§0. Introduction and Notations

One of the most powerful methods of studying representations of quadratic forms by forms is via theta-series. Many authors did a great deal of work in this direction after Siegel’s pioneering work[Si1]. Unfortunately, however, most of them worked in the case when the representing quadratic form has an even number of variables. The reason is simple: quadratic forms with even number of variables are associated to integral weight theta-series while those with odd number of variables to half integral weight theta-series whose transformation formulas involve branch problems.

In this article, we study the behavior of half integral weight theta-series under theta operators. Theta operators are very important in the study of theta-series in connection with Hecke operators. Andrianov[A1] proved that the space of integral weight theta-series is invariant under the action of theta operators. We prove that his statement can be extended for half integral weight theta-series with a slight modification. By using this result one can
prove that the space of theta-series is invariant under the action of Hecke operators as Andrianov did for integral weight theta-series [A1].

Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively.

For an $m \times m$ matrix $g$ and an $m \times n$ matrix $h$, let $g[h] = {}^tgh$, where $^t$ is the transpose of $h$. For a $2n \times 2n$ matrix $g$, let $A_g, B_g, C_g, D_g$ be the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners of $g$, respectively. Let $\text{diag}(N_1, N_2, \ldots, N_r)$ be the matrix with block matrices $N_1, N_2, \ldots, N_r$ on its main diagonal and zeroes outside. Let $\mathcal{N}_m$ be the set of all semi-positive definite (eigenvalues $\geq 0$), semi-integral (diagonal entries and twice of nondiagonal entries are integers), symmetric $m \times m$ matrices, and $\mathcal{N}_m^+$ be its subset consisting of positive definite (eigenvalues $> 0$) matrices.

Let

$$G_n = \text{GSp}_n^+(\mathbb{R}) = \{g \in M_{2n}(\mathbb{R}) : J_n[g] = rJ_n, \ r > 0\}$$

$$\Gamma^n = \text{Sp}_n(\mathbb{Z}) = \{M \in M_{2n}(\mathbb{Z}) : J_n[M] = J_n\},$$

where $J_n = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ and $r = r(g) \in \mathbb{R}$ is determined by $g$. And let

$$\mathcal{H}_n = \{Z = X + iY \in M_n(\mathbb{C}) : {}^tZ = Z, Y > 0\}$$

where $Y > 0$ means $Y$ is positive definite. For $g \in G_n$ and $Z \in \mathcal{H}_n$, we set

$$g\langle Z \rangle = (A_gZ + B_g)(C_gZ + D_g)^{-1} \in \mathcal{H}_n.$$

For $g \in M_n(\mathbb{C}), e(g) = \exp(2\pi i \sigma(g))$ where $\sigma(g)$ is the trace of $g$. Finally, we let $\langle n \rangle = n(n+1)/2$ for $n \in \mathbb{Z}$.

For other standard terminologies and basic facts, we refer the readers [A2], [M], [O].
§1. Siegel Modular Forms

Let $n, q$ be positive integers and $p$ be a prime relatively prime to $q$. Let

\[
\Gamma_0^n(q) = \{ M \in \Gamma^n \mid C_M \equiv 0 \, (\text{mod } q) \} \\
\Gamma^n = \{ M \in \Gamma^n \mid C_M = 0 \} \\
L^n = L^n_p = \{ g \in M_{2n}(\mathbb{Z}[p^{-1}]) \mid J_n[g] = p^\delta J_n, \delta \in \mathbb{Z} \} \\
L^n_0 = L^n_{0,p} = \{ g \in L^n \mid C_g = 0 \} \\
E^n = E^n_p = \{ g \in L^n \mid \delta(g) \in 2\mathbb{Z} \} \\
E^n_0 = E^n_{0,p} = E^n \cap L^n_0
\]

where $\delta = \delta(g) \in \mathbb{Z}$ is determined by $g$. Let

\[
\hat{G}_n = \{ (g, \alpha(Z)) \mid g \in G \}
\]

where $\alpha(Z)$ is a holomorphic function on $\mathcal{H}_n$ satisfying $\alpha(Z)^2 = t(\text{det } g)^{-1/2}$. Let $\det(C_g Z + D_g)$ for some $t \in \mathbb{C}$, $|t| = 1$. $\hat{G}_n$ is a multiplicative group under the multiplication $(g, \alpha(Z))(h, \beta(Z)) = (gh, \alpha(h(Z))\beta(Z))$ and is called the universal covering group of $G_n$. This group was introduced by Shimura[S] for $n = 1$ and then generalized by Zhuravlev[Z1,2] for arbitrary $n$. Let $\gamma : \hat{G}_n \to G_n$ be the projection $\gamma(g, \alpha(z)) = g$. We define an action of $\hat{G}_n$ on $\mathcal{H}_n$ by

\[
\zeta(Z) = \gamma(\zeta)(Z)
\]

for $\zeta \in \hat{G}_n$, $Z \in \mathcal{H}_n$.

From now on, we let $q$ be a positive integer such that $4 | q$. Let

\[
(1.1) \quad \theta^n(Z) = \sum_M e(t^T M M Z) = \sum_N e(Z[N]), \quad Z \in \mathcal{H}_n,
\]

where $M(N, \text{ resp.})$ runs over all the integral row (column, resp.) matrices of length $n$. $\theta^n(Z)$ is called the standard theta-function of degree $n$. For $M \in \Gamma_0^n(q)$, we define

\[
(1.2) \quad j(M, Z) = \theta^n(M(Z))/\theta^n(Z), \quad Z \in \mathcal{H}_n.
\]
It is well known [Z1] that $(M, j(M, Z)) \in \hat{G}_n$. So the map $j : \Gamma_0^n(q) \to \hat{G}_n$ defined by $j(M) = (M, j(M, Z))$ is a well defined injective homomorphism such that $\gamma \circ j$ is the identity map on $\Gamma_0^n(q)$. We denote $\hat{\Gamma}_0^n(q) = j(\Gamma_0^n(q))$, $\hat{\Gamma}_0^n = j(\Gamma_0^n)$ and $\hat{L}_0^n = \gamma^{-1}(L_0^n)$.

Let $\chi$ be a Dirichlet character (mod $q$) and $k$ be a positive half integer, i.e., $k = m/2$ for some positive odd integer $m$. For a complex valued function $F$ on $H_n$ and $\zeta = (g, \alpha(Z)) \in \hat{G}_n$, we set

$$ (1.3) \quad (F|_k \zeta)(Z) = r(g)^{(nk/2)-\langle n \rangle} \alpha(Z)^{-2k} F(g\langle Z \rangle), \quad Z \in H_n. $$

Since the map $Z \to g\langle Z \rangle$ is an analytic automorphism of $H_n$ and $\alpha(Z) \neq 0$ on $H_n$, $F|_k \zeta$ is holomorphic on $H_n$ if $F$ is. Also from the definition follows that $F|_{k1} \zeta_1 \zeta_2 = F|_{k} \zeta_1 \zeta_2$ for $\zeta_1, \zeta_2 \in \hat{G}_n$.

A function $F : H_n \to \mathbb{C}$ is called a Siegel modular form of degree $n$, weight $k$, level $q$, with character $\chi$ if the following conditions hold:

(i) $F$ is holomorphic on $H_n$,

(ii) $F|_k \hat{M} = \chi(\det D_M) \cdot F$ for every $\hat{M} = (M, j(M, Z)) \in \hat{\Gamma}_0^n(q)$, and

(iii) $F|_{k}(M, \alpha(z))$ is bounded as $\text{Im} \ z \to \infty$, $z \in H_1$, for every $(M, \alpha(z)) \in \gamma^{-1}(\Gamma_1)$ when $n = 1$.

It is known[Kö] that the boundedness condition (iii) follows from (i) and (ii) for $n \geq 2$. We denote the set of all such Siegel modular forms by $M^n_k(q, \chi)$.

This is known[Si2] to be a finite dimensional vector space over $\mathbb{C}$.

A function $F : H_n \to \mathbb{C}$ is called an even or odd modular form of degree $n$ if $F$ satisfies (i), (iii), and

$$(ii)' \quad (\det D_M)^s F(M\langle Z \rangle) = F(Z), \quad Z \in H_n \text{ for every } M \in \Gamma_0^n, \text{ where } s = 0 \text{ for even and } s = 1 \text{ for odd modular forms}.$$

We denote the set of all such even modular forms by $M^n_0$ and odd modular forms by $M^n_1$. They are also vector spaces over $\mathbb{C}$. 

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Let $F \in M^n_k(g, \chi)$ and $\chi(-1) = (-1)^s$ for $s = 0$ or $1$. For $M \in \Gamma_0^n$, we have $\hat{M} = (M, j(M, Z)) = (M, 1)$ and $\det D_M = \pm 1$. So, $F$ satisfies (ii)$'$ and hence

\begin{equation}
(1.4) \quad \mathcal{M}^n_k(q, \chi) \subset \mathcal{M}^n_s \text{ if } \chi(-1) = (-1)^s.
\end{equation}

Let $X = (\Gamma^n_0 g \Gamma^n_0)$ with $g \in L^n_0$. Then $X$ can be written as a disjoint union of left cosets $(\Gamma^n_0 g_i), g_i \in L^n_0, i = 1, 2, \ldots, \mu$. For $F \in \mathcal{M}^n_s$ and $X = (\Gamma^n_0 g \Gamma^n_0)$, we set

\begin{equation}
(1.5) \quad F|_{k, \chi} X = \sum_{i=1}^{\mu} \chi(\det A_i) \cdot F|_{k} \tilde{g}_i,
\end{equation}

where

\begin{equation}
(1.6) \quad \tilde{g}_i = (g_i, (\det g_i)^{-1/4} | \det D_i|^{1/2}) \in \hat{L}^n_0
\end{equation}

with $g_i = \begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix} \in L^n_0$ and $\chi(-1) = (-1)^s$.

If we let $\mathcal{L}^n_0$ be the vector space over $\mathbb{C}$ formally spanned by double cosets $(\Gamma^n_0 g \Gamma^n_0), g \in L^n_0$, then we can extend the action (1.5) to $\mathcal{L}^n_0$ by linearity. It follows from definitions that for $F \in \mathcal{M}^n_s$ and $X, Y \in \mathcal{L}^n_0$, we have $F|_{k, \chi} X \in \mathcal{M}^n_s$ and $F|_{k, \chi} X|_{k, \chi} Y = F|_{k, \chi} X \cdot Y$, where $\chi(-1) = (-1)^s$. So the action (1.5) is a well defined action of $\mathcal{L}^n_0$ on $\mathcal{M}^n_s$. The elements of $\mathcal{L}^n_0$ acting on $\mathcal{M}^n_s$ in this manner are called Hecke operators. It is well known[A2] that $\mathcal{L}^n_0$ is equipped with a ring structure and is called a Hecke ring.

§2. Theta-Series and Theta Operators

Let $Q \in \mathcal{N}^+_m$. The level $q_Q$ of $Q$ is defined to be the smallest positive integer satisfying that $q_Q(2Q)^{-1}$ is integral with even diagonals. It is known that $q_Q$ is divisible by $4$ if $m$ is odd. We define the theta-series of degree $n$ associated to $Q$ by

\begin{equation}
(2.1) \quad \theta^n(Z, Q) = \sum_{X \in M_{m,n}(Z)} e(Q[X]Z), \quad Z \in \mathcal{H}_n.
\end{equation}
If we let \( r(N, Q) = |\{X \in M_{m,n}(\mathbf{Z})|Q[X] = N\}| \) for each \( N \in \mathcal{N}_n \), then

\[
\theta^n(Z, Q) = \sum_{N \in \mathcal{N}_n} r(N, Q) e(NZ), \quad Z \in \mathcal{H}_n.
\]

(2.2)

It is known [K] that for odd \( m \)

\[
\theta^n(Z, Q) \in \mathcal{M}_k^n(q_Q, \chi_Q)
\]

where \( k = \frac{m}{2} \) is a half integer and \( \chi_Q \) is a Dirichlet character modulo \( q_Q \) defined by

\[
\chi_Q(M) = \left( \frac{2 \det 2Q}{|\det D_M|} \right)_{jac}, \quad \forall M \in \Gamma^n_0(q).
\]

(2.4)

For even \( m \), (2.3) was proved by Andrianov and Maloletkin [A-M].

Let \( \Theta^n_m \) be the vector space over \( \mathbf{C} \) spanned by \( \theta^n(Z, Q) \), \( Q \in \mathcal{N}_m^+ \) and let \( \Theta^n_m(q, d) \) be its subspace spanned by \( \theta^n(Z, Q) \), \( Q \in \mathcal{N}_m^+ \), \( \det 2Q = d \) and \( q_Q = q \) for given positive integers \( d, q \). It is easy to check that

\[
\Theta^n_m \subset \mathcal{M}^n_0 \quad \text{and} \quad \Theta^n_m(q, d) \subset \mathcal{M}^n_{m/2}(q, \chi),
\]

(2.5)

where \( \chi(M) = \left( \frac{2d}{|\det D_M|} \right)_{jac} \) for any \( M \in \Gamma^n_0(q) \).

Let \( \Theta^n_m[Q] \) be the subspace of \( \Theta^n_m \) spanned by \( \theta^n(Z, Q') \), \( Q' \in gen(Q) \) where \( gen(Q) \) is the genus of \( Q \). Clearly

\[
\Theta^n_m[Q] \subset \Theta^n_m(q, d) \subset \Theta^n_m
\]

(2.6)

if \( q = q_Q \) and \( d = \det 2Q \).

We now define theta operators following Andrianov [A2]: \( p \) is still a prime relatively prime to \( q \). Let \( \alpha : L^n_0 \to \mathbf{C}^\times \) be a character such that \( \alpha(\Gamma^n_0) = 1 \).

For \( X = (\Gamma^n_0, g_0 \Gamma^n_0) \in L^n_0 \) with \( g_0 = \left( \begin{array}{cc} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{array} \right) \in L^n_0 \) and \( \theta^n(Z, Q) \in \Theta^n_m \), we set

\[
\theta^n(Z, Q) \circ_\alpha X = \alpha(g_0) \sum_{D \in A \Delta_0 \Lambda / A} l_x(Q, D) \theta^n(Z, p^\delta Q[D^*])
\]

(2.7)
where $\Lambda = SL_m(\mathbb{Z})$ and

$$l_x(Q, D) = \sum_{B \in B_x(D) \mod D} e(QBD^{-1}).$$

Here

$$B_x(D) = \{B \in M_m(Q) ; \left( \begin{array}{cc} p^\delta D^* & B \\ 0 & D \end{array} \right) \in \Gamma_0^m g_0 \Gamma_0^m \}$$

and $B_1, B_2 \in B_x(D)$ are said to be congruent modulo $D$ on the right if $(B_1 - B_2)D^{-1} \in M_m(\mathbb{Z})$. This congruence is obviously an equivalent relation and the summation in (2.8) is over equivalent classes in $B_x(D)$ modulo $D$ on the right. We extend (2.7) by linearity to the whole space $\Theta^n_m$ and the whole ring $\mathcal{L}_0^m$.

Let

$$\mathcal{L}_0^{m,0} = \mathcal{L}_{0,p}^{m,0} = \{ \sum a_i (\Gamma_0^m g_i \Gamma_0^m) \in \mathcal{L}_0^m ; \delta_i m - 2b_i = 0 \}$$

where $g_i = \left( \begin{array}{cc} p^\delta D^*_i & B_i \\ 0 & D_i \end{array} \right) \in L_0^m$ and $b_i = \log_p |\det D_i|$, and let

$$\mathcal{E}_0^{m,0} = \mathcal{E}_{0,p}^{m,0} = \{ \sum a_i (\Gamma_0^m g_i \Gamma_0^m) \in \mathcal{L}_0^{m,0} ; \delta_i \in 2\mathbb{Z} \}.$$

§3. Main Theorem

We prove the following theorem:

**Theorem 3.1.** (1) The action (2.7) is a well-defined action of $\mathcal{L}_0^m$ on $\Theta^n_m$. The elements of $\mathcal{L}_0^m$ acting on $\Theta^n_m$ in this manner are called theta operators.

(2) $\Theta^n_m(q, d)$ is invariant under the theta operators of $\mathcal{L}_0^{m,0}$ if $p$ and $q$ are relatively prime.

(3) $\Theta^n_m[Q]$ is invariant under the theta operators of $\mathcal{E}_0^{m,0}$ if $p$ and $2q_Q$ are relatively prime.

**Proof.** This theorem is proved for the case $m$ even in [A1]. So, we restrict ourselves to the case $m$ odd here. Let

$$\varepsilon(Z, Q) = \sum_{U \in \Omega} e(Q[U]Z), \quad Z \in \mathcal{H}_m,$$
where $\Omega = GL_m(Z)$.

$\varepsilon(Z, Q)$ is called the $\varepsilon$-series of $Q$. For every $M = \begin{pmatrix} D^* & B \\ 0 & D \end{pmatrix} \in \Gamma_m^0$ with $D \in \Omega$, we have

\begin{equation}
\varepsilon(M\langle Z \rangle, Q) = \sum_{U \in \Omega} e(Q[U D^*]Z)e(Q[U]BD^{-1}) = \varepsilon(Z, Q)
\end{equation}

Note that $e(Q[U]BD^{-1}) = 1$ because $Q[U] \in \mathcal{N}_m^+$ and $BD^{-1}$ is integral symmetric $[M]$. From (3.2) and the definition of even modular forms follows that

$\varepsilon(Z, Q) \in \mathcal{M}_0^m$.

Let

\begin{equation}
\mathcal{A}_m = \{ \sum c_i \varepsilon(Z, Q_i) ; \ Q_i \in \mathcal{N}_m^+ \} \subset \mathcal{M}_0^m.
\end{equation}

Let $k = \frac{m}{2}$ and $\chi$ be a character satisfying $\chi(-1) = 1$. Let $X = (\Gamma_0^m g_0 \Gamma_0^m) \in L_0^m$ with $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{pmatrix} \in L_0^m$. Then

$$X = \sum_{D \in \Omega \setminus \Omega D_0 \Omega} \sum_{B \in B_x(D) \mod D} \chi(p^\delta D^*),$$

where $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in L_0^m$. See (2.9) for $B_x(D) \mod D$. From (1.3) and (1.6) follows

\begin{equation}
\varepsilon(Z, Q) = \sum_{U \in \Omega} e(Q[U]Z) = \sum_{U \in \Omega} e(QZ)|_{k\tilde{M}_U}
\end{equation}

where $M_U = \begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix} \in \Gamma_0^m$ and $\tilde{M}_U = (M_U, 1)$. Hence

\begin{equation}
\varepsilon(Z, Q)\mid_{k, \chi} X = \sum_{U \in \Omega} \sum_{D \in \Omega \setminus \Omega D_0 \Omega} \sum_{B \in B_x(D) \mod D} \chi(|p^\delta D^*e(QZ)|_{k\tilde{M}_U}|_{k\tilde{g}}},
\end{equation}
where $\tilde{g} = (g, p^{-\delta m/4} | \det D|^{1/2})$. Since $M_u g = \begin{pmatrix} p^\delta (UD)^* & U^* B \\ 0 & UD \end{pmatrix}$ and $U^* \{B_x(D)/\mod D\} = \{B_x(U D)/\mod UD\}$ for any $U \in \Omega$, we have

$$\varepsilon(Z, Q)|_{k, \chi} X = \sum_{D \in \Omega \cap \Omega_0} \sum_{B \in B_x(D)/\mod D} \chi(\det p^\delta D^*) e(QZ)|_{k, \tilde{g}}.$$

So, we may rewrite (3.5) as

(3.6) $$\varepsilon(Z, Q)|_{k, \chi} X = \sum_{D \in \Omega \cap \Omega_0} \sum_{B \in B_x(D)/\mod D} \chi(\det p^\delta D^*) e(QZ)|_{k, \tilde{g}} \tilde{M}_u.$$

We now consider

(3.7) $$\beta(Z, Q) = \sum_{B \in B_x(D)/\mod D} \chi(\det p^\delta D^*) e(QZ)|_{k, \tilde{g}}.$$

From (1.3) follows that

$$\beta(Z, Q) = \sum_{B \in B_x(D)/\mod D} \chi(p^{m-b}) p^{\delta(mk-\langle m \rangle)-bk} e(Qp^\delta Z[D^{-1}] + QBD^{-1})$$

$$= \chi(p^{m-b}) p^{\delta(mk-\langle m \rangle)-bk} e(p^\delta Q[D^*]Z) \sum_{B \in B_x(D)/\mod D} e(QBD^{-1}),$$

and so,

(3.8) $$\beta(Z, Q) = \alpha_{k, \chi}(g_0) e(p^\delta Q[D^*]Z) \sum_{B \in B_x(D)/\mod D} e(QBD^{-1}),$$

where $b = \log_p |\det D| = \log_p |\det D_0|$ and $\alpha_{k, \chi} : L_0^m \to \mathbb{C}^\times$ is a character defined by

(3.9) $$\alpha_{k, \chi}(g) = \chi(p^{m-b}) p^{\delta(mk-\langle m \rangle)-bk}$$

for any $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in L_0^m$. If we take $B + AD$ instead of $B$ as a representative in $B_x(D)/\mod D$ where $^tA = A \in M_m(\mathbb{Z})$, then $e(Q(B + \text{9})
\(ADD^{-1}) = e(QBD^{-1})e(QA) = e(QBD^{-1})\). So (3.8) is independent of the choice of representatives \(B\) of \(B_x(D)/\text{mod }D\). Let \(g_s = \left(\begin{array}{cc} I_m & S \\ 0 & I_m \end{array}\right) \in \Gamma_m^0\) with \(^t\!S = S \in M_m(\mathbb{Z})\). Then \(\tilde{g}_s = (g_s, 1)\) and \(gg_s = \left(\begin{array}{cc} p^\delta D^* & p^\delta D^*S + B \\ 0 & D \end{array}\right)\) so that \(\{B + p^\delta D^*S\}\) is a complete set of representatives of \(B_x(D)/\text{mod }D\) if \(\{B\}\) is. Therefore, from (3.7) follows

\[
\beta(Z, Q)|_{k\tilde{g}_s} = \beta(Z, Q)
\]

Applying \(|_{k\tilde{g}_s}\) on the right hand side of (3.8), we obtain

\[
\beta(Z, Q) = \alpha_{k,x}(g_0)e(p^\delta Q[D^*]Z)e(p^\delta Q[D^*]S)l_x(Q, D).
\]

So, if \(l_x(Q, D) \neq 0\), then \(e(p^\delta Q[D^*]S) = 1\) for any \(^t\!S = S \in M_m(\mathbb{Z})\).

This implies that \(p^\delta Q[D^*] \in N_m^+\). In other words, if \(p^\delta Q[D^*] \notin N_m^+\), then \(l_x(Q, D) = 0\). From this and (3.4), (3.6), (3.8) follows

\[
(3.10) \quad \varepsilon(Z, Q)|_{k,x} X = \alpha_{k,x}(g_0) \sum_{D \in \Omega D_0 \Omega/\Omega} \varepsilon(Z, p^\delta Q[D^*])l_x(Q, D) \in A_m.
\]

Choosing a complete set of representatives \(\{D_i\}\) of \(\Omega D_0 \Omega/\Omega\) such that \(\det D_i = \det D_0\), we may rewrite (3.10) as follows:

\[
(3.11) \quad \varepsilon(Z, Q)|_{k,x} X = \alpha_{k,x}(g_0) \sum_{D \in \Omega D_0 \Lambda/\Lambda} \varepsilon(Z, p^\delta Q[D^*])l_x(Q, D).
\]

We now define a linear map \(\vartheta = \vartheta_{m,n} : A_m \to \Theta_m^n\) by \(\vartheta(\varepsilon(Z, Q)) = \theta^n(Z_n, Q), Q \in N_m^+\), where \(Z = \left(\begin{array}{cc} Z_n & * \\ * & * \end{array}\right) \in \mathcal{H}_m, Z_n \in \mathcal{H}_n\). Obviously \(\vartheta\) is a well-defined epimorphism. From (3.11) and (2.7) follows

\[
\vartheta(\varepsilon(Z, Q)|_{k,x} X) = \theta^n(Z, Q) \circ X,
\]

for any \(X \in L_m^0\), where \(\alpha = \alpha_{k,x}\) as in (3.9).
\[ \varepsilon(Z, Q)|_{k,\chi} X|_{k,\chi} Y = \varepsilon(Z, Q)|_{k,\chi} X \cdot Y \text{ implies} \]

(3.12) \[ \theta^n(Z, Q) \circ_{\alpha} X \circ_{\alpha} Y = \theta^n(Z, Q) \circ_{\alpha} X \cdot Y. \]

From the surjectivity of \( \vartheta \), (3.11) and (3.12) follows (1).

Let \( p \) be relatively prime to \( q \) and let \( X \in \mathcal{L}^{m,0}_0 \). We may assume \( X = (\Gamma^m_0 g \Gamma^m_0) \), \( q = \left( \begin{array}{cc} p^\delta D^* & B \\ 0 & D \end{array} \right) \in \mathcal{L}^m_0 \) such that \( \delta m = 2b \) with \( b = \log_p |\det D| \).

To prove (2), it is enough to show that \( \det 2Q' = d \) and \( q_{Q'} = q \) for \( Q' = p^\delta Q[D^*] \in \mathcal{N}^+_m \), where \( d = \det 2Q \) and \( q = q_Q \). Clearly \( \det 2Q' = d \).

Let \( q' \) be the level of \( Q' \). Then \( q(2Q')^{-1}p^{\delta'} = q p^{\delta'-\delta}(2Q)^{-1}[D] \) is integral for some \( \delta' \geq 0 \). So, \( q' | q p^{\delta'} \), which implies \( q' | q \). Similarly \( q | q' \). This proves (2).

Finally, let \( \delta \) be even. Note that since we restrict ourselves to the case \( m \) odd, \( q = q_Q \) is divisible by 4. Let \( D' = p^{\frac{\delta}{2}} D^* \) so that \( \det D' = \pm 1 \). It is a well known fact [Og] that \( q \) and \( d = \det(2Q) \) have the same prime factors. So \( p \mid d \) and hence one can find \( U \in M_m(\mathbb{Z}) \) such that \( U \equiv D' \pmod{8d^3} \). Since \( 2Q' = 2Q[D'] \), we have \( 2Q' \equiv 2Q[U] \pmod{8d^3} \). Therefore [Si2] \( Q' \in \text{gen}(Q) \) if \( Q' = p^\delta Q[D^*] \in \mathcal{N}^+_m \) and this proves (3).

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