Desingularization of labeled graphs and their $C^*$-algebras

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In 1980, J. Cuntz and W. Krieger introduced the universal $C^*$-algebra $O_A$ generated by partial isometries associated with finite $\{0, 1\}$-matrices $A$. 
The $C^*$-algebra of a directed graph was introduced by A. Kumjian, D. Pask, I. Raeburn, J. Renault and Y. Watatani.
R. Exel and M. Laca introduced the Exel-Laca algebras $\mathcal{O}_A$ associated to infinite $\{0, 1\}$-matrices $A$. 

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\vdots & \ddots & \ddots
\end{pmatrix}
\]
T. Carlsen and K. Matsumoto generalized Cuntz-Krieger algebras as $C^*$-algebras $O_\Lambda$ (called Matsumoto algebras) associated to a two-sided subshift $\Lambda$ over a finite alphabet.
T. Bates and D. Pask introduced a class of labeled graph $C^*$-algebras to unify these different classes.
1. Desingularization (removing sinks and sources), ideal and quotient, and simplicity of graph $C^*$-algebras.

2. Desingularization (removing sinks and sources), ideal and quotient, and simplicity of labeled graph $C^*$-algebras.
Terminology

1. A (directed) graph \( E = (E^0, E^1, r, s) \) consists of a countable set \( E^0 \) of vertices, a countable set \( E^1 \) of edges, and \( r, s : E^1 \to E^0 \), range and source maps.

2. A path \( \alpha \) of length \( n \) in \( E \) is a sequence of edges \( \alpha_1 \alpha_2 \cdots \alpha_n \) such that \( r(\alpha_i) = s(\alpha_{i+1}) \) for \( i = 1, 2, \cdots, n-1 \).

3. For nonnegative integer \( n \), we denote the set of paths in \( E \) of length \( n \) by \( E^n \), and the set of all vertices and paths by \( E^* = \bigcup_{n=0}^{\infty} E^n \).

4. A vertex \( v \) in \( E^0 \) is a sink if it emits no edges.

5. A vertex \( v \) in \( E^0 \) is a source if it receives no edges.
Definition (Kumjian, Pask, Raeburn, 1998)

A Cuntz-Krieger $E$-family $\{s_e, p_v : e \in E^1, v \in E^0\}$ consists of partial isometries $s_e$ with mutually orthogonal ranges and mutually orthogonal projections $p_v$ such that

(i) $s_e^*s_e = p_{r(e)}$,
(ii) $s_es_e^* \leq p_{s(e)}$,
(iii) if the number of edges starting at $v$ is finite, then

$$p_v = \sum_{s(e) = v} s_es_e^*.$$ 

The graph algebra $C^*(E)$ is the universal $C^*$-algebra generated by a CK $E$-family.

From the Cuntz-Krieger relation, $C^*(E) = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*, r(\alpha) = r(\beta)\}.$

where $s_\alpha = s_{\alpha_1} \ldots s_{\alpha_n}$. 

Theorem (Bates, Pask, Raeburn, Szymanski, 2000)

Using the process of adding a tail to a sink and adding a head to a source, one can obtain a new graph \( E' \) with no sinks or sources from any graph \( E \). Then two \( C^* \)-algebras \( C^*(E) \) and \( C^*(E') \) are Morita equivalent.

For example, given a graph \( E \) with a sink \( v \) and a source \( w \),

\[
\begin{array}{c}
\bullet & \rightarrow & \bullet & \rightarrow & \bullet \\
W & & & & V
\end{array}
\]

a new graph \( E' \) is given by

\[
\begin{array}{c}
\cdots & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \cdots \\
W & & & & & & & & & & & & & & V
\end{array}
\]
By universal property of $C^*(E) = C^*(s_e, p_v)$, there exists a strongly continuous action $\gamma : \mathbb{T} \rightarrow Aut(C^*(E))$, called the gauge action, such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$.

Let $H$ be a subset of $E^0$.

1. $H$ is hereditary if $v \in H$ implies $r(e) \in H$ for all edges $e$ starting at $v$.
2. $H$ is saturated if $r(e) \in H$ for all edges $e$ starting at $v$ implies $v \in H$.
3. For a saturated hereditary subset $H$ of $E^0$, we define $I_H$ to be an ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$. 
Theorem (Bates, Pask, Raeburn, Szymanski, 2000)

Let \( E \) be a row-finite directed graph.

(i) The map \( H \mapsto I_H \) is a lattice isomorphism of saturated hereditary subsets of \( E^0 \) onto the lattice of gauge-invariant ideals of \( C^*(E) \).

(ii) \( C^*(E)/I_H \) is isomorphic to \( C^*(F) \) where \( F \) is a subgraph of \( E \) such that \( F^0 := E^0 \setminus H \) and \( F^1 := \{ e \in E^1 : r(e) \notin H \} \).

Furthermore, it is well-known that if a directed graph \( E \) satisfies Condition (K), then all ideals in \( C^*(E) \) are gauge-invariant (BPRS).
Consider the following directed graph $E$:

There exist five hereditary saturated subsets of $E^0$;

$\emptyset$, $\{v_2, v_3\}$, $\{v_4\}$, $\{v_2, v_3, v_4\}$ and $E^0$.

Thus the $C^*$-algebra $C^*(E)$ has five gauge-invariant ideals;

$\mathcal{I}_0 = 0$, $\mathcal{I}_{\{v_2, v_3\}}$, $\mathcal{I}_{\{v_4\}}$, $\mathcal{I}_{\{v_2, v_3, v_4\}}$, and $\mathcal{I}_{E^0} = C^*(E)$.

Since the graph $E$ satisfies Condition (K), $C^*(E)$ has only five ideals.
Furthermore, the quotient $C^*$-algebra $C^*(E)/I_{\{v_4\}}$ is isomorphic to $C^*(F)$, where the graph $F = E \setminus \{v_4\}$ is given by
Simplicity of graph algebras

Theorem (Kumjian, Pask, Raeburn, Renault, 1997)

Graph $C^*$-algebra $C^*(E)$ is simple if and only if

1. $E$ satisfies condition (K) and has no saturated hereditary subset except $\emptyset$ and $E^0$,

2. $E$ satisfies condition (K) and cofinal, and

3. $E$ satisfies condition (L) and cofinal.

We say that $E$ satisfies condition (L) if every loop has an exit. That is, $vE^\infty \neq \{\alpha^\infty\}$ for any vertex $v$. 
**Definition** (Bates, Pask, 2007)
A labeled graph \((E, \mathcal{L})\) over a countable alphabet set \(\mathcal{A}\) consists of a directed graph \(E\) and a labeling map \(\mathcal{L} : E^1 \rightarrow \mathcal{A}\).

1. For a finite path \(\lambda = \lambda_1 \cdots \lambda_n \in E^n\), put \(\mathcal{L}(\lambda) = \mathcal{L}(\lambda_1) \cdots \mathcal{L}(\lambda_n)\).

2. The range of a labeled path \(\alpha \in \mathcal{L}(E_{\geq 1})\) is a subset of \(E^0\) defined by \(r(\alpha) = \{r(\lambda) \mid \lambda \in E_{\geq 1}, \mathcal{L}(\lambda) = \alpha\}\).

3. The relative range of \(\alpha \in \mathcal{L}(E_{\geq 1})\) with respect to \(A \subset E^0\) is defined to be \(r(A, \alpha) = \{r(\lambda) \mid \lambda \in E_{\geq 1}, \mathcal{L}(\lambda) = \alpha, s(\lambda) \in A\}\).
Consider the following labeled graph \((E, \mathcal{L})\):

\[
\begin{array}{c}
\bullet & \xrightarrow{a} & v_1 & \xrightarrow{b} & v_3 \\
\bullet & \xrightarrow{a} & v_2 & \xleftarrow{b} & \bullet
\end{array}
\]

\[
r(a) = \{v_1, v_2\}, \quad r(ab) = \{v_2, v_3\} \neq r(b) = \{v_1, v_2, v_3\}.
\]
We call $\mathcal{B} \subset 2^{E^0}$ an **accommodating set** for $(E, \mathcal{L})$ if

1. $r(\alpha) \in \mathcal{B}$,
2. $r(A, \alpha) \in \mathcal{B}$, and
3. $A \cap B, A \cup B \in \mathcal{B}$

for $A, B \in \mathcal{B}$ and $\alpha \in \mathcal{L}(E \geq 1)$. If $\mathcal{B}$ is an accommodating set for $(E, \mathcal{L})$, the triple $(E, \mathcal{L}, \mathcal{B})$ is called a **labeled space**.

Let $\overline{\mathcal{E}}$ be the smallest one among the accommodating sets $\mathcal{B}$ for $(E, \mathcal{L})$ such that $A \setminus B \in \mathcal{B}$ whenever $A, B \in \mathcal{B}$.

In this talk, we shall focus on the labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$. 
Labeled graph $C^*$-algebra

Definition (Bates, Pask, 2007)

(When $E$ has no sinks or sources)
A representation of $(E, \mathcal{L}, \overline{E})$ consists of projections $\{p_A : A \in \overline{E}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ such that for $A, B \in \overline{E}$ and $a, b \in \mathcal{A}$,

(i) $p_A p_B = p_{A \cap B}$, $p_\emptyset = 0$ and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$,
(ii) $p_A s_a = s_a p_{r(A,a)}$,
(iii) $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$,
(iv) for $A \in \overline{E}$, if $\mathcal{L}(AE^1)$ is finite and non-empty, then
$$p_A = \sum_{a \in \mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^*.$$ 

There exists a $C^*$-algebra $C^*(E, \mathcal{L}, \overline{E})$ generated by a universal representation $\{s_a, p_A\}$ of $(E, \mathcal{L}, \overline{E})$. We call $C^*(E, \mathcal{L}, \overline{E})$ the labeled graph $C^*$-algebra of a labeled space $(E, \mathcal{L}, \overline{E})$. Then
$$C^*(E, \mathcal{L}, \overline{E}) = \overline{\text{span}\{s_\alpha p_A s_\beta^* | \alpha, \beta \in \mathcal{L}(E^{\geq 1}), A \in \overline{E}\}}.$$
This labeled graph presents the Dyck shift $D_2$. The labeled graph $C^*$-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is unital, simple, purely infinite and $K_0(C^*(E, \mathcal{L}, \overline{\mathcal{E}})) \cong \mathbb{Z}_2 \oplus C(\mathcal{C}, \mathbb{Z})$, $K_1(C^*(E, \mathcal{L}, \overline{\mathcal{E}})) \cong 0$. Since $K_0$-group is not finitely generated, $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is not isomorphic to a graph $C^*$-algebra.
If a graph $E$ has sinks or sources, then we have many choices of an accommodating set to define labeled graph $C^*$-algebras, considering how to handle the sinks or sources;

(i) $E_1 = \{ r(\alpha) : \alpha \in L^* (E) \} \cup \{ \{ v \} : v \in E_{\text{sink}}^0 \} \
\hspace{2cm} \cup \{ \{ v \} : v \in E_{\text{source}}^0 \}$

(ii) $E_2 = \{ r(\alpha) : \alpha \in L^* (E) \} \cup \{ r(\alpha)_{\text{sink}} : \alpha \in L^* (E) \} \
\hspace{2cm} \cup \{ \{ v \} : v \in E_{\text{source}}^0 \}$

(iii) $E_3 = \{ r(\alpha) : \alpha \in L^* (E) \} \cup \{ \{ v \} : v \in E_{\text{sink}}^0 \}$

(iv) $E_4 = \{ r(\alpha) : \alpha \in L^* (E) \} \cup \{ r(\alpha)_{\text{sink}} : \alpha \in L^* (E) \}$

where $A_{\text{sink}} = A \cap E_{\text{sink}}^0$. And for each smallest accommodating set $\overline{E}_i$ containing $E_i$, we have non-isomorphic labeled graph $C^*$-algebras $C^*(E, L, \overline{E}_i)$ for $i = 1, \cdots, 4$. 
(i) $C^*(E, \mathcal{L}, \overline{\mathcal{E}_1})$ is not defined,
(ii) $C^*(E, \mathcal{L}, \overline{\mathcal{E}_2})$ is not defined,
(iii) $C^*(E, \mathcal{L}, \overline{\mathcal{E}_3}) \cong M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$,
(iv) $C^*(E, \mathcal{L}, \overline{\mathcal{E}_4}) \cong M_4(\mathbb{C})$. 
Theorem (K.)

For any choice of an accommodating set for \((E, \mathcal{L})\), there exists a labeled graph \((F, \mathcal{L}_F)\) obtained by adding a head and adding a tail such that \(C^*(E, \mathcal{L}, \mathcal{E})\) is Morita equivalent to \(C^*(F, \mathcal{L}_F, \overline{E})\).

Therefore we may assume that labeled graph has no sinks or sources (up to Morita equivalence).
Definition (Jeong, K., Park, 2012)

Let $H$ be a subset of an accommodating set $\overline{E}$. $H$ is said to be *hereditary* if $H$ satisfies the following:

1. $r(A, a) \in H$ for all $A \in H$, $a \in A$,
2. $A \cup B \in H$ for all $A, B \in H$,
3. $B \in H$ if $A \in H$, $B \in \overline{E}$ with $B \subset A$.

A hereditary subset $H$ of $\overline{E}$ is called *saturated* if for any $A \in \overline{E}$, $\{r(A, a) : a \in A\} \subset H$ implies that $A \in H$.

Theorem (Jeong, K., Park, 2012)

Let $(E, \mathcal{L}, \overline{E})$ be a labeled space. Then the map $H \mapsto I_H$ is a bijection from the set of nonempty saturated hereditary subsets of $\overline{E}$ into the set of gauge-invariant ideals of $C^*(E, \mathcal{L}, \overline{E})$. 

Definition

Let \((E, \mathcal{L}, \mathcal{E})\) be a labeled space and \(H\) be a nonempty saturated hereditary subsets of \(\mathcal{E}\). Define an equivalence relation \(\sim_I\) on \(\mathcal{E}\) by

\[A \sim_I B \iff A \cup W = B \cup W \text{ for some } W \in H\]

\((\iff p_A + I_H = p_B + I_H \text{ on } C^*(E, \mathcal{L}, \mathcal{E})/I_H).\)

Denote the equivalence class of \(A \in \mathcal{E}\) by \([A]\) and define the equivalence classes \([\mathcal{E}]_I := \{[A] : A \in \mathcal{E}\}\). We call a triple \((E, \mathcal{L}, [\mathcal{E}]_I)\) a quotient labeled space of \((E, \mathcal{L}, \mathcal{E})\). There exists a \(C^*\)-algebra generated by a universal representation of \((E, \mathcal{L}, [\mathcal{E}]_I)\). We call \(C^*(E, \mathcal{L}, [\mathcal{E}]_I)\) the quotient labeled graph \(C^*\)-algebra.

Theorem (Jeong, K., Park, 2012)

Let \(I\) be a nonzero gauge-invariant ideal of \(C^*(E, \mathcal{L}, \mathcal{E})\). Then there exists an isomorphism of \(C^*(E, \mathcal{L}, [\mathcal{E}]_I)\) onto the quotient algebra \(C^*(E, \mathcal{L}, \mathcal{E})/I\).
Quotients of labeled graph \( C^* \)-algebras

**Definition (K.)**

1. We call \([A] \in \overline{[E]} \) minimal if \([A] \neq [\emptyset] \) and \([A] \cap [B] \) is either \([\emptyset] \) or \([A] \) for all \([B] \in \overline{[E]} \).

2. We say that the quotient labeled space \((E, \mathcal{L}, \overline{[E]} \) is minimal if for any nonempty set \([B] \in \overline{[E]} \) there exist finite minimal sets \([A_i] \)’s for \(i = 1, \ldots, n\) such that \([B] = \bigcup_{i=1}^{n}[A_i] \).

**Theorem (K.)**

(i) If a quotient labeled space \((E, \mathcal{L}, \overline{[E]} \) is minimal, then \(C^*(E, \mathcal{L}, \overline{[E]} \) is isomorphic to a labeled graph \( C^* \)-algebra.

(ii) If an alphabet set \(A_1 \) is a finite set, then \(C^*(E, \mathcal{L}, \overline{[E]} \) is isomorphic to a Matsumoto algebra.
Theorem (Bates, Pask, 2009)

If a labeled space \((E, \mathcal{L}, \overline{E})\) is disagreeable and strongly cofinal (analogue of condition \((L)\) and cofinal), then \(C^*(E, \mathcal{L}, \overline{E})\) is simple.

Theorem (Jeong, K., 2012)

When every singleton set \(\{v\} \in \overline{E}\), \(C^*(E, \mathcal{L}, \overline{E})\) is simple implies that \((E, \mathcal{L}, \overline{E})\) is disagreeable and strongly cofinal.
Definition (Bates, Pask, 2009)

A labeled path $\alpha \in \mathcal{L}(E \geq 1)$ with $s(\alpha) \cap [v]_l \neq \emptyset$ is said to be agreeable for $[v]_l$ if $\alpha = \beta \alpha' = \alpha' \gamma$ for some $\alpha', \beta, \gamma \in \mathcal{L}(E \geq 1)$ with $|\beta| = |\gamma| \leq l$. Otherwise $\alpha$ is said to be disagreeable. We call $[v]_l$ disagreeable if there is an $N > 0$ such that for all $n > N$ there is an $\alpha \in \mathcal{L}(E \geq n)$ that is disagreeable for $[v]_l$. The labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ is disagreeable if for every $v \in E^0$ there is an $L_v > 0$ such that $[v]_l$ is disagreeable for all $l > L_v$.

Theorem (K.)

The following are equivalent:

1. $(E, \mathcal{L}, \overline{\mathcal{E}})$ is disagreeable,
2. $\mathcal{L}(AE^\infty) \neq \{\alpha^\infty\}$ for all $A \in \overline{\mathcal{E}}$. 
Example

$x = \cdots ba^n b \cdots ba^3 ba^2 bab^2 ab^3 a \cdots ab^n a \cdots$

\[
\begin{array}{cccccccc}
\ldots \bullet & a & a & a & a & a & a & a \\
\downarrow & b & b & b & b & b & b & b \\
\end{array}
\]

The $C^*$-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(s_a, s_b)$ is the universal $C^*$-algebra with relations

$s_a^* s_a + s_b^* s_b = 1 = s_a s_a^* + s_b s_b^*$

and $\cdots$. Since the number of saturated hereditary set is infinite, it is not isomorphic to a row-finite graph $C^*$-algebra.
THANK YOU