Note on paranormal operators and operator equations \(ABA = A^2\) and \(BAB = B^2\)

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Fredholm Operators

**Definitions**

Let $T \in B(H)$.

- $T$ is called **upper semi-Fredholm** if $R(T)$ is closed and $\alpha(T) < \infty$,
- $T$ is called **lower semi-Fredholm** if $\beta(T) < \infty$.
- $T$ is called **Fredholm** if $\alpha(T) < \infty$ and $\beta(T) < \infty$, in this case, the **index** is defined by
  $$i(T) := \alpha(T) - \beta(T).$$
- $T$ is called **Weyl** if it is Fredholm of index zero.
- $T$ is called **Browder** if it is Fredholm of finite ascent and descent.
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- $T$ is called Weyl if it is Fredholm of index zero.
- $T$ is called Browder if it is Fredholm of finite ascent and descent.
The smallest nonnegative integer $p$ such that $N(T^p) = N(T^{p+1})$ is called the ascent of $T$ and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer $q$ such that $R(T^q) = R(T^{q+1})$ is called the descent of $T$ and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$. 
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Several Spectrums

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Local Spectrum

Given an arbitrary $T \in B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, the local resolvent set $\rho_T(x)$ of $T$ at the point $x \in \mathcal{H}$ is defined as the union of all open subsets $U$ of $\mathbb{C}$ for which there is an analytic function $f : U \to \mathcal{H}$ which satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. 

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We define the local spectral subspaces of $T$ by

$$H_T(F) := \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\} \text{ for all sets } F \subseteq \mathbb{C}.$$
Localized Single Valued Extension Property

**Definitions [1952, N. Dunford]**

$T \in B(X)$ has the **single valued extension property** at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0$) if for every open neighborhood $U$ of $\lambda_0$ the only analytic function $f : U \rightarrow X$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on $U$. The operator $T$ is said to have SVEP if $T$ has SVEP at every $\lambda \in \mathbb{C}$. 
Well Known Facts

\[ p(T - \lambda) < \infty \implies T \text{ has SVEP at } \lambda \]

\[ q(T - \lambda) < \infty \implies T^* \text{ has SVEP at } \lambda \]

It is well known that if \( T - \lambda \) is semi-Fredholm, then these implications are equivalent.
Operator equations $ABA = A^2$ and $BAB = B^2$

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\[ABA = A^2 \quad \text{and} \quad BAB = B^2. \tag{1.1}\]

[A and B are self-adjoint operators satisfying the operator equations (1.1) if and only if \(A = PP^*\) and \(B = P^*P\) for some idempotent operator \(P\).]
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[2006, C. Schmoeger]

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The common spectral properties of the operators $A$ and $B$ satisfying the operator equations (1.1).

[2011, B.P. Duggal]

It is possible to relate the several spectrums, the single-valued extension property and Bishop’s property ($\beta$) of $A$ and $B$. 
Note.

(1) If \( \lambda \neq 0 \), then
\[
N(A - \lambda I) = N(AB - \lambda I) = A(N(B - \lambda I)),
\]
\[
N(B - \lambda I) = N(BA - \lambda I) = B(N(A - \lambda I)),
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and

$$\alpha(A - \lambda I) = \alpha(AB - \lambda I) = \alpha(BA - \lambda I) = \alpha(B - \lambda I).$$
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Moreover, if \( \lambda \neq 0 \), then

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p(A - \lambda I) = p(AB - \lambda I) = p(BA - \lambda) = p(B - \lambda I) \text{ and}
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q(A - \lambda I) = q(AB - \lambda I) = q(BA - \lambda) = q(B - \lambda I)
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\[(2) \quad \sigma_x(A) = \sigma_x(AB) = \sigma_x(BA) = \sigma_x(B),\]

where $\sigma_x = \sigma, \sigma_p, \sigma_a, \sigma_{SF+}, \sigma_{SF-}, \sigma_e, \sigma_w,$ or $\sigma_b$. 
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where \( \sigma_x = \sigma, \sigma_p, \sigma_a, \sigma_{SF+}, \sigma_{SF-}, \sigma_e, \sigma_w \), or \( \sigma_b \).

(3) \( A \) has SVEP iff \( AB \) has SVEP iff \( BA \) has SVEP iff \( B \) has SVEP.
Question!

Q. When $A$ is paranormal (respectively, normal), is $AB$, $BA$, or $B$ also a paranormal (respectively, normal) operator?
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**Definitions**

$T \in B(\mathcal{H})$ is **normal** if $T^* T = TT^*$ and $T$ is **paranormal** if

$$\| Tx \|^2 \leq \| T^2 x \| \| x \|$$ for all $x \in \mathcal{H}$. 
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$$\{\text{Normal }\} \subseteq \{\text{Paranormal }\} \subseteq \{\text{Polynomial roots of paranormal operators }\}$$
Answer!

A. No, it isn’t.
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**Example 1**

let $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix}$ in $B(\mathcal{H} \oplus \mathcal{H})$. Then $P^2 = P$ and $Q^2 = Q$. If $A := PQ$ and $B := QP$, then $(A, B)$ is a solution of the operator equations (1.1). Since $B^* = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}$, a straightforward calculation shows that
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\[
B^2 B^2 - 2 \lambda B^* B + \lambda^2 I = \begin{pmatrix} (2 - 4 \lambda + \lambda^2) I & 0 \\ 0 & \lambda^2 I \end{pmatrix},
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$$B^2 B^2 - 2\lambda B^* B + \lambda^2 \nless 0.$$ 

Therefore $B$ is neither paranormal nor normal. On the other hand, $A$ is normal, so that it is a paranormal operator.
Example 2

If $P = \begin{pmatrix} I & 2I \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ are in $B(\mathcal{H} \oplus \mathcal{H})$, then both $P$ and $Q$ are idempotent operators. Also, $A := PQ$ and $B := QP$ satisfy the operator equations (1.1). Since $B^*A^* = \begin{pmatrix} I & 0 \\ 2I & 0 \end{pmatrix}$, a straightforward calculation shows that

However, $(4 - 8\lambda + \lambda^2)I$ is not a positive operator for $\lambda = 1$, hence $AB$ is neither paranormal nor normal. On the other hand, $A$ is normal, so that it is a paranormal operator.
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$$(AB)^{2*}(AB)^2 - 2\lambda (AB)^*(AB) + \lambda^2 I = \begin{pmatrix} (1 - 2\lambda + \lambda^2)I & (2 - 4\lambda)I \\ (2 - 4\lambda)I & (4 - 8\lambda + \lambda^2)I \end{pmatrix}.$$

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However, \((4 - 8\lambda + \lambda^2)I\) is not a positive operator for \( \lambda = 1 \), hence \( AB \) is neither paranormal nor normal. On the other hand, \( A \) is normal, so that it is a paranormal operator.
Main Results 1

Let a pair \((A, B)\) denote the solution of the operator equations (1.1) throughout this talk.
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**Q.** Suppose \(A\) is paranormal. How can the operators \(AB\), \(BA\), or \(B\) be paranormal or normal?
Theorem

Let $A$ be a paranormal operator on $\mathcal{H}$ and $N(A) = N(AB)$. 

(1) If $\dim \mathcal{H} < \infty$, then $AB$ is a normal operator.

(2) If $\dim \mathcal{H} < \infty$ and $N(A - \lambda) = N(B - \lambda)$ for each $\lambda \in \mathbb{C}$, then all of $A$, $AB$, $BA$, and $B$ are normal operators.
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Main Result 1

Given $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the commutator $C(S, T) \in B(B(\mathcal{H}, \mathcal{K}))$ is the mapping defined by

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The iterates $C(S, T)^n$ of the commutator are defined by

$C(S, T)^0(A) := A$ and

$$C(S, T)^n(A) := C(S, T)(C(S, T)^{n-1}(A))$$

for all $A \in B(\mathcal{H}, \mathcal{K})$ and $n \in \mathbb{N}$; they are often called the higher order commutators.
Main Results 1

There is the following binomial identity. It states that

\[ C(S, T)^n(A) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A T^k, \]

which is valid for all \( A \in B(H, \mathcal{K}) \) and all \( n \in \mathbb{N} \cup \{0\} \).
Let $A$ be paranormal with $N(A) = N(AB)$. If $\dim \mathcal{H} < \infty$ and $\alpha$ is a real number, then the following statements hold:

1. $\alpha AB + (1 - \alpha)A$ is a solution $X$ of the operator equations $C(A, X)_n(A^*) = 0$ for all $n \in \mathbb{N}$.

2. $\sigma_A(\alpha AB + (1 - \alpha)A_x) \subseteq \sigma_{\alpha AB + (1 - \alpha)A_x}$ for all $x \in \mathcal{H}$.

3. $A_x^*H_{\alpha AB + (1 - \alpha)A}(F) \subseteq H_{A(F)}$ for every set $F$ in $C$. 

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2. $\sigma_A(A^*x) \subseteq \sigma_{\alpha AB+(1-\alpha)A}(x)$ for all $x \in \mathcal{H}$.

3. $A^*\mathcal{H}_{\alpha AB+(1-\alpha)A}(F) \subseteq \mathcal{H}_A(F)$ for every set $F$ in $\mathbb{C}$. 
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1. If $\lambda = 0$, then $B^2 = 0$.

2. If $\lambda \neq 0$, then $\lambda = 1$ and $A = B = I$. 
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(1) If $A$ is quasinilpotent, then $AB$, $BA$, and $B$ are nilpotent.

(2) If $A - I$ is quasinilpotent, then $B$ is the identity operator, that is, $AB - \lambda$, $BA - \lambda$, and $B - \lambda$ are invertible for all $\lambda \in \mathbb{C} \setminus \{1\}$. 
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Corollary

If $A$ is a paranormal operator, then $\text{iso} \ \sigma(T) \subseteq \{0, 1\}$ where $T \in \{A, AB, BA, B\}$. 
If $T$ is a paranormal operator and $\lambda_0$ is an isolated point of $\sigma(T)$, then the Riesz idempotent $E_{\lambda_0}(T) := \frac{1}{2\pi i} \int_{\partial D} (\lambda - T)^{-1} d\lambda$, where $D$ is the closed disk of center $\lambda_0$ which contains no other points of $\sigma(T)$, satisfies

$$R(E_{\lambda_0}(T)) = N(T - \lambda_0).$$

Here, if $\lambda_0 \neq 0$, then $E_{\lambda_0}(T)$ is self-adjoint and $N(T - \lambda_0)$ reduces $T$. 

[2006, Uchiyama]
Lemma 1

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If $A$ is paranormal and $\lambda_0$ is a nonzero isolated point of $\sigma(AB)$, then for the Riesz idempotent $E_{\lambda_0}(A)$ with respect to $\lambda_0$, we have that

$$R(E_{\lambda_0}(A)) = N(AB - \lambda_0) = N(A^*B^* - \overline{\lambda_0}).$$
Main Results 3

We denote the set $\mathcal{C}$ by the collection of every pair $(A, B)$ of operators as the following:
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\mathcal{C} := \{(A, B) : A \text{ and } B \text{ are solutions of the operator equations } (1.1) \text{ with } N(A - \lambda) = N(B - \lambda) \text{ for } \lambda \neq 0\}.
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We denote the set $C$ by the collection of every pair $(A, B)$ of operators as the following:

$$C := \{(A, B) : A \text{ and } B \text{ are solutions of the operator equations (1.1) with } N(A - \lambda) = N(B - \lambda) \text{ for } \lambda \neq 0\}.$$ 

**Lemma 2**

Suppose that $(A, B) \in C$ and $A$ is paranormal. If $\lambda_0 \in \text{iso } \sigma(BA) \setminus \{0\}$, then for the Riesz idempotent $E_{\lambda_0}(A)$ with respect to $\lambda_0$, we have that

$$R(E_{\lambda_0}(A)) = N(BA - \lambda_0) = N(A^*B^* - \overline{\lambda_0}).$$
Let \((A, B) \in \mathcal{C}\) and \(A\) be a paranormal operator.
Theorem

Let \((A, B) \in \mathcal{C}\) and \(A\) be a paranormal operator.

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Let \((A, B) \in \mathcal{C}\) and \(A\) be a paranormal operator.

(1) If \(\lambda_0\) is a nonzero isolated point of \(\sigma(BA)\), then the range of \(BA - \lambda_0\) is closed.

(2) If \(B^*\) is injective and \(\lambda_0 \in \text{iso } \sigma(T) \setminus \{0\}\), then \(N(T - \lambda_0)\) reduces \(T\), where \(T \in \{AB, B\}\).
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It was shown by [Djor, Lemma 1] that for every $\lambda \in \pi_{00}(T)$, $H_T(\{\lambda\})$ is finite dimensional if and only if $R(T - \lambda)$ is closed. Furthermore we can easily prove that

It was shown by [Djor, Lemma 1] that for every $\lambda \in \pi_{00}(T)$, $\mathcal{H}_T(\{\lambda\})$ is finite dimensional if and only if $R(T - \lambda)$ is closed. Furthermore we can easily prove that

$$\pi_{00}(A) \setminus \{0\} = \pi_{00}(AB) \setminus \{0\} = \pi_{00}(BA) \setminus \{0\} = \pi_{00}(B) \setminus \{0\}.$$
Main Result 3


It was shown by [Djor, Lemma 1] that for every $\lambda \in \pi_{00}(T)$, $\mathcal{H}_T(\{\lambda\})$ is finite dimensional if and only if $R(T - \lambda)$ is closed. Furthermore we can easily prove that

$$\pi_{00}(A) \setminus \{0\} = \pi_{00}(AB) \setminus \{0\} = \pi_{00}(BA) \setminus \{0\} = \pi_{00}(B) \setminus \{0\}.$$

**Corollary**

Let $(A, B) \in \mathcal{C}$ and $A$ be a paranormal operator. If $\lambda_0 \in \pi_{00}(BA) \setminus \{0\}$, then $\mathcal{H}_{BA}(\{\lambda_0\})$ is finite dimensional.
Main Result 3

Remark

Let $(A, B) \in \mathcal{C}$ and one of $A$, $BA$, $AB$, or $B$ be paranormal. If $\lambda_0$ is a nonzero isolated point in the spectrum of one of them, then all of the ranges of $A - \lambda_0$, $BA - \lambda_0$, $AB - \lambda_0$, and $B - \lambda_0$ are closed. Moreover, if $\lambda_0$ is a nonzero isolated eigenvalue of the spectrum of one of them with finite multiplicity, then all of the spectral manifolds $\mathcal{H}_A(\{\lambda_0\})$, $\mathcal{H}_{AB}(\{\lambda_0\})$, $\mathcal{H}_{BA}(\{\lambda_0\})$, and $\mathcal{H}_B(\{\lambda_0\})$ are finite dimensional.
Main Result 4


It is well known that every polynomial roots of paranormal operators satisfy generalized Weyl’s theorem.
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Now, we would like to show that if $A$ is paranormal, then Weyl’s theorem holds for $T$, where $T \in \{AB, BA, B\}$. More generally, we prove that if $A$ or $A^*$ is a polynomial root of paranormal operators, then generalized Weyl’s theorem holds for $f(T)$ for $f \in H(\sigma(T))$, where $T \in \{AB, BA, B\}$. 
Definitions [2001, M. Berkani]

Let $T \in B(H)$.

For a nonnegative integer $n$ define $T_n$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$).
\textbf{B-Fredholm Operators}

\textbf{Definitions [ 2001, M. Berkani ]}

Let $T \in B(\mathcal{H})$.

- For a nonnegative integer $n$ define $T_n$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$).
- $T$ is called \textbf{upper} (resp., \textbf{lower}) \textbf{semi-}B-Fredholm if for some integer $n$ the range $R(T^n)$ is closed and $T_n$ is upper (resp., lower) semi-Fredholm.
- $T$ is called \textbf{semi-}B-Fredholm if it is upper or lower semi-B-Fredholm.
**B-Fredholm Operators**

**Definitions [ 2001, M. Berkani ]**

Let $T \in B(\mathcal{H})$.

- For a nonnegative integer $n$ define $T_n$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$).
- $T$ is called **upper** (resp., **lower**) semi-$B$-Fredholm if for some integer $n$ the range $R(T^n)$ is closed and $T_n$ is upper (resp., lower) semi-Fredholm.
- $T$ is called **semi-$B$-Fredholm** if it is upper or lower semi-$B$-Fredholm.
- $T$ is called **$B$-Fredholm** if $T_n$ is Fredholm.
- $T$ is called **$B$-Weyl** if it is $B$-Fredholm with index 0.
Well Known Facts

[Berk, Theorem 2.7] $T \in B(\mathcal{H})$ is $B$-Fredholm if and only if

$$T = T_1 \oplus T_2,$$

where $T_1$ is Fredholm and $T_2$ is nilpotent.

Let $T \in B(H)$.

$$\sigma_{BF}(T) := \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Fredholm} \}$$

$$\sigma_{BW}(T) := \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl} \}$$

$$\pi_0(T) := \{ \lambda \in \text{iso}\sigma(T) : \alpha(T) > 0 \}$$
Concepts of Generalized Weyl type theorems

Definitions [2003, Berkani and Koliha]

Generalized Weyl’s theorem holds for $T$, in symbol $(g\mathcal{W})$, if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T).$$

$g$-Weyl’s theorem $\iff$ Weyl’s theorem
Main Result 4

Lemma

We have the following properties:

\begin{enumerate}
\item \( \pi_0(A) = \pi_0(AB) = \pi_0(BA) = \pi_0(B) \).
\item \( A \) is isoloid if and only if \( AB \) is isoloid if and only if \( BA \) is isoloid if and only if \( B \) is isoloid.
\end{enumerate}

Theorem

Suppose that \( A \) or \( A^* \) is a polynomial root of paranormal operators. Then \( f(T) \in g(W) \) for each \( f \in H(\sigma(T)) \), where \( T \in \{AB, BA, B\} \).
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We have the following properties:

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Theorem

Suppose that \( A \) or \( A^* \) is a polynomial root of paranormal operators. Then \( f(T) \in g\mathcal{W} \) for each \( f \in H(\sigma(T)) \), where \( T \in \{AB, BA, B\} \).
Corollary

Suppose that \((A, B) \in \mathcal{C}\) and \(A\) is a compact paranormal operator. Then we have that

\[
BA = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} \text{ on } N(BA - I) \oplus N(BA - I)^\perp,
\]

where \(Q\) is quasinilpotent.
Thank You!