ON NEW WARING'S PROBLEM

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Abstract

There are several directions of generalizing the famous Four Square Theorem of Lagrange that every positive integer is a sum of four integer squares. Among them we consider the followings:
(1) Ramanujan's universal forms
(2) Mordell's five square thereom
(3) Maass' three square theorem

In the talk, I will introduce a brief history regarding the three directions, and then discuss recent developments and some open problems.

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0. NOTATIONS AND TERMINOLOGIES

(∗) F : a totally real (number) field

F^+ : the set of all (totally) positive elements of F

$\mathcal{O} = \mathcal{O}_F$: the ring of algebraic integers of F

$U = U_F$: the group of units of $\mathcal{O}$

$\mathcal{O}^+ := \mathcal{O} \cap F^+$

$U^+ := U \cap F^+$

(∗) $Q(x_1, \ldots, x_m) = \sum_{1\le i,j\le m} a_{ij}x_ix_j$: an integral quadratic form in m variables ($a_{ij} = a_{ji} \in \mathcal{O}$)

$M = M_Q = (a_{ij}) \in M_{m\times m}(\mathcal{O})$: an integral symmetric $m \times m$ matrix such that ${}^tXMX = Q(x_1, \ldots, x_m)$ for $X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in M_{m\times 1}(\mathcal{O})$

$L = L_Q = \mathcal{O}v_1 + \cdots + \mathcal{O}v_m$: an m-ary $\mathcal{O}$-lattice, i.e., a free $\mathcal{O}$-module of rank m equipped with a symmetric bilinear form $B(v_i, v_j) = a_{ij}$ such that $Q(v) = Q(x_1, \ldots, x_m)$, where $Q(v) := B(v, v)$ for $v = x_1v_1 + \cdots + x_mv_m \in L$

(∗) We write $L \simeq M$ in the basis $v_1, \ldots, v_m$, or simply $L \simeq M$ in the above case

$dL := \det M$: the <u>determinant</u> of L, which is well defined up to U^2

$L = L_1 \perp L_2$ if $L = L_1 \oplus L_2$ and $B(L_1, L_2) = 0$

$(*)$ L : <u>positive (definite)</u> if $Q(v)$ is positive for all $v \in L^{\times}$

L : <u>diagonal</u> if M is a diagonal matrix, and we write
$L \simeq \langle a_1 \rangle \perp \cdots \perp \langle a_m \rangle \simeq \langle a_1, \ldots, a_m \rangle$, where
$a_i := Q(v_i)$

L <u>represents</u> K (K <u>is represented by</u> L) :
$\exists$ an isometry $\sigma : FK \to FL$ such that $\sigma(K) \subset L$,
i.e., $\exists\ X \in M_{m\times k}(\mathcal{O})$ such that $N = {}^tXMX$,
where L (K, resp.) is a m-ary (k-ary, resp.)
$\mathcal{O}$-lattice with $L \simeq M$ ($K \simeq N$, resp.)
and we write $K \to L$ in this case

$L \simeq K$ (L and K are <u>isometric</u>) if $\sigma(K) = L$
The <u>class</u> of L, denoted by $\mathrm{cls}(L)$, is the set of
all K's isometric to L
The <u>genus</u> of L, denoted by $\mathrm{gen}(L)$, is the set of
all K's isometric to L locally everywhere

L : <u>k-universal</u> if L is positive and represents every
positive k-ary $\mathcal{O}$-lattice (<u>universal</u> = 1-universal)

$(*)$ Lagranges four square theorem (1770/72) :
$I_4 = \langle 1, 1, 1, 1 \rangle$ is universal, i.e., $\langle n \rangle \to I_4$ for every
positive integer n, i.e., $x_1^2 + x_2^2 + x_3^2 + x_4^2 = n$ is
solvable in integers for all positive integers n

1. BRIEF HISTORY

(1-1) Ramanujan's universal forms

($*$) Liouville (1856) found all pairs (a, b) of positive integers with $0 < a \le b$ for which $\langle 1, a, b, ab \rangle$ are universal :
(1,1) (1,2) (1,3) (2,2) (2,3) (2,4) (2,5)

Pepin (1890) added six more to the list :
$\langle 1,1,1,2 \rangle$ $\langle 1,2,2,2 \rangle$ $\langle 1,1,1,4 \rangle$
$\langle 1,1,2,4 \rangle$ $\langle 1,2,4,4 \rangle$ $\langle 1,1,2,8 \rangle$

($*$) Ramanujan (1917) claimed : $\exists$ exactly 55 universal quaternary diagonal $\mathbf{Z}$-lattices up to isometry

$\langle 1,1,1,1 \rangle$	$\langle 1,1,1,2 \rangle$	$\langle 1,1,2,2 \rangle$	$\langle 1,2,2,2 \rangle$
$\langle 1,1,1,3 \rangle$	$\langle 1,1,2,3 \rangle$	$\langle 1,2,2,3 \rangle$	$\langle 1,1,3,3 \rangle$
$\langle 1,2,3,3 \rangle$	$\langle 1,1,1,4 \rangle$	$\langle 1,1,2,4 \rangle$	$\langle 1,2,2,4 \rangle$
$\langle 1,1,3,4 \rangle$	$\langle 1,2,3,4 \rangle$	$\langle 1,2,4,4 \rangle$	$\langle 1,1,1,5 \rangle$
$\langle 1,1,2,5 \rangle$	$\langle 1,2,2,5 \rangle$	$\langle 1,1,3,5 \rangle$	$\langle 1,2,3,5 \rangle$
$\langle 1,2,4,5 \rangle$	$\langle 1,1,1,6 \rangle$	$\langle 1,1,2,6 \rangle$	$\langle 1,2,2,6 \rangle$
$\langle 1,1,3,6 \rangle$	$\langle 1,2,3,6 \rangle$	$\langle 1,2,4,6 \rangle$	$\langle 1,2,5,6 \rangle$
$\langle 1,1,1,7 \rangle$	$\langle 1,1,2,7 \rangle$	$\langle 1,2,2,7 \rangle$	$\langle 1,2,3,7 \rangle$
$\langle 1,2,4,7 \rangle$	$\langle 1,2,5,7 \rangle$	$\langle 1,1,2,8 \rangle$	$\langle 1,2,3,8 \rangle$
$\langle 1,2,4,8 \rangle$	$\langle 1,2,5,8 \rangle$	$\langle 1,1,2,9 \rangle$	$\langle 1,2,3,9 \rangle$
$\langle 1,2,4,9 \rangle$	$\langle 1,1,5,9 \rangle$	$\langle 1,1,2,10 \rangle$	$\langle 1,2,3,10 \rangle$
$\langle 1,2,4,10 \rangle$	$\langle 1,2,5,10 \rangle$	$\langle 1,1,2,11 \rangle$	$\langle 1,2,4,11 \rangle$
$\langle 1,1,2,12 \rangle$	$\langle 1,2,4,12 \rangle$	$\langle 1,1,2,13 \rangle$	$\langle 1,2,4,13 \rangle$
$\langle 1,1,2,14 \rangle$	$\langle 1,2,4,14 \rangle$	$\langle 1,2,5,5 \rangle^*$	

($*$) Dickson (1927) confirmed Ramanujan's claim except

$\langle 1,2,5,5\rangle$, which fails to represent 15, and hence :
$\exists$ exactly 54 such $\mathbf{Z}$-lattices

He and Morrow (1927/28) extended Ramanujan's result to the non-diagonal case

Willerding (1947) found all 124 universal quaternary non-diagonal $\mathbf{Z}$-lattices up to isometry and hence :
$\exists$ exactly 178 universal quaternary $\mathbf{Z}$-lattices

There exists no universal ternary $\mathbf{Z}$-lattice

<u>Proof</u>

Suppose L is a positive universal ternary $\mathbf{Z}$-lattice
Then $\langle 1\rangle$ splits L, and hence $L = \langle 1\rangle \perp K$
for some binary sublattice K of L
Let $K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ s.t. $a = \min(K) \leq 2$
$D = d(K) = d(L) = ac - b^2$
We may assume $L^a \subset \langle a, 1, D\rangle$,
where L^a is the scaling of L by a
So, $\langle a, 1, D\rangle$ represents all the multiples of a
Let am be the smallest mutiple of a that is not represented by $\langle a, 1\rangle$ so that $D \leq am$
If $a = 1$ then $m = 3$, and if $a = 2$ then $m = 5$

So, $D \leq 10$ and hence no positive ternary $\mathbf{Z}$-lattice of with $d(L) > 10$ can be universal

For those finitely many positive ternary $\mathbf{Z}$-lattices with $d(L) \leq 10$, one can check their non-universality by using, for example, the table of Brandt-Intrau $\square$

(∗) Kloosterman (1926) determined all quadruples (a, b, c, d) of positive integers with $0 < a \leq b \leq c \leq d$ for which $\langle a, b, c, d\rangle$ are <u>almost universal</u>, i.e., $\langle n\rangle \to \langle a, b, c, d\rangle \quad \forall\, n \gg 0$, i.e., $\langle a, b, c, d\rangle$ represents all but finitely many positive integers

He left the following four **Z**-lattices undetermined : $\langle 1, 2, 11, 38\rangle$ $\langle 1, 2, 17, 33\rangle$ $\langle 1, 2, 19, 22\rangle$ $\langle 1, 2, 19, 38\rangle$

Pall (1946) showed the almost universality for the remaining **Z**-lattices above

Pall and Ross (1946) extended Kloosterman's result to the non-diagonal case

Halmos (1938) found all 88 quaternay diagonal **Z**-lattices that represent all positive integers except exacltly one
The largest exception is 15

(1-2) Mordell's five square thereom

(∗) Waring's Problem - Waring (1782) asked :
In order to represent every positive integer as a sum of k-th powers of non-negative integers, how many k-th powers are necessary ?

(∗) Hilbert (1909) proved : $\forall\ k,\ \exists\ g$ s.t.
$x_1^k + \cdots + x_g^k = n$ is solvable in non-negative integers for all positive integers n

(∗) $g(k)$: the smallest such g for a given k

$G(k)$: the smallest g for a given k s.t.
$x_1^k + \cdots + x_g^k = n$ is solvable in non-negative integers for all positive integers $n \gg 0$

$g(1) = 1,\ g(2) = 4$ (Lagrange)
$g(3) = 9$ (Wieferich, 1909)
$g(4) = 19$ (Balasubramanian et. al., 1986)
$g(5) = 37$ (Chen, 1964)
$g(6) = 73$ (Pillai, 1940)
$g(k) = [(3/2)^k] + 2^k - 2$ holds for almost all $k \geq 7$
(Dickson, Rubugunday, Niven, Mahler, 1936-57)

$G(k) \leq g(k)$ for all $k \in \mathbf{Z}^+$
No value of $G(k)$ other than
$G(1) = 1, G(2) = 4, G(4) = 16$ is yet known
$4 \leq G(3) \leq 7,\ 6 \leq G(5) \leq 21,\ 9 \leq G(6) \leq 31,\ \ldots$

(∗) New Waring's Problem

Mordell proved the five square theorem :
I_5 is 2-universal, i.e.,
I_5 represents all positive binary $\mathbf{Z}$-lattices

Ko (1937) proved : I_{k+3} is k-universal for $3 \leq k \leq 5$

Thus, Mordell and Ko naturally expected :
I_{k+3} is k-universal for all $k \in \mathbf{Z}^+$

But Mordell (1937) found E_6 cannot be represented by I_N for any positive integer N

So, the question for $k \geq 6$ should be modified :
Is every positive k-ary $\mathbf{Z}$-lattice that can be represented by a sum of squares represented by I_{k+3} ?

Ko (1939) conjectured :

(1) Every positive senary $\mathbf{Z}$-lattice that can be represented by a sum of squares is represented by the sum of nine squares I_9 (wrong!)

(2) Every positive senary $\mathbf{Z}$-lattice that cannot be represented by a sum of squares is represented by $E_6 \perp I_3$ (?)

(1-3) Maass' three square thereom

($*$) Maass (1941) proved the following remarkable theorem :
I_3 is a universal $\mathcal{O}$-lattice, where $F = \mathbf{Q}(\sqrt{5})$, i.e., every element of $\mathcal{O}^+$ can be written as a sum of three squares (of elements of $\mathcal{O}$)

($*$) Siegel (1945) proved : For a totally real field F, every element of $\mathcal{O}^+$ can be written as a sum of squares if and only if $F = \mathbf{Q}$ or $\mathbf{Q}(\sqrt{5})$

($*$) Note that there exists no totally real field F that admits binary universal $\mathcal{O}$-lattice

($*$) (Integral) Pythagoras numbers $g\{F\}$:
F : a totally real field
$P := P_F$ is the subset of $\mathcal{O}$ consisting of all elements that can be written as sums of squares

$g\{F\}$: the smallest positive integer g s.t. every element of P can be written as a sum of g squares

$g\{\mathbf{Q}\} = 4$ (Lagrange), $g\{\mathbf{Q}(\sqrt{5})\} = 3$ (Maass)

Cohn (1960-61) proved :
$g\{\mathbf{Q}(\sqrt{2})\} = 3,\ g\{\mathbf{Q}(\sqrt{3})\} = \underline{3}$ or 4

Cohn and Pall (1962), and Peters (1973) independently proved : $g\{F\} \le 5,\ \forall$ totally real quadratic field F

Peters (1974) further conjectured :
$g\{F\} \le 5$ for any totally real field F
But this turns out to be wrong (Scharlau, 1980)

2. RECENT RESULTS AND OPEN QUESTIONS

(2-1) k-universal $\mathbf{Z}$-lattices

($*$) 2-universal $\mathbf{Z}$-lattices :

Kim-K-Raghavan : There exist exactly five 2-universal quinary diagonal $\mathbf{Z}$-lattices up to isometry :

$$I_3 \perp \langle a, b \rangle$$

for $(a,b) = (1,1), (1,2), (1,3), (2,2), (2,3)$

The five $\mathbf{Z}$-lattices above coincide with those $\mathcal{O}$-lattices, introduced by Peters, that represent all elements of P for any real quadratic field F

Kim-K : There exist exactly six 2-universal quinary non-diagonal $\mathbf{Z}$-lattices up to isometry :

$$I_2 \perp \begin{pmatrix} a & b & 0 \\ b & 2 & 1 \\ 0 & 1 & c \end{pmatrix}$$

for $(a,b,c) = (1,0,2), (1,0,3), (2,0,2)^*$
$(3,0,2), (2,1,2), (2,1,3)$

So, there are eleven 2-universal quinary $\mathbf{Z}$-lattices

$I_2 \perp \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ is of class number two

There exists no quaternary 2-universal $\mathbf{Z}$-lattice

(∗) A positive **Z**-lattice L is called <u>k-regular</u> if L represents all positive k-ary **Z**-lattices that are represented by L locally everywhere

Recently, Earnest proved that there are only finitely many primitive positive quaternary **Z**-lattices, up to isometry, which are 2-regular.

See Hsia's paper for more about regular **Z**-lattices

(∗) k-universal diagonal **Z**-lattices ($k \geq 3$) :

Kim-K : 3-universal senary **Z**-lattices are

$$I_4 \perp \langle a, b \rangle$$

$$\text{for } (a, b) = (1,1), (1,2), (1,3), (2,2), (2,3)^*,$$

4-universal septanary **Z**-lattices are

$$I_5 \perp \langle a, b \rangle \text{ for } (a, b) = (1,1), (1,2),$$

and 5-universal octanary **Z**-lattices are

$$I_6 \perp \langle a, b \rangle \text{ for } (a, b) = (1,1), (1,2)^*$$

The two **Z**-lattices with (∗) are of class number two

There exists no k-universal $(k+2)$-ary **Z**-lattice, $\forall\ k$

There exists no k-universal diagonal **Z**-lattice $\forall\ k \geq 6$

(∗) k-universal non-diagonal **Z**-lattices ($k \geq 3$) ?

(2-2) almost k-universal Z-lattices

($*$) $\exists$ only three positive quinary diagonal **Z**-lattices that represent all but one positive binary **Z**-lattice :

$$\langle 1,1,1,2,4\rangle,\ \langle 1,1,1,2,5\rangle,\ \langle 1,1,2,2,3\rangle$$

and the exceptions are $\langle 3,3\rangle, \langle 3,3\rangle, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, resp.

In order for a positive quinary diagonal **Z**-lattice L to be almost 2-universal, L should be one of the following forms :
$\langle 1,1,1,1,a\rangle$ $\langle 1,1,1,2,a\rangle$ $\langle 1,1,1,3,a\rangle$ $\langle 1,1,2,2,a\rangle$
$\langle 1,1,2,3,a\rangle$ $\langle 1,1,2,4,a\rangle$ $\langle 1,1,2,5,a\rangle$
a's are yet to be determined

($*$) Almost k-universal $(k+3)$-ary **Z**-lattices for $k \geq 2$?

($*$) Recall Halmos' 88 positive quaternary diagonal **Z**-lattices that represent all but one positive integer
The largest exception was 15 by $\langle 1,2,5,5\rangle$

Conway and Schneebeger recently proved so called the 15-Theorem, which says : If a positive **Z**-lattice L represents $1, 2, \ldots, 15$, then L is universal

They also conjectured the 290-Theorem, which is for the universality of <u>non-classic</u> positive **Z**-lattices

See also a very interesting article in Notices by Duke

($*$) Can we obtain a similar criterion for the k-universality when $k \geq 2$?

(2-3) New Waring's Problem

(∗) Icaza (1992) proved : $\forall\ k,\ \exists\ g$ s.t. $K \to I_g$
for all positive k-ary $\mathbf{Z}$-lattices K (that can be represented by I_N for some positive integer N)

The condition in the parenthesis is for $k \geq 6$

(∗) $g[k]$: the smallest such g for a given k

$G[k]$: the smallest g for a given k s.t. $K \to I_g$ for all positive k-ary $\mathbf{Z}$-lattices K with $\min(K) \gg 0$

$g[1] = 4$ (Lagrange), $g[2] = 5$ (Mordell)
$g[3] = 6,\ g[4] = 7,\ g[5] = 8$ (Ko)

$k + 3 \leq G[k] \leq g[k]$ for all $k \in \mathbf{Z}^+$
$G[k] = g[k] = k + 3$ for $1 \leq k \leq 5$

Hsia-Kitaoka-Kneser's result (1978) can be applied to show : $G[k] \leq 2k + 3$ for all $k \in \mathbf{Z}^+$

(∗) K-Oh : $g[k] \geq \left[\dfrac{3k}{2}\right] + 1,\ \forall\ k \in \mathbf{Z}^+$
In particular, $g[k] > k + 3$ for all $k \geq 6$,
which disproves Ko's conjecture

Proof

Let $K = A^m(2,2) \perp A(2,3)$, where $A(a,b) = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$
and $A^m(a,b)$ is the orthogonal sum of m copies of it
Then the result follows from $K \to I_{3m+4},\ K \to I_{3m+3}$
and $K \perp \langle 1 \rangle \to I_{3m+5},\ K \perp \langle 1 \rangle \to I_{3m+4}$ $\square$

(∗) Baeza and Icaza(1992-1996) provided explicit upper bounds for $g[k]$

(∗) K-Oh : $g[6] = 10$, i.e., every positive senary $\mathbf{Z}$-lattice that can be represented by a sum of squares is represented by I_{10}, the sum of ten squares, no less

Proof

K : a positive senary $\mathbf{Z}$-lattice

(1) dK : even $\Rightarrow$ $K \to I_9$

(2) dK : odd

(2-1) $Q(K) \not\subseteq 2\mathbf{Z}$, $K \to I_N \Rightarrow K \to I_{10}$

(2-2) $Q(K) \subseteq 2\mathbf{Z}$ implies $dK \equiv 7, 3 \pmod 8$

(2-2-1) $dK \equiv 7 \pmod 8 \Rightarrow K \to I_9$

(2-2-2) $dK \equiv 3 \pmod 8$, $K \to I_N \Rightarrow K \to I_{11}$

From these and the previous lower bound follows

$$10 \leq g[6] \leq 11$$

Improve (2-2-2) to obtain $K \to I_{10}$ for the case when $dK \equiv 3 \pmod 8$, $Q(K) \subseteq 2\mathbf{Z}$, and $K \to I_N$, which is the hard part, and then conclude $g[6] = 10$ $\square$

sketchy proof of (1) : $dK \equiv 0 \pmod 2$

By a suitable change of basis, we may write
$K = \mathbf{Z}v_1 + \mathbf{Z}v_2 + \cdots + \mathbf{Z}v_6$, $B(v_1, K) \subseteq 2\mathbf{Z}$

$\mathrm{gen}(I_9) = \mathrm{cls}(I_9) \cup \mathrm{cls}(\Phi_8 \perp \langle 1 \rangle)$ (disjoint union)
If $K \to I_9$, then $K \to \Phi_8 \perp \langle 1 \rangle$
because K is represented by I_9 locally everywhere,
(i.e., $K \to \mathrm{gen}(I_9)$)

So, $\exists\ a_1, a_2, \cdots, a_6 \in \mathbf{Z}$ s.t. $\tilde{K} \to \Phi_8$, where $\tilde{K}$ is a semi-positive $\mathbf{Z}$-lattice corresponding to the integral quadratic form $\tilde{Q} := Q - (a_1x_1 + a_2x_2 + \cdots + a_6x_6)^2$

$B(v_1, K) \subseteq 2\mathbf{Z}, Q(\Phi_8) \subseteq 2\mathbf{Z}$ imply $d\tilde{K} \equiv 0 \pmod 2$

So, $\tilde{K}_2$ is not unimodular and hence $\tilde{K}_2 \to (I_8)_2$

$(\Phi_8)_p \simeq (I_8)_p$ implie $\tilde{K}_p \to (I_8)_p$ for all $p \neq 2$

Since $\text{gen}(I_8) = \text{cls}(I_8)$, we have $\tilde{K} \to I_8$

From this follows $K \to I_9$, as desired $\square$

(∗) The proof for the fact that $g[k] = k + 3$ for $1 \leq k \leq 5$ is quite simple because $\text{gen}(I_{k+3}) = \text{cls}(I_{k+3})$ for those k

The most important ingredient for the $g[6]$ case is that

$$\text{gen}(I_9) = \text{cls}(I_8 \perp \langle 1 \rangle) \cup \text{cls}(\Phi_8 \perp \langle 1 \rangle)$$
$$\text{gen}(I_8) = \text{cls}(I_8),\ \text{gen}(\Phi_8) = \text{cls}(\Phi_8)$$

Such a property, however, is not available for $k \geq 7$

(∗) $g[k]$ for $k \geq 7$?

(∗) The minimal rank of k-universal $\mathbf{Z}$-lattices for $k \geq 6$?

(2-4) universal $\mathcal{O}$-lattices

($*$) Chan-K-Raghavan : $\exists$ only five universal ternary $\mathcal{O}$-lattices for $F = \mathbf{Q}(\sqrt{5})$ (three diagonal ones and two non-diagonal ones)

$$\langle 1,1,1\rangle \quad \langle 1,1,2\rangle \quad \langle 1,1,2+\epsilon\rangle$$

$$\langle 1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2+\epsilon \end{pmatrix} \quad \langle 1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2+\epsilon' \end{pmatrix}$$

where $\epsilon = (1+\sqrt{5})/2$ is the fundamental unit of $\mathbf{Q}(\sqrt{5})$ and ϵ' is its conjugate

We further proved that there are exactly three real quadratic fields F that admit universal ternary $\mathcal{O}$-lattices, namely, $F = \mathbf{Q}(\sqrt{2}), \mathbf{Q}(\sqrt{3}), \mathbf{Q}(\sqrt{5})$

$\mathbf{Q}(\sqrt{2})$ admits four such :

$$\langle 1,1,1+\epsilon\rangle \quad \langle 1\rangle \perp \begin{pmatrix} 1+\epsilon & 1 \\ 1 & 1+\epsilon' \end{pmatrix}$$

$$\langle 1\rangle \perp \begin{pmatrix} 1+\epsilon & 1 \\ 1 & 3 \end{pmatrix} \quad \langle 1\rangle \perp \begin{pmatrix} 1+\epsilon' & 1 \\ 1 & 3 \end{pmatrix}$$

where $\epsilon = 1+\sqrt{2}$ is the fundamental unit of $\mathbf{Q}(\sqrt{2})$

$\mathbf{Q}(\sqrt{3})$ admits two such : $\langle 1,1,\epsilon\rangle \quad \langle 1,\epsilon,\epsilon\rangle$

where $\epsilon = 2+\sqrt{3}$ is the fundamental unit of $\mathbf{Q}(\sqrt{3})$

($*$) For a totally real field F, $K \to I_N$ for some N for every positve binary $\mathcal{O}$-lattice K if and only if $F = \mathbf{Q}$

($\because$) For $F = \mathbf{Q}(\sqrt{5})$, the binary $\mathcal{O}$-lattice $\begin{pmatrix} 2 & 1 \\ 1 & 4+3\epsilon \end{pmatrix}$ cannot be represented by I_N for any N $\quad\square$

($*$) We briefly indroduce the strategy of our proof for the universality in C-K-R, K-K-R, and K-K :

(1) Eliminate others except the listed $\mathcal{O}$-lattices by norm and trace comparison, etc

(2) Show the listed are locally universal at all places by O'Meara and Riehm's local representation theory

(3) Compute the class numbers of the listed by using Siegel's mass formula

If the class number is one, we are lucky
Otherwise, we need more time and better idea
See, for example, Duke and Schulze-Pillot's result

($*$) Kitaoka suggested that there may be only finitely many totally real fields F that admit universal ternary $\mathcal{O}$-lattices (?)

This is true if one restricts F to real quadratic fields by C-K-R

If $[F : \mathbf{Q}]$ is odd, then F admits no universal ternary $\mathcal{O}$-lattice

Kim recently proved the followings :

(1) For a fixed n, $\exists$ finitely many totally real fields F with $[F : \mathbf{Q}] = n$ s.t. F admits universal ternary $\mathcal{O}$-lattices

(2) If $n = 2$ or odd, then $\exists$ finitely many totally real fields F with $[F : \mathbf{Q}] = n$ s.t. F admits universal quaternary $\mathcal{O}$-lattices

(∗) He also observed : $\exists$ infinitely many real quadratic fields that admit universal octanary $\mathcal{O}$-lattices

For every square-free m, $F = \mathbf{Q}(\sqrt{m^2-1})$ admits the universal octanary $\mathcal{O}$-lattice $\langle 1,1,1,1,\epsilon,\epsilon,\epsilon,\epsilon\rangle$, where $\epsilon = m + \sqrt{m^2-1}$ is the fundamental unit of F

This is very interesting since it seems that there are only finitely many real quadratic fields that admit universal $\mathcal{O}$-lattices of rank up to 7

(∗) Pythagoras numbers for real quadratic fields :

$g\{\mathbf{Q}(\sqrt{2})\} = 3$ (Cohn)
$g\{\mathbf{Q}(\sqrt{3})\} = 3$ (Cohn)
$g\{\mathbf{Q}(\sqrt{5})\} = 3$ (Maass)
$g\{\mathbf{Q}(\sqrt{6})\} = g\{\mathbf{Q}(\sqrt{7})\} = 4$ (Cohn-Pall)
$g\{\mathbf{Q}(\sqrt{m})\} = 5$ for all square-free $m \geq 10$ (Kim)*

<u>Proof</u> $m \geq 10$

For $m \equiv 2,3 \pmod 4$, suppose that $(m+8)+2\sqrt{m}$ can be written as a sum of four squares
Since $m+8 < 2m$, we may write $(m+8)+2\sqrt{m}$ $= (a_1 + b_1\sqrt{m})^2 + a_2^2 + a_3^2 + a_4^2$ and hence $|\,a_1| = |\,b_1|$ $= 1$ and $a_2^2 + a_3^2 + a_4^2 = 7$, which is impossible

$m \equiv 1 \pmod 4$ case can be treated similarly

(∗) Can we obtain an analogy of Conway-Schneeberger's 15-Theorem for the universality of positive $\mathcal{O}$-lattices when F is a totally real field ?