{AB}$ has LFRP.

- The right full-rank property (RFRP) can be similarly defined.
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- So $\rho_{AB}$ has no LFRP, and $\sigma_{AB}$ has LFRP.

- The right full-rank property (RFRP) can be similarly defined.

- **Strategy of proof.** Prove that $\rho_{AB}$ is 1-distillable when
  (1) $\rho_{AB}$ has no LFRP or RFRP, and
  (2) $\rho_{AB}$ has LFRP and RFRP.
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- (1) $\rho_{AB}$ has no LFRP or RFRP.
  Using the matrix decomposition of semidefinite positive matrix $\rho = C^\dagger C$, where

  $$C = (C_1, \ldots, C_i, \ldots, C_M)$$

  and each matrix $C_i$ is of size $(\text{rank } \rho) \times N$.  

When $M \times N$ NPT states of rank $\max\{M, N\}$
(1) \( \rho_{AB} \) has no LFRP or RFRP.

Using the matrix decomposition of semidefinite positive matrix \( \rho = C^\dagger C \), where

\[
C = (C_1, \ldots, C_i, \ldots, C_M)
\]

and each matrix \( C_i \) is of size \((\text{rank } \rho) \times N\).

Project \( \rho \) to the following state by using the projector

\[
P = |1\rangle\langle 1| + |i\rangle\langle i|
\]

\[
\rho_{1,i} = (P \otimes I_B)\rho(P \otimes I_B)
\]

\[
= (C_1, C_i)^\dagger \cdot (C_1, C_i) = \begin{pmatrix}
C_1^\dagger C_1 & C_1^\dagger C_i \\
C_i^\dagger C_1 & C_i^\dagger C_i
\end{pmatrix}
\]
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- We split each $C_i$ into four blocks $C_i = \begin{pmatrix} C_{i1} & C_{i2} \\ C_{i3} & C_{i4} \end{pmatrix}$ with $C_{i1}$ square of size $r_1$, where $C_1 = I_{r_1} \oplus 0$ because of $\rho$ has no LFRP or RFRP. We have

$$\rho_{1,i} = \begin{pmatrix} I_{r_1} & 0 & \cdots & C_{i1} & C_{i2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{i1}^\dagger & 0 & \cdots & * & * \\ C_{i2}^\dagger & 0 & \cdots & * & * \end{pmatrix},$$

where $i > 1$ and the asterisk stands for an unspecified block.
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, where $i > 1$ and the asterisk stands for an unspecified block.

- If some $C_{i2} \neq 0$, then $\rho$ is 1-distillable. Thus we may assume that all $C_{i2} = 0$. 

...
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

Now $\rho = C^\dagger C$ where

$$C = \begin{bmatrix}
(I_{r_1} & 0) , & (C_{21} & 0) , & \cdots , & (C_{M1} & 0) \\
0 & 0 \\
C_{23} & C_{24} \\
C_{M3} & C_{M4}
\end{bmatrix}$$
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- Now $\rho = C^\dagger C$ where

\[
C = \begin{bmatrix}
\left( I_{r_1} \ 0 \right), \left( C_{21} \ 0 \right), \ldots, \left( C_{M1} \ 0 \right)
\end{bmatrix}
\]

- Since $\rho$ has no LFRP or RFRP, the linear combination of $C_{21}, \ldots, C_{N1}$ is of deficient rank. We may assume

\[
C_{24} = \begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix}
\]

and

\[
C_{i4} = \begin{pmatrix} C_{i41} & C_{i42} \\ C_{i43} & C_{i44} \end{pmatrix}
\]
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- Project $\rho$ to the state $(C')^\dagger C'$ where

$$C' = \begin{bmatrix}
    \left( I_r \quad 0 \right), \\
    \left( C_{341} \quad C_{342} \right), \ldots, \\
    \left( C_{M41} \quad C_{M42} \right)
\end{bmatrix}$$
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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$$C' = \begin{bmatrix}
    \begin{pmatrix}
        I_r & 0 \\
        0 & 0
    \end{pmatrix},
    \begin{pmatrix}
        C_{341} & C_{342} \\
        C_{343} & C_{344}
    \end{pmatrix},
    \ldots,
    \begin{pmatrix}
        C_{M41} & C_{M42} \\
        C_{M43} & C_{M44}
    \end{pmatrix}
\end{bmatrix}$$

- Repeating the above argument one can show the blocks $C_{i42} = 0$. 
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- Project $\rho$ to the state $(C')^\dagger C'$ where

$$C' = \begin{bmatrix}
(I_{r_2} & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
C_{341} & C_{342} \\
C_{343} & C_{344}
\end{bmatrix}, \ldots, \begin{bmatrix}
C_{M41} & C_{M42} \\
C_{M43} & C_{M44}
\end{bmatrix}\]$$

- Repeating the above argument one can show the blocks $C_{i42} = 0$.

- Then we have $\rho = C^\dagger C$ where $C$ is

$$\begin{bmatrix}
(I_{r_1} & 0 & 0) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
C_{21} & 0 & 0 \\
C_{221} & I_{r_2} & 0 \\
C_{223} & 0 & 0
\end{bmatrix}, \begin{bmatrix}
C_{31} & 0 & 0 \\
C_{321} & C_{341} & 0 \\
C_{323} & C_{343} & C_{344}
\end{bmatrix}, \ldots, \begin{bmatrix}
C_{M1} & 0 & 0 \\
C_{M21} & C_{M41} & 0 \\
C_{M23} & C_{M43} & C_{M44}
\end{bmatrix}\]
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- The process continues and the facts $C_{i2} = C_{i42} = \cdots = 0$ implies that $\rho$ has RFRP. It is a contradiction and we obtain that the process must terminate.
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So \( \rho \) is distillable when it has no LFRP or RFRP.
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- (2) $\rho_{AB}$ has LFRP and RFRP.
  
  Using the matrix decomposition of semidefinite positive matrix $\rho = C^\dagger C$, we have

  $$\rho = (C_1, \ldots, C_{M-1}, I_N)^\dagger \cdot (C_1, \ldots, C_{M-1}, I_N)$$
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Since $\rho$ is NPT, there exist $i, j$ such that $[C_i, C_j] \neq 0$. 


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- Since $\rho$ is NPT, there exist $i, j$ such that $[C_i, C_j] \neq 0$.
- One can show that $(xC_i + C_j, I_N)^\dagger \cdot (xC_i + C_j, I_N)$ is distillable for some complex number $x$. 
Conclusion 1: the LFRP (RFRP) is a key property for the distillation.
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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- Since any state lacking LFRP or RFRP is distillable, we have
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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**Corollary**

*All non-distillable states, e.g., bipartite PPT states possess LFRP and RFRP.*
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**Corollary**

All non-distillable states, e.g., bipartite PPT states possess LFRP and RFRP.

**Corollary**

The bipartite state of rank four is separable if and only if it is PPT and its range contains at least one product state.
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- Application 1:
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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Lemma

For a tripartite pure state $\rho = |\psi\rangle\langle\psi|$, the bipartite reduced density operators $\rho_{AB}$ and $\rho_{AC}$ are PPT if and only if $|\psi\rangle = \sum_{i} |a_{i}\rangle|i\rangle$ up to local unitary operations.
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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**Lemma**

For a tripartite pure state $\rho = |\psi\rangle\langle\psi|$, the bipartite reduced density operators $\rho_{AB}$ and $\rho_{AC}$ are PPT if and only if $|\psi\rangle = \sum_i |a_i\rangle|ii\rangle$ up to local unitary operations.

- So

$$\rho_{AB} = \rho_{AC} = \sum_i |a_i, i\rangle\langle a_i, i|$$

are both separable states.
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- Application 2: In quantum information, the following six criteria are extensively useful for studying bipartite states $\rho_{AB}$ in the space $\mathcal{H}_A \otimes \mathcal{H}_B$. 

1. Separability.
2. PPT condition.
4. Reduction criterion: $\rho_{A \otimes I} \geq \rho_{AB}$ and $I_{A} \otimes \rho_{B} \geq \rho_{AB}$, Horodecki et al, 1999.
5. Majorization criterion: $\rho_A \succ \rho_{AB}$ and $\rho_B \succ \rho_{AB}$, Hiroshima, 2003.
6. Conditional entropy criterion: $H(\rho_B | A) = H(\rho_{AB}) - H(\rho_A) \geq 0$ and $H(\rho_A | B) = H(\rho_{AB}) - H(\rho_B) \geq 0$, where $H$ is the von Neumann entropy.
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*Horodecki et al, 1999.*
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Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- Masahito Hayashi and LC, 2011.
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**Theorem**

*For a tripartite state $|\psi\rangle_{ABC}$ with a non-distillable reduced state $\rho_{BC}$ namely condition (3), then conditions (1)-(6) are equivalent for $\rho_{AB}$.***
Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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**Theorem**

For a tripartite state $|\psi\rangle_{ABC}$ with a non-distillable reduced state $\rho_{BC}$ namely condition (3), then conditions (1)-(6) are equivalent for $\rho_{AB}$.

- It is a way of unifying the six well-known conditions.
The distillability problem and entanglement distillation

$M \times N$ NPT states of rank $\max\{M, N\}$

Two-qutrit NPT states of rank four and five

Open problems
Distilling two-qutrit NPT states of rank four

- Entanglement distillation of $M \times N$ states $\rho$ of rank bigger than $\max\{M, N\}$ turns out to be much harder.
Distilling two-qutrit NPT states of rank four

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- For example $\rho$ can be the Werner state.
Distilling two-qutrit NPT states of rank four

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- **Facts:** $2 \times N$ NPT states are distillable, and $M \times N$ NPT states of rank $\max\{M, N\}$ are distillable.
Distilling two-qutrit NPT states of rank four

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- **Facts:** $2 \times N$ NPT states are distillable, and $M \times N$ NPT states of rank $\max\{M, N\}$ are distillable.

- Hence, the first unsolved problem is to distill $3 \times 3$ NPT states of rank four.
Distilling two-qutrit NPT states of rank four

- LC and DZ, 2016.
Distilling two-qutrit NPT states of rank four

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**Theorem**

*If \( \rho \) is a two-qutrit NPT state and \( \rho^\Gamma \) has at least two non-positive eigenvalues counting multiplicities, then \( \rho \) is 1-distillable.*
Distilling two-qutrit NPT states of rank four

- LC and DZ, 2016.

**Theorem**

*If ρ is a two-qutrit NPT state and ρΓ has at least two non-positive eigenvalues counting multiplicities, then ρ is 1-distillable.*

**Proof.**

By the hypothesis, there exist two eigenvectors of ρΓ, say |α⟩ and |β⟩ with matrices A and B, such that ρΓ|α⟩ = λ|α⟩, λ < 0, ρΓ|β⟩ = µ|β⟩, µ ≤ 0, and ⟨α|β⟩ = 0. If A is not invertible, then its rank is 2 and so ρ is 1-distillable.

If $N := A^{-1}B$ is not nilpotent, then det$(I_3 + tN)$ is a nonconstant polynomial in $t$ and we can choose $t$ so that this determinant is 0. Thus $A + tB$ is singular, and $|φ⟩ := |α⟩ + t|β⟩$ satisfies

$⟨φ|ρΓ|φ⟩ = λ∥α∥^2 + µ|t|^2∥β∥^2 < 0$. Hence ρ is 1-distillable. The case that $N$ is nilpotent is similar. □
Distilling two-qutrit NPT states of rank four

- From the theorem we have
Distilling two-qutrit NPT states of rank four

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**Corollary**

*If the kernel of a two-qutrit NPT state $\rho$ contains a product state, then $\rho$ is 1-distillable.*
Distilling two-qutrit NPT states of rank four

- From the theorem we have

**Corollary**

*If the kernel of a two-qutrit NPT state \( \rho \) contains a product state, then \( \rho \) is 1-distillable.*

**Proof.**

We can assume that \( |0, 0\rangle \in \ker \rho \). Consequently, the first diagonal entry of \( \rho \) is 0, and the same is true for \( \rho^\Gamma \). If the first column of \( \rho^\Gamma \) is not 0, then \( \rho \) is 1-distillable by projecting to a \( 2 \times 3 \) NPT state. Otherwise \( |0, 0\rangle \in \ker \rho^\Gamma \) and \( \rho \) is 1-distillable by last Theorem. \( \square \)
The distillability problem and entanglement distillation

\[ M \times N \text{ NPT states of rank } \max\{M, N\} \]

Distilling two-qutrit NPT states of rank four

**Theorem**

*Any bipartite NPT state of rank at most four is 1-distillable.*
Distilling two-qutrit NPT states of rank four

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Any bipartite NPT state of rank at most four is 1-distillable.

**Corollary**

If $\rho$ is a 1-undistillable two-qutrit NPT state, then $\ker \rho$ is a completely entangled space, and $\rho^\Gamma$ has exactly one negative and eight positive eigenvalues. Consequently, $\text{rank } \rho > 4$ and $\det \rho^\Gamma \neq 0$. 
The distillability problem and entanglement distillation

$M \times N$ NPT states of rank $\max\{M, N\}$

Distilling two-qutrit NPT states of rank four

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- So the minimum rank of 1-undistillable NPT states is at least five.
Distilling two-qutrit NPT states of rank four

**Theorem**
Any bipartite NPT state of rank at most four is 1-distillable.

**Corollary**
If \( \rho \) is a 1-undistillable two-qutrit NPT state, then \( \ker \rho \) is a completely entangled space, and \( \rho^\Gamma \) has exactly one negative and eight positive eigenvalues. Consequently, \( \text{rank} \rho > 4 \) and \( \det \rho^\Gamma \neq 0 \).

- So the minimum rank of 1-undistillable NPT states is at least five.
- We construct an example below.
The following state $\sigma$ is an edge PPT entangled state of birank $(5, 8)$ constructed by Kye and Osaka, 2012.

\[
\begin{bmatrix}
2 \cos \theta & 0 & 0 & 0 & -\cos \theta & 0 & 0 & 0 & -\cos \theta \\
0 & \frac{1}{b} & 0 & -e^{-i\theta} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & -e^{i\theta} & 0 & 0 \\
0 & -e^{i\theta} & 0 & b & 0 & 0 & 0 & 0 & 0 \\
-\cos \theta & 0 & 0 & 0 & 2 \cos \theta & 0 & 0 & 0 & -\cos \theta \\
0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & -e^{-i\theta} & 0 \\
0 & 0 & -e^{-i\theta} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\
0 & 0 & 0 & 0 & -e^{i\theta} & 0 & b & 0 & 0 \\
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\end{bmatrix},
\]
Distilling two-qutrit NPT states of rank four

The following state $\sigma$ is an edge PPT entangled state of birank $(5, 8)$ constructed by Kye and Osaka, 2012.

\[
\frac{1}{N} \begin{bmatrix}
2\cos \theta & 0 & 0 & 0 & -\cos \theta & 0 & 0 & 0 & -\cos \theta \\
0 & \frac{1}{b} & 0 & -e^{-i\theta} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & -e^{i\theta} & 0 & 0 \\
0 & -e^{i\theta} & 0 & b & 0 & 0 & 0 & 0 & 0 \\
-\cos \theta & 0 & 0 & 0 & 2\cos \theta & 0 & 0 & 0 & -\cos \theta \\
0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & -e^{-i\theta} & 0 \\
0 & 0 & -e^{-i\theta} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -e^{i\theta} & 0 & b & 0 \\
-\cos \theta & 0 & 0 & 0 & -\cos \theta & 0 & 0 & 0 & 2\cos \theta
\end{bmatrix},
\]

where

\[N = 3(2\cos \theta + b + 1/b),\]

and the two parameters $b > 0$ and $0 < |\theta| < \pi/3.$
Distilling two-qutrit NPT states of rank four

- Since $\text{rank } \sigma = 5$ and $\sigma$ is an edge state, $\mathcal{R}(\sigma)$ contains a product state $|f, g\rangle$ such that $|f^*, g\rangle \notin \mathcal{R}(\sigma^\Gamma)$.
Distilling two-qutrit NPT states of rank four

- Since rank $\sigma = 5$ and $\sigma$ is an edge state, $\mathcal{R}(\sigma)$ contains a product state $|f, g\rangle$ such that $|f^*, g\rangle \notin \mathcal{R}(\sigma^\Gamma)$.
- For sufficiently small $\epsilon > 0$, the matrix

$$\rho = \frac{1}{1 - \epsilon}(\sigma - \epsilon|f, g\rangle\langle f, g|)$$

is a two-qutrit NPT state of rank five.
Distilling two-qutrit NPT states of rank four

- Since \( \text{rank } \sigma = 5 \) and \( \sigma \) is an edge state, \( \mathcal{R}(\sigma) \) contains a product state \( |f, g\rangle \) such that \( |f^*, g\rangle \notin \mathcal{R}(\sigma^\Gamma) \).
- For sufficiently small \( \epsilon > 0 \), the matrix
  \[
  \rho = \frac{1}{1 - \epsilon} (\sigma - \epsilon |f, g\rangle\langle f, g|)
  \]
  is a two-qutrit NPT state of rank five.
- The kernel of \( \sigma^\Gamma \) is spanned by the two-qutrit maximally entangled state \( |\Psi\rangle \). Let \( p \) be the minimum positive eigenvalue of \( \sigma^\Gamma \).
Distilling two-qutrit NPT states of rank four

- Since rank $\sigma = 5$ and $\sigma$ is an edge state, $\mathcal{R}(\sigma)$ contains a product state $|f, g\rangle$ such that $|f^*, g\rangle \notin \mathcal{R}(\sigma^\Gamma)$.
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is a two-qutrit NPT state of rank five.

- The kernel of $\sigma^\Gamma$ is spanned by the two-qutrit maximally entangled state $|\Psi\rangle$. Let $p$ be the minimum positive eigenvalue of $\sigma^\Gamma$.
- For any pure state $|\psi\rangle$ of Schmidt rank two, we have

$$\langle \psi | \rho^\Gamma | \psi \rangle \propto \langle \psi | (\sigma^\Gamma - \epsilon |f^*, g\rangle\langle f^*, g|) | \psi \rangle > p/3 - \epsilon \geq 0.$$
Distilling two-qutrit NPT states of rank four

- Since \( \text{rank} \sigma = 5 \) and \( \sigma \) is an edge state, \( \mathcal{R}(\sigma) \) contains a product state \( |f, g\rangle \) such that \( |f^*, g\rangle \notin \mathcal{R}(\sigma^\Gamma) \).
- For sufficiently small \( \epsilon > 0 \), the matrix
  \[
  \rho = \frac{1}{1 - \epsilon} (\sigma - \epsilon |f, g\rangle \langle f, g|)
  \]
  is a two-qutrit NPT state of rank five.
- The kernel of \( \sigma^\Gamma \) is spanned by the two-qutrit maximally entangled state \( |\Psi\rangle \). Let \( p \) be the minimum positive eigenvalue of \( \sigma^\Gamma \).
- For any pure state \( |\psi\rangle \) of Schmidt rank two, we have
  \[
  \langle \psi | \rho^\Gamma | \psi \rangle \propto \langle \psi | (\sigma^\Gamma - \epsilon |f^*, g\rangle \langle f^*, g|) | \psi \rangle > p/3 - \epsilon \geq 0.
  \]
- Hence \( \rho \) is 1-undistillable.
Distilling two-qutrit NPT states of rank four

Lemma
For any integer $n$, and sufficiently small $\epsilon = \epsilon(n) > 0$, the two-qutrit NPT state

$$\rho = 1 - \epsilon (\sigma \otimes \mathbf{1}_n - |f, g\rangle \langle f, g|)$$

is $n$-undistillable.

Proof.
For any pure state $|\psi\rangle$ of Schmidt rank two, we have

$$(1 - \epsilon) \langle \psi | \rho^{\otimes n} | \psi \rangle := \langle \psi | (\sigma \otimes \mathbf{1}_n) \otimes \mathbf{1}_n | \psi \rangle + n \sum_{k=1}^{c_k \epsilon_k} \geq \langle \psi | (\mathbf{1}_n - |\Psi\rangle \langle \Psi|) \otimes \mathbf{1}_n | \psi \rangle + n \sum_{k=1}^{c_k \epsilon_k}$$

where $c_k$ are complex numbers and $p$ is the minimum positive eigenvalue of $\sigma \otimes \mathbf{1}_n$. Since the first summand is positive and has nothing to do with $\epsilon$, the assertion holds.

$\Box$
The distillability problem and entanglement distillation

$M \times N$ NPT states of rank $\max\{M, N\}$

Distilling two-qutrit NPT states of rank four

Lemma

For any integer $n$, and sufficiently small $\epsilon = \epsilon(n) > 0$, the two-qutrit NPT state $\rho = \frac{1}{1-\epsilon}(\sigma - \epsilon|f, g\rangle\langle f, g|)$ is $n$-undistillable.
Distilling two-qutrit NPT states of rank four

Lemma

For any integer \( n \), and sufficiently small \( \epsilon = \epsilon(n) > 0 \), the two-qutrit NPT state \( \rho = \frac{1}{1-\epsilon}(\sigma - \epsilon|f, g\rangle\langle f, g|) \) is \( n \)-undistillable.

Proof.

For any pure state \( |\psi\rangle \) of Schmidt rank two, we have

\[
(1 - \epsilon)^n \langle \psi | (\rho^\Gamma)^\otimes n |\psi\rangle := \langle \psi | (\sigma^\Gamma)^\otimes n |\psi\rangle + \sum_{k=1}^{n} c_k \epsilon^k
\]

\[
\geq p^n \langle \psi | (I_9 - |\Psi\rangle\langle \Psi|)^\otimes n |\psi\rangle + \sum_{k=1}^{n} c_k \epsilon^k
\]

where \( c_k \) are complex numbers and \( p \) is the minimum positive eigenvalue of \( \sigma^\Gamma \). Since the first summand is positive and has nothing to do with \( \epsilon \), the assertion holds. \( \square \)
Distilling two-qutrit NPT states of rank four

- The following auxiliary lemma is used in the previous proof.
Distilling two-qutrit NPT states of rank four

- The following auxiliary lemma is used in the previous proof.

**Lemma**

\[
\min_{\psi \in \mathcal{S}_{2}} \langle \psi | (I_9 - |\Psi\rangle\langle \Psi|)^{\otimes n} |\psi\rangle \geq \frac{1}{3^n},
\]

where \( \mathcal{S}_{2} \) is the set of bipartite pure states of Schmidt rank two, and \( |\Psi\rangle \) is the two-qutrit maximally entangled state.
The comparison between our suspicious two-qutrit NPT states $\rho$ of rank five and the “critical” NPT Werner states $\rho_w = \frac{2}{15} \left( I_9 - \frac{1}{2} \sum_{i,j=1}^{3} |ij\rangle\langle ji| \right)$.

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Comparing with Werner states

- The comparison between our suspicious two-qutrit NPT states \( \rho \) of rank five and the “critical” NPT Werner states 
  \( \rho_w = \frac{2}{15} (I_9 - \frac{1}{2} \sum_{i,j=1}^{3} |ij\rangle \langle ji|) \).

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- Whether there is a “critical” \( \rho \) is unknown.
Comparing with Werner states

- The comparison between our suspicious two-qutrit NPT states $\rho$ of rank five and the “critical” NPT Werner states $\rho_w = \frac{2}{15}(I_9 - \frac{1}{2} \sum_{i,j=1}^{3} |ij\rangle\langle ji|)$. 

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- Whether there is a “critical” $\rho$ is unknown.
- The condition of rank nine prevents the further investigation in both cases.
Open problems

- Can we distill more NPT states satisfying LFRP and RFRP?
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- Can we distill more NPT states satisfying LFRP and RFRP?
- Distill $3 \times N$ NPT states of rank $N + 1$ for $N \geq 4$. 
Open problems

- Can we distill more NPT states satisfying LFRP and RFRP?

- Distill $3 \times N$ NPT states of rank $N + 1$ for $N \geq 4$.

- Is there an undistillable suspicious two-qutrit NPT state
  
  $\rho = \frac{1}{1-\epsilon}(\sigma - \epsilon |f, g\rangle\langle f, g|)$

  by a constant $\epsilon > 0$?
The distillability problem and entanglement distillation $M \times N$ NPT states of rank $\max\{M, N\}$

Distilling two-qutrit NPT states

End

Thanks for your attention!