Classification of $\mathcal{C}^*$-algebras and $\mathcal{W}^*$-algebras

Classification theorems for amenable $\mathcal{C}^*$-algebras and Connes' fundamental work for injective factors

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The fundamental result for AFD

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- 1986, S. Popa
  gave a short proof for Connes’ fundamental result (in the case of type II_1),

↓ H. Lin introduced the definition of Tracial AF algebra.
Sketch of the proof by Connes and Haagerup

Let $M$ be an injective II$_1$-factor with a separable predual.

By using his previous results of automorphisms, Connes showed that $M^\omega$ has an outer automorphism, $M^\omega \supset M_2(\mathbb{C})$ unitally.
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$\therefore M \bar{\otimes} R \cong M, \quad R$: the AFD $\text{II}_1$ factor.

Now, there are two unital embeddings.

$\iota : M \hookrightarrow M \bar{\otimes} R \subset (M \bar{\otimes} R)^\omega, \quad x \mapsto x \otimes 1_R.\quad \varphi : M \hookrightarrow R^\omega \subset (M \bar{\otimes} R)^\omega.$
Sketch of the proof by Connes and Haagerup

In the alternative proof given by Haagerup, he showed the following condition:
for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ and $\exists a_i \in (M \tilde{\otimes} \mathcal{R})^\omega$, $i = 0, 1, \ldots, N$ such that

$$\iota(x)a_i = a_i \varphi(x) \text{ for all } x \in M, \quad \sum_i a_i^*a_i \approx_\varepsilon 1 \approx_\varepsilon \sum_i a_ia_i^*.$$ 

Define, $\mathcal{M}_0 := (M \tilde{\otimes} \mathcal{R})^\omega \otimes M_2$,

$$\pi(x) := \begin{bmatrix} \varphi(x) & 0 \\ 0 & \iota(x) \end{bmatrix} \in \mathcal{M}_0, \quad A_i := \begin{bmatrix} 0 & 0 \\ a_i & 0 \end{bmatrix} \in \mathcal{M}_0,$$

$$p := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_0, \quad q := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{M}_0.$$

$$(\iota : M \hookrightarrow M \tilde{\otimes} \mathcal{R} \subset (M \tilde{\otimes} \mathcal{R})^\omega, \quad \varphi : M \hookrightarrow \mathcal{R}^\omega \subset (M \tilde{\otimes} \mathcal{R})^\omega).$$
Classifications of $C^*$-algebras and $W^*$-algebras

Sketch of the proof by Connes and Haagerup

Let $\mathcal{N}_0 := \pi(M)' \cap M_0$. It follows that $p, q \in \mathcal{N}_0$. By the condition of $a_i$, we also have $A_i \in \mathcal{N}_0$. Then for any normal tracial state $\tau$ on $\mathcal{N}_0$,

$$\tau(p) \approx_\varepsilon \tau \left( \sum_{i=0}^{N} A_i^* A_i \right) = \tau \left( \sum_{i=0}^{N} A_i A_i^* \right) \approx_\varepsilon \tau(q).$$

which implies that $\tau(p) = \tau(q)$. Then $p$ and $q$ are Murray-von Neumann equivalent, there exist $\nu \in \mathcal{N}_0$ and $u \in (M \otimes R)^\omega$ such that $\nu^* \nu = p$, $\nu \nu^* = q$,

$$\nu = \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix} \in M_0, \quad u^* u = 1_{(M \otimes R)^\omega} = uu^*.$$ 

Since $\nu \in \pi(M)'$ we conclude that

$$\iota(x) = \text{Ad} u(\varphi(x)), \quad \text{for any } x \in M.$$
A brief survey of $C^*$-algebra classification result

- **1989, G. A. Elliott**: showed a classification of amenable $C^*$-algebras called AT algebra of real rank zero by their K-groups, and initiated a program to classify amenable $C^*$-algebras via K-theoretic invariants.
A brief survey of $C^*$-algebra classification result

- 1989, G. A. Elliott showed a classification of amenable $C^*$-algebras called AT algebra of real rank zero by their $K$-groups, and initiated a program to classify amenable $C^*$-algebras via $K$-theoretic invariants.

- 2003, 2005, J. Villadsen, M. Rørdam, A. Toms constructed amenable $C^*$-algebras which have rather pathological property, i.e., can’t be classified by $K$-theoretic invariants.

It is necessary to determine regularity properties for classifiable $C^*$-algebras.
Toms-Winter Conjecture

Conjecture (Toms-Winter, 2008)

Let $A$ be a unital separable simple amenable $C^*$-algebra with $\dim(A) = \infty$. Then the following conditions are equivalent.

(i) $A$ has strict comparison,

(ii) $A \otimes \mathbb{Z} \cong A$,

(iii) $\dim_{\text{nuc}}(A) < \infty$ (resp. $\text{dr}(A) < \infty$ for stably finite cases).

(i) $\iff$ for positive elements $a, b \in A \otimes M_k$ satisfying that

$$\lim_n \tau(a^{1/n}) < \lim_n \tau(b^{1/n})$$

for any tracial state $\tau$ on $A$, there exists $v_n \in A \otimes M_k$ such that $\|v_n b v_n^* - a\| \to 0$, 1982, 2004, Blackadar, Rørdam.
Let $A$ be a unital separable simple amenable C*-algebra with $\dim(A) = \infty$. Then the following conditions are equivalent.

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- **(i)** $\iff$ for positive elements $a, b \in A \otimes M_k$ satisfying that
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  for any tracial state $\tau$ on $A$, there exists $v_n \in A \otimes M_k$ such that $\|v_n b v_n^* - a\| \to 0$, 1982, 2004, Blackadar, Rørdam.

- **(iii)** $\dim_{\text{nuc}}(A)$ (resp. $\text{dr}(A)$) is the smallest number $N \in \mathbb{Z}_+$ satisfying that;
  \[\exists F_{i,n} : \text{finite dim. C*-algebras}, i = 0, 1, \ldots, N, \exists \varphi_n : A \to \bigoplus_{i=0}^N F_{i,n} : \text{c.p.c}, \]
  \[\exists \psi_{i,n} : F_{i,n} \to A : \text{order zero (disjointness preserving) c.p.c, s.t.} \]
  \[
  \| (\sum_{i=0}^N \psi_{i,n}) \circ \varphi_n(a) - a \| \to 0, \forall a \in A, \quad (\text{and } \| \sum_{i=0}^N \psi_{i,n} \| \leq 1),
  \]
Previous results for TW conjecture

- 2004, M. Rørdam,  \((ii) \Rightarrow (i)\).
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- Under the assumption of unique tracial state,
  2013, Matui-S., (i) + QD $\Rightarrow$ (iii) ($\text{dr}(A) \leq 3$).

  2014, White-Winter-S., (i) $\Rightarrow$ (iii) ($\text{dim}_{\text{nuc}}(A) \leq 3$).
Classification of unital simple \( \mathcal{Z} \) absorbing \( \mathbb{C}^* \)-algebras

The following classification is a direct consequence of the above partial answer to TW-conjecture and Lin-Niu and Winter’s classification theorems.

Theorem (Matui-S., Lin-Niu, Winter)

Let \( A, B \) be unital separable simple amenable \( \mathbb{C}^* \)-algebras, with a unique tracial state. Assume that \( A, B \) satisfy (i), QD, UCT. Then \( A \cong B \) if and only if

\[
(K_0(A), K_0(A)_+, [1_A]_0, K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B]_0, K_1(B)).
\]
Applications

Application 1, (2013, Matui-S.)

We obtain a counter example to the Powers-Sakai conjecture.

Application 2, (2014, Ozawa-Rørdam-S.)

The Rosenberg conjecture has an affirmative answer in the class of elementary amenable groups.