Constructing the duals of quantum groups and semigroups without the Haar weight

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Pontryagin’s duality: $G \simeq \hat{G}$

where $G$ is a locally compact abelian group and $\hat{G}$ is the dual group

If $G$ is non-commutative: $\hat{G}$ is no more a group
so we pass to group algebras: $C_0(G)$ and $C^*(G)$,
    or $L^\infty(G)$ and $L(G)$, or another pair

Kac algebras (1970s: Vainerman, Kac, Enock, Schwartz)
Locally compact quantum groups (LCQG) 2000s, Kustermans, Vaes
I am not giving a historical overview!
Duality for locally compact quantum groups

Von Neumann algebraic LCQG is a von Neumann algebra $M$ with a comultiplication $\Delta : M \to M \otimes M$, which is a normal unital *-homomorphism such that $(\mathbb{I} \otimes \Delta)\Delta = (\Delta \otimes \mathbb{I})\Delta$; and a pair of Haar (invariant) weights $\phi, \psi$

Classical example: $M = L^\infty(G), \hat{M} = \mathcal{L}(G)$

Duality (Kustermans–Vaes)

If $M$ is a von Neumann algebraic LCQG, then the following diagram commutes:

\[
\begin{array}{ccc}
    M & \xrightarrow{\text{predual}} & M_* \\
    \lambda_\phi & \downarrow & \lambda_\phi \\
    \hat{M}_* & \xleftarrow{\text{predual}} & \hat{M}
\end{array}
\]

The weight $\phi$ is used both to define the dual algebra and to guarantee the duality: $\hat{M} \simeq M$
Definition of algebras we are working with

\( M \) is a \textbf{VN bialgebra with antipode (VNBA)} if:

1. \( M \) is a von Neumann algebra;
2. there exists a comultiplication and an antipode \( S : D(S) \subset M \to M \) such that
   - \( S : D(S) \to M \) is an anti-homomorphism;
   - \( (\ast S)^2 = \mathbb{I} \);
   - \( D(S) \) is \( \sigma \)-weakly dense in \( M \);
3. \( S \) is a “coalgebra morphism”: if \( \mu, \nu \in M_* \) are such that \( \mu \circ S, \nu \circ S \in M \), then for \( x \in D(S) \)
   \[
   (\Delta(x))(\nu \circ S \otimes \mu \circ S) = (\Delta S(x))(\mu \otimes \nu).
   \]

Example: LCQG, but also quantum semigroups
If \( S \) is bounded, \( M \) is a coinvolutive Hopf-VNA, a class which includes Kac algebras
Duality without the Haar measure

Multiplicative unitaries:
Baaj, Skandalis: regular multiplicative unitaries
Woronowicz: manageable multiplicative unitaries

Recall that for a LCQG, both $M$ and $\hat{M}$ are generated by a unitary $W \in M \tilde{\otimes} \hat{M}$:

\[
M = \{ \omega \otimes \text{id})(W) : \omega \in \hat{M}* \}
\]

\[
\hat{M} = \{ \omega \otimes \text{id})(W) : \omega \in M* \}
\]

One can start with a $W$ and define $M$, $\hat{M}$ as above. If $W$ is "good" then these are Hopf algebras

but how to construct such unitaries?
Representations of $M_*$

Let $M$ be a von Neumann bialgebra with an antipode. Recall that $M_*$ is a Banach algebra, with

$$(\mu \ast \nu)(x) = (\mu \otimes \nu)(\Delta(x))$$

for $\mu, \nu \in M_*, x \in M$

The natural involution is defined by $\mu^*(x) = \overline{\mu((Sx)^*)}$, and is defined only for a subset $M^{**} \subset M_*$.

Kustermans, 2000: if $M = L^\infty(G)$ is a LCQG, then $M^{**} = L^1(G)^\#$ is dense in $M_*$.

In general, $M^{**}$ might not be dense in $M_*$, but one can pass to $M^r = M/(M^{**})^\perp$, then $M^r_* = [M^{**}]$ and $M^r$ is still a VNBA.

A $^*$-representation of $M_*$ is by definition repn which is involutive on $M^{**}$. We consider only representations $\pi : M_* \to B(H)$ which are completely bounded, and this is equivalent to the existence of $U \in M \widehat{\otimes} B(H)$ (generator) such that

$U(\mu, \omega) = \omega(\pi(\mu))$ for all $\mu \in M_*, \omega \in B(H)^\ast$. 
Unitary representations

Definition

Call a *-representation $\pi$ of $M_*$ on a Hilbert space $H$ unitary if (i) or (ii) holds:

(i) the generator $U$ is unitary; (ii), equivalently, in some basis $(e_\alpha)$ of $H$, with $\pi_{\alpha \beta}(\mu) = \langle \pi(\mu)e_\beta, e_\alpha \rangle$,

$$
\sum_\gamma \pi^*_{\gamma \alpha} \cdot \pi_{\gamma \beta} = \sum_\gamma \pi_{\alpha \gamma} \cdot \pi^*_{\beta \gamma} = \begin{cases} 
1, & \alpha = \beta \\
0, & \alpha \neq \beta 
\end{cases}
$$

for every $\alpha, \beta$, the series converging absolutely in the $M_*$-weak topology of $M$. 
If $M_*$ is commutative, then its irreducible representations are characters;

$\pi$ is unitary $\iff$ it is a unitary element in $M$:

$\pi^* \pi = \pi \pi^* = 1$

Consider $M_* = B(G) = C^*(G)$, the Fourier–Stieltjes algebra of $G$.
$M = B(G)^* = W^*(G) = C^*(G)^*$, the “big group algebra” of J. Ernest

**Theorem (M. Walter)**

For a character $u$ of $B(G)$ TFAE:

- $u$ is unitary in $W^*(G)$
- $u(f) = f(t)$ for some $t \in G$
- $u|_{A(G)} \not\equiv 0$. 
Representations of the measure algebra

Theorem
An irreducible representation $\pi$ of $M(G)$ is unitary if and only if it is generated by a continuous unitary representation $\tilde{\pi}$ of $G$ by the integral formula

$$\pi(\mu) = \int_G \tilde{\pi}(t) d\mu(t), \quad (*)$$
The unitary dual and the universal $C^*$-algebra

Theorem (E. Kirchberg, 1977)

There is a functor $M \mapsto W^* U(M)$ on the category of CHvNA such that

\[ \{ \text{normal reps of } W^* U(M) \} \leftrightarrow \{ \text{unitary reps of } M_* \} \]

If $M$ has a Haar weight, then $\hat{M} = W^* (M_*)$ and $\hat{M} \simeq \hat{\hat{M}}$.

$C^*$-version : Ch.-K. Ng (2002)

Theorem (J. Kustermans, 2000)

Let $G$ be a LCQG. Then there exists a $C^*$-algebra $C_u^*(\widehat{G})$ whose *-representations are in bijection with unitary *-representations of $L_1(G)$.
Absolutely continuous ideal

Definition

\[ M^0_\ast = \bigcap \{ \ker \pi : \pi \text{ is irreducible and non-unitary} \} \]

By definition, if an irrep \( \pi \not\equiv 0 \) on \( M^0_\ast \) then \( \pi \) is unitary.

If \( M \) is weakly separable, by disintegration one can show this for general \( \pi \) which is non-degenerate on \( M^0_\ast \).

We need more:

if \( \pi \) is unitary (on \( M_\ast \)) then it is non-degenerate on \( M^0_\ast \) (*)

Solution: reset \( M^0_\ast = 0 \) if this does not hold; then either (*) or \( M^0_\ast = 0 \).

In fact: set \( M^0_\ast = 0 \) if the ideal generated by \((M^0_\ast)\perp\) equals \( M \).

Finally: if \( M^0_\ast \neq 0 \) then

\( \pi \) is unitary iff \( \pi \) is non-degenerate on \( M^0_\ast \)

If \( M \) is a LCQG: \( M^0_\ast = M_\ast \)

If \( M = L^\infty(S) \) for a semigroup \( S \) with only \( e \) invertible, then \( M^0_\ast = 0 \).
Group case

If $M_* = M(G)$, this ideal is the common kernel of all reps which annihilate $L_1(G)$ and has thus the same representations as $L_1(G)$; $C^*(M^0_*) = C^*(G)$.

J. Taylor denoted it by $L^{1/2}(G)$ and proved that $M^0_* \neq L_1(G)$ unless $G$ is discrete.

If $M_* = B(G)$, it is the common kernel of all non-evaluation characters; has the same characters as $A(G)$, i.e. evaluations at points of $G$; $C^*(M^0_*) = C_0(G)$
Duality : definition

Assume $M$ is weakly separable

**Definition**

Set $\hat{M} = W^*(M^0)$: the enveloping von Neumann algebra of $C^*(M^0)$. Then $\hat{M}$ has a canonical structure of a VNBA and is called the dual algebra of $M$.

**Theorem**

The map $M \mapsto \hat{M}$ is a functor on the category of weakly separable VNBA.

If $M$ is a LCQG, then $\hat{M} = W^*(M_*)$;

if $M$ is a Kac algebra then $\hat{M} \simeq \hat{M}$ (probably true for all LCQG).
Other examples

\[ M = \hat{N} \text{ and } \begin{cases} \hat{M} \neq 0 \text{ is commutative } & \Rightarrow M \simeq C_0(G)^{**} \\ \hat{\hat{M}} \neq 0 \text{ is commutative } & \Rightarrow M \simeq C^*(G)^{**} \end{cases} \] for some \( G \) (\( G \) is a locally compact group)

Various semigroups \( M \) give:
- \( \hat{M} = 0 \) or \( \hat{M} = \mathbb{C} \)
- \( W^*(\mathbb{R}_+) \), \( L^\infty(\mathbb{R}) \) with \( Sf = f \),
- \( SU_q(N) \) with a bounded antipode;
- \( C^*(\Sigma) \) where \( \Sigma \) is a discrete sub-semigroup of a group
  with \( x \in \Sigma \) invertible iff \( x = e \);
for \( M = L^\infty(\mathbb{R}^2) \) with \( Sf(s,t) = f(s,-t) \), we have \( \hat{M} = C_0(\mathbb{R})^{**} \)

Conjecture: if the canonical map \( D_M : \hat{\hat{M}} \to M \) is surjective, then \( M \) carries a Haar weight

(satisfied for example if \( S \) is bounded and \( \hat{M} \neq 0 \))
Theorem

There is a functor $A \mapsto \hat{A}$ on the category of separable $C^*$-bialgebras with antipode defined as $\hat{A} = \hat{M}$, where $M = A^{**}$ is the enveloping VNA of $A$ with induced structure.

$A = \hat{B}$ and
\[
\left\{ \begin{array}{l}
A \neq 0 \text{ is commutative} \Rightarrow A \simeq C_0(G) \\
\hat{A} \neq 0 \text{ is commutative} \Rightarrow A \simeq C^*(G)
\end{array} \right.
\]
for some $G$ ($G$ is a locally compact group)

For every LCG $G$, the following diagram commutes:

\[
\begin{array}{ccc}
\widehat{C^*_u(G)} & \simeq & C_0(G) \\
\widehat{B(G)} & \simeq & C^*_u(G)
\end{array}
\]

\[
\begin{array}{ccc}
\widehat{C^*_u(G)} & \simeq & C_0(G) \\
\widehat{B(G)} & \simeq & C^*_u(G)
\end{array}
\]

\[
\begin{array}{c}
\downarrow C^*_u-env \\
\uparrow C^*_u-env
\end{array}
\]

\[
\begin{array}{cc}
\widehat{C^*_u(G)} & \simeq & C_0(G) \\
\widehat{B(G)} & \simeq & C^*_u(G)
\end{array}
\]