Toeplitz operators and their binormality

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Study

- Eungil Ko and Ji Eun Lee, *Characterizations of binormal Toeplitz operators on the Hardy space $H^2$*, preprint.
Binormal operator

- $\mathcal{H}$: a complex (separable) Hilbert space
- $\mathcal{L}(\mathcal{H})$: the algebra of all bounded linear operators on $\mathcal{H}$.

**Definition**

- $T$ is *normal* if $T^*T = TT^*$.
- $T$ is *quasinormal* if $T^*T$ and $T$ commute.
- $T$ is *binormal* if $T^*T$ and $TT^*$ commute.
- $T$ is *subnormal* if $\exists$ a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator $N$ on $\mathcal{K}$ such that $NH \subset \mathcal{H}$ and $T = N|_H$.
- $T$ is *hyponormal* if $T^*T - TT^* \geq 0$.

$normal \Rightarrow quasinormal \Rightarrow binormal.$
Binormal operator

- $\mathcal{M}$ is *nontrivial* if it is different from $(0)$ and $\mathcal{H}$.
- A closed subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* for $T$ if $T\mathcal{M} \subset \mathcal{M}$.

**Known fact, Campbell(1972)**

- If $T \in \mathcal{L}(\mathcal{H})$ is hyponormal and binormal, then $T$ has a non-trivial invariant subspace.
Hardy space

- $L^2 := L^2(\partial\mathbb{D})$ is the usual Lebesque space on the unit circle $\partial\mathbb{D}$.
- $L^\infty$ is the Banach space consisting of all essentially bounded functions on $\partial\mathbb{D}$.
- $\{z^n : n = 0, \pm1, \pm2, \pm3, \cdots\}$ is an orthonormal basis for $L^2$.
- $H^2 = \{f \mid f : \text{analytic functions on } \mathbb{D} \text{ with } f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$

: Hilbert Hardy space.

- $H^2 = \text{span}\{z^n : n = 0, 1, 2, 3, \cdots\}$.
- $H^\infty$ is the space of bounded analytic functions on $\mathbb{D}$. 
Toeplitz operator

**Definition**

For any \( \varphi \in L^\infty \), the *Toeplitz operator* \( T_\varphi : H^2 \to H^2 \) is defined by the formula

\[
T_\varphi f = P(\varphi f), \quad f \in H^2
\]

where \( P \) denotes the orthogonal projection of \( L^2 \) onto \( H^2 \).

- \( T_\varphi \) is bounded if and only if \( \varphi \in L^\infty \) and \( \| T_\varphi \| = \| \varphi \|_\infty \).

- \( T_\varphi \) is a Toeplitz operator if and only if \( S^* T_\varphi S = T_\varphi \) where \( S \) is the unilateral shift on \( H^2 \), i.e., \( Sf(z) = zf(z) \) for \( f \in H^2 \).
Toeplitz operator

**Known facts**

- (In 1963, A. Brown and P. R. Halmos) 
  \( T_\varphi \) is normal if and only if \( \varphi = \alpha + \beta \rho \) where \( \rho \) is a real valued function in \( L^\infty \) and \( \alpha, \beta \in \mathbb{C} \).

- (In 1975, I. Amemiya, T. Ito, and T. K. Wong) 
  Every quasinormal Toeplitz operator is either normal or analytic.

- (In 1988, C. Cowen) 
  For \( \varphi \in L^\infty \), let \( \varphi = f + \overline{g} \) where \( f \) and \( g \) are in \( H^2 \). Then the Toeplitz operator \( T_\varphi \) is hyponormal if and only if \( g = c + T_{\overline{h}} f \) for some constant \( c \) and some function \( h \in H^\infty \) with \( \| h \|_\infty \leq 1 \).
Binormal Toeplitz

Question

• When are Toeplitz operators binormal?

Notation

(i) $A = T^* \varphi^2$, $B = \varphi T^* \varphi$, and $S$ denotes the unilateral shift.
(ii) $\varphi_+ = S^* T^* \varphi e_0 = \sum_{n=0}^{\infty} \hat{\varphi}(n+1)e_n$.
(iii) $\varphi_- = S^* T^* \varphi e_0 = \sum_{n=0}^{\infty} \hat{\varphi}(-n-1)e_n$.
(iv) $F = (S^* A e_0 \otimes S^* B e_0) + (\varphi_- \otimes B \varphi_-) + (S^* A S \varphi_+ \otimes \varphi_+)$.
(v) $E_1 = S^* A e_0 \otimes S^* B e_0$, $E_2 = \varphi_- \otimes B \varphi_-$, and $E_3 = S^* A S \varphi_+ \otimes \varphi_+$. 
Binormal Toeplitz

Theorem

Let $T_{\varphi}$ be a Toeplitz operator on $H^2$ with a symbol $\varphi \in L^\infty$. Then the following statements are equivalent:

(i) $T_{\varphi}$ is binormal.

(ii) The operator

$$F = (S^* A e_0 \otimes S^* B e_0) + (\varphi_- \otimes B \varphi_-) + (S^* A S \varphi_+ \otimes \varphi_+)$$

is a self-adjoint operator.
Binormal Toeplitz

Corollary

Let $T_\varphi$ be Toeplitz operator with a symbol $\varphi \in L^\infty$. Assume that one of the following statements holds:

(i) $T_\varphi$ is normal.
(ii) $T_\varphi$ is analytic and $\varphi = \lambda u$ for an inner function $u$ and $\lambda \in \mathbb{C}$.
(iii) $T_\varphi$ is coanalytic and $\varphi = \lambda \overline{u}$ for an inner function $u$ and $\lambda \in \mathbb{C}$.
(iv) $B = aA + bl$ for some nonzero real $a$ and real $b$.
(v) $A$ and $B$ are a linear combination of $T_\varphi$ and the identity $I$.
(vi) $A$ and $B$ are a linear combination of $T_\varphi^*$ and the identity $I$.
(vii) $A = aT_\varphi + \overline{a}T_\varphi^*$ and $B = bT_\varphi + \overline{b}T_\varphi^*$ for some $a, b \in \mathbb{C}$.

Then $F$ is self-adjoint. Hence, in this case, $T_\varphi$ is binormal.
Examples

Example

Assume that one of the followings holds.
(i) $\varphi(z) = az^2 + b\bar{z}$ or $\varphi(z) = az + b\bar{z}^2$ for any nonzero $a, b \in \mathbb{C}$.
(ii) $\varphi(z) = z + z^2$ or $\varphi(z) = \bar{z} + \bar{z}^2$.
Hence, by Theorem 1, $T_\varphi$ is not binormal.

Example

Let $\varphi(z) = az^2 + bz + c + d\bar{z} + e\bar{z}^2$ where $a, b, d, e$ are nonzeros.
If $|a| = |e|$ and $ad = \overline{be}$, then, by [Lee], $T_\varphi$ is binormal.

Example

Let $\varphi(z) = (1 + 2i)z^7$ or $\varphi(z) = (i - 1)\bar{z}^6$. Then by Theorem 1, $T_\varphi$ is binormal.
Corollary

Suppose that \( \varphi(z) = az^n + b + c\bar{z}^n \) for some \( a, b, c \in \mathbb{C} \) and \( n \in \mathbb{N} \). Then \( T_\varphi \) is binormal if and only if \( |a| = |c|, a = c = 0, \) \( a = b = 0 \), or \( b = c = 0 \).

e.g. If \( \varphi(z) = z + \frac{1}{2}\bar{z} \), then \( T_\varphi \) is not binormal. But it is hyponormal and not subnormal.
I. One of $E_i$ is nonzero and the others are zero

**Theorem**

Let $T_\varphi \in \mathcal{L}(H^2)$ be a Toeplitz operator on $H^2$ and let $E_i$ be rank-one operators. Assume that one of $E_i$ is nonzero and the others are zero. Then $T_\varphi$ is binormal if and only if one of the following statements holds:

1. $T_\varphi$ is normal.
2. $T_\varphi$ is analytic and $\varphi = \lambda u$ for an inner function $u$ and $\lambda \in \mathbb{C}$.
3. $T_\varphi$ is coanalytic and $\varphi = \lambda \overline{u}$ for an inner function $u$ and $\lambda \in \mathbb{C}$.
4. $e_0$ is an eigenvector of $A$ with respect to real $\lambda$, $\varphi_+$ is an eigenvector of $S^*AS$ with respect to nonzero real $\mu$, and $\varphi_-$ is an eigenvector of $B$ with respect to 0.
I. One of $E_i$ is nonzero and the others are zero

**Theorem**

(5) $e_0$ and $\varphi_-$ are eigenvectors of $B$ with respect to real $\lambda_1$ and 0, respectively, and $\varphi_+$ is an eigenvector of $S^*AS$ with respect to nonzero real $\mu$.

(6) $e_0$ is an eigenvector of $A$ with respect to real $\lambda$, $\varphi_+$ is an eigenvector of $S^*AS$ with respect to 0, and $\varphi_-$ is an eigenvector of $B$ with respect to nonzero real $\mu$.

(7) $e_0$ and $\varphi_-$ are eigenvectors of $B$ with respect to real $\lambda_1$ and $\mu$, respectively, and $\varphi_+$ is an eigenvector of $S^*AS$ with respect to 0.

(8) $e_0$ is an eigenvector of $A - \mu B$ with respect to real $\lambda$, $\varphi_+$ is an eigenvector of $S^*AS$ with respect to 0, and $\varphi_-$ is an eigenvector of $B$ with respect to 0.
II. Two of $E_i$ are nonzero

**Theorem**

Let $T_\varphi \in \mathcal{L}(H^2)$ be a Toeplitz operator on $H^2$. Suppose that **two of $E_i$ are nonzero** for $i = 1, 2, 3$.

(a) If $E_1 + E_2$ is **rank one**, then $T_\varphi$ is **binormal** if and only if one of the following statements holds:

1. $T_\varphi$ is coanalytic and $\varphi = \lambda \overline{u}$ for an inner function $u$ and $\lambda \in \mathbb{C}$.
2. $\varphi_+$ is an eigenvector of $S^*AS$ corresponding to the eigenvalue 0 and one of the following assertions holds:
   \[
   \begin{cases}
   (i) & B\varphi_+ = a\varphi_+ \text{ for some } a \in \mathbb{C}. \\
   (ii) & \varphi_- = aS^*Be_0 + bB\varphi_- \text{ for some } a \in \mathbb{C} \text{ and } b \in \mathbb{R} \setminus \{0\}. \\
   (iii) & \varphi_- + aS^*Ae_0 = bB\varphi_- \text{ for some } a \in \mathbb{C} \text{ and } b \in \mathbb{R} \setminus \{0\}.
   \end{cases}
   \]
II. Two of $E_i$ are nonzero

Theorem

(b) If $E_1 + E_3$ is rank one, then $T_\varphi$ is binormal if and only if one of the following arguments holds:

(1) $T_\varphi$ is analytic and $\varphi = \lambda u$ for an inner function $u$ and $\lambda \in \mathbb{C}$.

(2) $\varphi_-$ is an eigenvector of $B$ corresponding to the eigenvalue 0 and one of the following assertions holds:

\[
\begin{cases}
    (i) \varphi_+ = aS^*AS\varphi_+ \text{ for some } a \in \mathbb{C}.
    \\
    (ii) S^*AS\varphi_+ = aS^*Be_0 + b\varphi_+ \text{ for some } a \in \mathbb{C} \text{ and } b \in \mathbb{R} \setminus \{0\}.
    \\
    (iii) aS^*Ae_0 + S^*AS\varphi_+ = b\varphi_+ \text{ for some } a \in \mathbb{C} \text{ and } b \in \mathbb{R} \setminus \{0\}.
\end{cases}
\]
II. Two of $E_i$ are nonzero

**Theorem**

(c) If $E_2 + E_3$ is **rank one**, then $T_\varphi$ is **binormal** if and only if $e_0$ is an eigenvector of $A$ with respect to the real eigenvalue $\lambda$ or $e_0$ is an eigenvector of $B$ with respect to the real eigenvalue $\lambda_1$ and one of the following statements holds:

\[
\begin{align*}
(i) \quad & \varphi_+ = aS^*AS\varphi_+ \text{ for some } a \in \mathbb{C}. \\
(ii) \quad & S^*AS\varphi_+ = aB\varphi_- + b\varphi_+ \text{ for some } a \in \mathbb{C} \text{ and nonzero } b \in \mathbb{R}. \\
(iii) \quad & a\varphi_- + S^*AS\varphi_+ = b\varphi_+ \text{ for some } a \in \mathbb{C} \text{ and nonzero } b \in \mathbb{R}.
\end{align*}
\]
II. Two of \( E_i \) are nonzero

Theorem

Let \( T_\varphi \in \mathcal{L}(H^2) \) be a Toeplitz operator on \( H^2 \) with a symbol \( \varphi \in L^\infty \) and let two of \( E_i \) be nonzero for \( i = 1, 2, 3 \).

(a) If \( E_1 + E_2 \) is rank two, then \( T_\varphi \) is binormal if and only if \( \varphi_+ \) is an eigenvector of \( S^*AS \) corresponding to the eigenvalue 0 and

\[
S^*B_0 = aS^*A_0 + b\varphi_- \quad \text{and} \quad B\varphi_- = \overline{b}S^*A_0 + c\varphi_-
\]

for some nonzero constants \( a, c \in \mathbb{R} \) and \( b \in \mathbb{C} \).
II. Two of $E_i$ are nonzero

**Theorem**

(b) If $E_1 + E_3$ is rank two, then $T_\varphi$ is binormal if and only if $\varphi_-$ is an eigenvector of $B$ corresponding to the eigenvalue 0 and $S^*Ae_0 = aS^*Be_0 + b\varphi_+$ and $S^*AS\varphi_+ = \overline{b}S^*Be_0 + c\varphi_+$ for some nonzero constants $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$.

(c) If $E_2 + E_3$ is rank two, then $T_\varphi$ is binormal if and only if $e_0$ is an eigenvector of $A$ with respect to the real eigenvalue $\lambda$ or $e_0$ is an eigenvector of $B$ with respect to the real eigenvalue $\lambda_1$ and

$$B\varphi_- = a\varphi_- + bS^*AS\varphi_+ \quad \text{and} \quad \varphi_+ = \overline{b}\varphi_- + cS^*AS\varphi_+$$

for some nonzero constants $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$. 
III. All of $E_i$ are nonzero

**Theorem**

Let $T_\varphi \in \mathcal{L}(H^2)$ be a Toeplitz operator on $H^2$ and let $E_i$ be nonzero rank-one operators.

(a) If $F$ is rank one, then $T_\varphi$ is binormal if and only if

$S^*Ae_0 = a\varphi_+$, $S^*Be_0 = b\varphi_+$, $S^*AS\varphi_+ = c\varphi_+$, $\varphi_- = d\varphi_+$, and $B\varphi_- = e\varphi_+$ for some $a, b, c, d, e \in \mathbb{C}$. 
III. All of $E_i$ are nonzero

**Theorem**

(b) If $F$ is rank two, then $T_\varphi$ is binormal if and only if one of the following statements holds:

\[
\begin{cases}
(i) \quad S^* A e_0 = a \varphi_+ + b \varphi_-, S^* B e_0 = c \varphi_+, \\
\quad \quad S^* A S \varphi_+ = d S^* A e_0, \text{ and } B \varphi_- = e \varphi_+ + f \varphi_-.

(ii) \quad S^* A e_0 = a \varphi_-, S^* B e_0 = b \varphi_+ + c \varphi_-,
\quad \quad B \varphi_- = d S^* B e_0, \text{ and } S^* A S \varphi_+ = e \varphi_+ + f \varphi_-.

(iii) \quad S^* A e_0 = a \varphi_+ + b \varphi_-, S^* B e_0 = c \varphi_+ + d \varphi_-,
\quad \quad S^* A S \varphi_+ = e \varphi_-, \text{ and } B \varphi_- = f \varphi_+
\end{cases}
\]

for some $a, b, c, d, e, f \in \mathbb{C}$. 
III. All of $E_i$ are nonzero

**Theorem**

(c) If $F$ is rank three, then $T_\varphi$ is binormal if and only if

\[
\begin{align*}
S^*B_{e_0} &= aS^*A_{e_0} + b\varphi_- + cS^*AS\varphi_+,
B\varphi_- &= \overline{b}S^*A_{e_0} + d\varphi_- + eS^*AS\varphi_+,
\varphi_+ &= \overline{c}S^*A_{e_0} + \overline{e}\varphi_- + fS^*AS\varphi_+
\end{align*}
\]

for some nonzero $a, d, f \in \mathbb{R}$ and $b, c, e \in \mathbb{C}$. 
IV. All of E_i are zero

Theorem

Let \( T_\varphi \in \mathcal{L}(H^2) \) be a Toeplitz operator on \( H^2 \) and let \( E_1 = 0, E_2 = 0, \) and \( E_3 = 0. \) Then \( T_\varphi \) is binormal if and only if one of the following statements holds:

(1) \( T_\varphi \) is normal, i.e., \( T_\varphi = T_{\varphi_0} \) where \( \varphi_0 \) is the 0-th Fourier coefficient of \( \varphi. \)

(2) \( e_0 \) is an eigenvector of \( A \) with respect to real eigenvalue \( \lambda, \varphi_+ \) is an eigenvector of \( S^*AS \) with respect to the eigenvalue 0, and \( \varphi_- \) is an eigenvector of \( B \) with respect to 0.

(3) \( e_0 \) is an eigenvector of \( B \) with respect to real eigenvalue \( \lambda_1, \varphi_+ \) is an eigenvector of \( S^*AS \) with respect to the eigenvalue 0, and \( \varphi_- \) is an eigenvector of \( B \) with respect to 0.
Binormal Toeplitz

**Corollary**

If \( \varphi(z) = \sum_{n=-m}^{N} a_n z^n \) is a trigonometric polynomial where one of \( a_{-m} \) and \( a_N \) is nonzero, then \( T_\varphi \) is binormal if and only if it has one of the following symbols:

(i) \( \varphi(z) = \sum_{n=-m}^{m} a_n z^n \) where \( |a_{-m}| = |a_m| \) with \( a_m \neq 0 \) and

\[
\begin{pmatrix}
a_{-1} \\
a_{-2} \\
\vdots \\
a_{-m}
\end{pmatrix}
= a_m
\begin{pmatrix}
\overline{a_1} \\
\overline{a_2} \\
\vdots \\
\overline{a_m}
\end{pmatrix},
\]

\[
(1)
\]

(ii) \( \varphi(z) = a_N z^N \),

(iii) \( \varphi(z) = a_{-m} z^m \),

(iv) \( \varphi(z) = a_0 \).
Corollary

Let $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ be a trigonometric polynomial where one of $a_{-m}$ and $a_N$ is nonzero. If $T_\varphi$ is binormal with $T_\varphi \neq \lambda I$, then it has a nontrivial invariant subspace.
Conditions of binormality

**Theorem**

Let \( \varphi = \varphi_+ + \overline{\varphi_-} \) where \( \varphi_+ \) and \( \varphi_- \) are in \( zH^2 \) and let \( \varphi_- = \lambda \varphi_+ \) for some \( \lambda \in \mathbb{C} \) with \( \varphi_+ \neq 0 \). Suppose that one of the following assertions holds:

(i) \( \varphi_- = \lambda \varphi_+ \) for some \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \).

(ii) \( S^*Be_0 = \mu S^*Ce_0 \) for some \( \mu \in \mathbb{R} \setminus \{0\} \) and \( S^*BS - |\lambda|^2B = (|\lambda|^2 - 1)S^*CS \).

(iii) \( S^*Be_0 = \mu S^*Ce_0 \) and \( [S^*BS - |\lambda|^2B + (1 - |\lambda|^2)S^*CS] \varphi_+ = \mu_1 \varphi_+ \) for some \( \mu, \mu_1 \in \mathbb{R} \setminus \{0\} \), where \( B = T_\varphi T_\varphi^* \) and \( C = [T_\varphi^*, T_\varphi] \) is self-commutator of \( T_\varphi \). Then \( T_\varphi \) is binormal.
Corollary

Let $\varphi(z) = \varphi_+ + \lambda \overline{\varphi_+}$ where $\varphi_+ \in H^\infty$ and $\lambda \in \mathbb{C}$. If $T_\varphi$ is binormal and $[T_\varphi^*, T_\varphi]$ has a non-trivial kernel, then $T_\varphi$ is normal.

Corollary

Let $\varphi(z) = \varphi_+ + \overline{\varphi_-}$ where $\varphi_+$ and $\varphi_-$ are in $zH^2$. If $\varphi_- = \lambda \varphi_+$ for some complex number $\lambda$ with $|\lambda| < 1$ and the conditions (ii) or (iii) in Theorem 8 hold, then $T_\varphi$ has a nontrivial invariant subspace.
Nontrivial invariant subspace

Theorem

Let \( \varphi(z) = \varphi_+ + \overline{\varphi_-} \) where \( \varphi_+, \varphi_- \in zH^2 \) with \( \| \varphi_+ \| = \| \varphi_- \| \). Suppose that \( \varphi_+ = u \varphi_- \) for some inner function \( u \). If an operator

\[
(\varphi_- \otimes B \varphi_-) + (S^* A S u \varphi_- \otimes u \varphi_-)
\]

is self-adjoint, then \( T_{\varphi} \) has a nontrivial invariant subspace.
Nontrivial invariant subspace

Corollary

Let \( \varphi(z) = \varphi_+ + \overline{\varphi_-} \) where \( \varphi_+, \varphi_- \in zH^2 \) with \( \|\varphi_+\| = \|\varphi_-\| \).

Suppose that \( \varphi_+ = \frac{a-z}{1-\overline{a}z} \varphi_- \) for every \( a \in \mathbb{D} \). If an operator

\[
(\varphi_- \otimes B\varphi_-) + \left( S^* AS \frac{a-z}{1-\overline{a}z} \varphi_- \otimes \frac{a-z}{1-\overline{a}z} \varphi_- \right)
\]

is self-adjoint, then \( T_\varphi \) has a nontrivial invariant subspace.
Nontrivial invariant subspace

Example

If $\varphi_+ = z \varphi_-$ and an operator

$$(\varphi_- \otimes B \varphi_-) + (S^* A S \varphi_- \otimes z \varphi_-)$$

is self-adjoint, then $T_\varphi$ has a nontrivial invariant subspace by Corollary.

Even if $\varphi(z) = \varphi_+ + \overline{\varphi_-}$ with $\varphi_+(z) = z^2$ and $\varphi_-(z) = z$ in the above example, then, since $\varphi_+ = z \varphi_-$, $T_\varphi$ is hyponormal by [GS]. But $T_\varphi$ is not binormal by Example.
Binormal Toeplitz and CSO

- A conjugation on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$.
- An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be complex symmetric if there exists a conjugation $C$ on $\mathcal{H}$ such that $T = CT^*C$.

Corollary

Let $\varphi(z) = az^n + b\overline{z}^n$ for some nonzero $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$. Then the following assertions are equivalent.
(i) $T_\varphi$ is binormal.
(ii) $T_\varphi$ is complex symmetric.
(iii) $T_\varphi$ is normal.
(iv) $|a| = |b|$.
Binormal Toeplitz and CSO

**Proposition**

Let \( \varphi \in L^\infty \) and let \( C\left( \sum_{k=0}^{\infty} a_k z^k \right) = \sum_{k=0}^{\infty} (-1)^k \bar{a}_k z^k \) for \( \sum_{k=0}^{\infty} a_k z^k \in H^2 \). Then the following statements are equivalent.

1. \( T_\varphi \) is complex symmetric with the conjugation \( C \).
2. \( T_\varphi \) is normal.
3. \( \hat{\varphi}(-n) = \hat{\varphi}(n)(-1)^n \) for all \( n \).
Binormal Toeplitz and CS0

**Proposition**

Let $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ where $N \geq m > 0$ and $a_n \in \mathbb{C}$ with nonzero $a_{-m}, a_N$ and let $C(\sum_{k=0}^{\infty} a_k z^k) = \sum_{k=0}^{\infty} (-1)^k \bar{a}_k z^k$ for $\sum_{k=0}^{\infty} a_k z^k \in H^2$. Then $T_\varphi$ is complex symmetric with the conjugation $C$ if and only if $m = N$ and $a_{-n} = a_n (-1)^n$ for all $n$.

Let $\varphi(z) = z^2 + z - \bar{z} + \bar{z}^2$. Then $T_\varphi$ is complex symmetric with the same conjugation $C$ by Proposition or [GZ]. However, $T_\varphi$ is not binormal by Example 7.
Reference

Reference

Thank you for your attention!
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