A brief survey of C*-algebra classification theory

The subject arguably began with the work of Năstăianu and Naimark, giving an abstract characterization of concrete C*-algebras (useful for instance for showing that a quotient of a concrete C*-algebra may also be viewed as a concrete C*-algebra), and giving a concrete description of abstract commutative C*-algebras. (The spectrum of an arbitrary C*-algebra is an important invariant?)
The subject began in earnest with the work of Black on WTH algebras (1960), in which he showed that infinite tensor products $M_n$ and $M_{n'}$ (where $n = 2^{k_2}3^{k_3}5^{k_5}\ldots$ and $n' = 2^{k_1}3^{k_3}5^{k_5}\ldots$ are supernatural numbers) are isomorphic if (obviously) and only if $n = n'$. This parametrized the isomorphism classes by means of a Cantor set—the infinite Cartesian product of copies of the totally disconnected compact metric space $\mathbb{Z}_0 \cup \{1, 2, \ldots, +\infty\}$. (So far, so good!)
Dixmier soon showed that this was a fool's paradise (not to cast aspersions on Glimm, who was awarded the U.S. National Medal of Science for this among other results) (JFA Vol. 1, 1967). (G. Kuperman, PJM, Vol. 1.) More precisely, Dixmier showed that, after tensoring \( M_n \) and \( M_{n'} \) above by the \( C^* \)-algebra of compact operators on an infinite-dimensional separable Hilbert space, the resulting (stabilized \( \mathcal{U}(n) \)) algebra are isomorphic iff \( n = \frac{p}{q} n' \) for some \( p \in \mathbb{Q} \).
Since this equivalence relation has (many) dense orbits, Dixmier's very nicely behaved space of moduli flies apart, and becomes completely chaotic (a non-standard Borel space—not absolutely the worst of these, but not very useful!).

With hindsight—and there are even intimations of this in Dixmier's paper—one sees that Dixmier's classification can be described in a very simple, elegant, and natural way with the $K_0$ functor:

\[ \text{Stabilized UHF C* Algebra} \rightarrow K_0 \]
Namely, this functor (defined on this category) is what might be called a classification functor—it distinguishes isomorphism classes.

This prophetic aspect of Dixmier's work did not become fully clear until after Bratteli's generalization of UHF algebras to AF (approximately finite-dimensional) $C^*$-algebras. Bratteli's generalization of Glimm's and Dixmier's classification theorem was more
along the combinatorial lines suggested by the supernatural number approach—a supernatural number is an easy special case of a Bratteli diagram, equivalence classes of which (in a sense close to that of Dye and Dixmier) Bratteli showed classified AF algebras.

Reading between the lines of all these papers, I showed (1976) that (stabilized) AF algebra could be classified by the \( K_0 \) functor:

\[
\text{Matte AF algebra} \quad K_0 \rightarrow \quad \text{Ordered abelian group (countable, torsion-free, \( \mathbb{R} \)-valued)}
\]
Now it was important to consider the order structure on the $K_0$-group — arising of course naturally from the definition of this as the enveloping group of the Murray-von Neumann semigroup (the lion's share of the prophecy perhaps due to these last authors).

The $K_0$-group can be constructed explicitly as the inductive limit of the Bratteli diagram, if the latter is understood as a sequence of finite ordered group direct sums of copies of $Z$. 
The nice thing about the $K_0$ functor is that it is defined in complete generality—in particular for any $C^*$-algebra. Given this, one might wonder why the extension of the $K_0$ classification of AF algebras to the analogous classification of simple AF (approximate unitary) algebras (1992) took over fifteen years.

The answer is that the additional invariant consisting of the tracial simplex (in the unital case) had still to be grasped. Attention was drawn...
to this invariant, and how to deal with it, by Thomson who constructed simple unital AF algebras with $K_0$ equal to $A$ and with convex set of tracial states an arbitrary (metrizable) simplex. With the techniques of Thomson and of the four Romanians Dadarlat, Nagy, Nemethi, and Pastine (simple inductive limits of matrix algebras over spaces of bounded dimension have stable rank one — i.e., dense sets of invertible elements), it was feasible to go from $A\mathcal{F}$ to (simple) $A\mathcal{I}$. 
To go from $A_1 \times A_1$ (approximate circle) algebras took another five years (1987). This was not so much that the invariant had to be extended again—adjoining $C^*$-algebras (or Banach algebras) $K_1$ to the previous $K_0$ and $T$ (tracial simplex)—but that, in formulating the so-called\underline{\text{theorems}} which has almost always so far underlain the isomorphism theorem obtained by the so-called Elliott intertwining argument in a given context, the uniqueness part of it, as opposed
to the existence part (but this is a joke, or the existence part has to match the uniqueness part!), necessarily involves not only $C^*$-algebraic $K_1$, but in fact also algebraic $K_1$! (This is not quite clearly expressed in my paper, but was made completely explicit in a paper by Thomsen and Nielsen.)

With this triple of invariants, ordered $K_0$, $K_1$, and $T$ (the tracial simplex), the stage was now set for further action, which rapidly took place!
Several streams formed quite soon—to begin with, two of classification—one proceeding (rapidly) with the inductive limit approach—indeed, it is possible that all stably finite simple amenable (i.e., nuclear) $C^*$-algebras are inductive limits of matrix algebras over spaces, or subalgebras of them—and one passing over the trace simplex to the case that this is empty, and the order structure, or rather, pre-order structure, on $K_0$ is trivial—everything is positive.
in addition, and one stream of what might be called non-classification! — the construction and study of examples arising in the two streams above but exhibiting more complicated behaviour and not lending themselves to the naive $K_1-\text{trace}$ classification pattern. (Examples of Villadsen had additional $K_0$ structure — holes in the positive cone — but not only did not yield to a $K_0$ classification, but also were modified by Toms to escape completely from the naive classification pattern. Also examples of Rordam
also based on Villadsen's—combined with these, in the setting (as traces), to indicate that the naive invariants $K_0, K_1$, and $T$ would have either to be modified, to be substantially reinforced, or—hope only as a last resort—abandoned entirely!

The dividing line between classifiable and, apparently (or at least with naive invariants, and present techniques however masterful!), non-classifiable, is quite sharp.
I refer to the Toms-Winter Conjecture, which predicts that several quite different forms of good behaviour for an amenable separable simple unital C*-algebra are equivalent, and that this (Toms-Winter) class is in fact classifiable in terms of the invariants that have been successful so far. (The second part of the conjecture is perhaps implicit!) The evidence for this conjecture is very strong, and in the case of either zero traces (Kirchberg) or exactly one trace (...
(the first part)

White: Winter it is known completely. (Vianos 16
implications for almost all parts, it are known either in general or in
interesting cases.)

(A recent breakthrough which led to the full result in
the setting of quite one trace was due to Matui and Solts
—roughly speaking, they established the
conjecture in this setting but assuming quasidiagonality
at one point—relatively harmless since in the tracial
setting, quasidiagonality may hold for any amenable
(simple) C*-algebra.) (Already the work of Matui and
Solts made it possible to show that certain examples
of C*-algebras are classifiable.)
Both classification streams—finite and infinite simple algebras—while starting out looking at quite concrete examples, consisting of inductive limits of quite concrete (usually non-simple) building blocks, at a more or less well defined point, became abstract in the sense of considering axiomatically determined classes.

Thus, after considerable work in the infinite case, based on study of the von Neumann algebras and use of them tensored with commutative algebras (mainly finite-dimensional, interval, or circle algebras)—and all of this based, it should be mentioned (at the very least!)
on the absolutely remarkable paper of Bratteli, Kishimoto, Rordam, and Størmer proving the
Rokhlin property for the shift on the (two-sided) infinite tensor product of 2x2 matrix algebras
more or less out of the blue, Kirchberg and Phillips classified the purely infinite simple separable
C*-algebras (with the UCT). In terms of the countable groups K_0 and K_1 — arbitrary countable abelian groups.
It is now known that these algebras belong to the
Toms—Winter class. The range of the invariant was the
same as for previously classified cases, but the scope now formidable.
In very much the same way in the finite case, Lin axiomatized the classification by Gong and me of real rank zero simple inductive limits of matrix algebras over compact metric spaces of bounded dimension—which he called (UCT) TAF algebras. This class had interesting applications. (For instance, it settled the Powers–Sakai Problem for AF algebras if not for UHF— the UHF case yielded finally to the results of Matui & Sato.)

Subsequently, Lin’s axiomatic approach expanded to encompass the non real rank zero generalization of this class dealt with by Gong, Li, and me (based on a very important decomposition theorem of Gong).
The TAI algebra introduced by Lin to cover this EGL class (and which included all of this) by Long's decomposition theorem evolved into various more general TA(something) classes. (The first one after TAI, TA(splitting interval), dealt with by Niu in his Ph.D. thesis, was notable in that it was an axiomatic class that had no constructive limit counterpart, and so the logistics of the proof had to use in an essential way a certain aspect of the strategy of Lin (TAI, TAI). Namely, prove an abstract member of the class is isomorphic to the model... any model!)
The ultimate TA(something) class has now been dealt with by Dong, Lin, and Niu (see abstract of Lin).

Indeed, GLN can classify the class of “rationally” TA(point-line) algebras — the tensor product with the universal UHF algebra (Mn with n = 2, 3, 5, ...) is TA(point-line) — where point-line means subhomogeneous with spectrum a finite union of lines with a finite set of points at infinity (in some way). The range of the invariant (again, point K0, K1, and traces) is now completely general. It will be useful simple separable stably finite amenable C*-algebras, but generalizing say to just the ASH algebras in this class (all of the class?) will need a Dong-like decomposition theorem.
A more complete survey of classification theory would of course have to pay more attention to technique.

Almost the only technique I have mentioned, besides my intertwining argument, based on a homomorphism theorem for building blocks, are the Rokhlin property (which has been important for the study of automorphism to classify them, not just to classify C*-algebras), Thomson’s trace simplex techniques (including an interesting non-Krein-Milman theorem), Lin’s one-size-fits-all model technique (not used by him for TAF or TAI, but used by Neu and in all later TA (something) theorem, including the GLN TA (point-line), and algebraic K).

A more thorough treatment would have to deal with the Basic Homotopy Lemma, the Winter–Lin–Niez-path one-parameter deformation isomorphism leap-frog technique, and, among other things, the Cuntz semigroup.

The Cuntz semigroup wavering between being a powerful hidden tool and an explicitly formulated additional invariant—perhaps even a critical case for pushing classification beyond the Toms–Winter class.

Good behaviour of the Cuntz semigroup (Blackadar
"strict comparability") is one of the Tone-Winter criteria - thus appearing explicitly - but also appearing indirectly in the recent implication due to $\mathcal{WW}$ from another of the criteria (frang-stability) to yet another one (finite nuclear dimension).

Bad behaviour of the Cuntz semigroup was how Tone first showed my invariants were not sufficient. Any such example works, by remaining with the frame of algebra does not change my invariants, then but does change the Cuntz semigroup making it well behaved.)
The Cuntz semigroup works very well hand-in-hand with the large subalgebra technique, introduced by Paterson in his pioneering analysis of crossed products, and used in almost every crossed product setting, as well as other settings—e.g., the extended rotation algebra $\mathcal{O}$ of Kerr and co-authors. Phillips and Phillips and Archey, have systematically intertwined this technique with the criteria of well-behaved Cuntz semigroups and Jiang-Su stability. In this way, Niu and I have managed to extend the major minimal homomorphic results of Toms-Winter to mean dimension...