Bost-Connes system for local fields of characteristic zero

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  - $Y_K = \hat{\mathcal{O}}_K \times \hat{\mathcal{O}}^* \cdot G^\text{ab}_K$
  - $I_K$: ideal semigroup of $K$. 
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  - \( f \mapsto f \) on \( C(Y_K) \),
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- \( G_K \) acts by right multiplication on \( Y_K \) and trivially on \( I_K \).
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- Partition function $= \zeta_K$. 
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$A_K$ has a $K$-subalgebra $A_K^{\text{arith}}$ which satisfies the following:

- $\phi(A_K^{\text{arith}}) = K_{ab}$ for any extremal KMS $1$-state $\phi$.
- $\phi(ga) = \phi(a)g$ for any extremal KMS $1$-state $\phi$, $a \in A_K^{\text{arith}}$ and $g \in G_{ab}$.

$A_K^{\text{arith}} \otimes_K C$ is dense in $A_K$. $A_K^{\text{arith}}$ is called an arithmetic subalgebra.
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Let \( Y_K = \mathcal{O}_K \times \mathcal{O}_K \). The semigroup \( \mathbb{N} \) acts on \( Y_K \) by \( k \cdot (\rho, g) = (\rho \pi^k, \pi^k g) \) for \( (\rho, g) \in Y_K \).

Let \( A_K = \mathbb{C}(Y_K) \rtimes \mathbb{R} \) acts on \( A_K \) by \( f \mapsto f \) on \( \mathbb{C}(Y_K) \), \( \mu_n \mapsto q^n \) for \( t \in \mathbb{R} \).

\[ q = p_{f, f} : \text{inertia degree of } K / \mathbb{Q} \]
Let $Y_K = \mathcal{O}_K \times \mathcal{O}_K \mathcal{G}^{ab} K$.

The semigroup $N$ acts on $Y_K$ by $k \cdot ([\rho, g]) = ([\rho \pi^k K, \pi \pi^k K], [\pi \pi^k g])$ for $[\rho, g] \in Y_K$.

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$R$ acts on $A_K$ by $f \mapsto f$ on $C(Y_K)$, $\mu \mapsto q^K \mu$ for $t \in R$, $q^K = p_f$, $f$: inertia degree of $K/Q_p$.

Definition 1 (T'14) $(A_K, \sigma_t)$ is called the Bost-Connes system for $K$. 

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$$k \cdot ([\rho, g]) = [\rho \pi_K^k, [\pi_K^k]^{-1} g] \quad \text{for } [\rho, g] \in Y_K.$$ Let $A_K = C(Y_K) \rtimes \mathbb{N}$. 

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Theorem 1 (T ’14)

For $0 < \beta \leq \infty$, there is one-to-one correspondence between extremal KMS-states of $(A_K, \sigma_t)$ and $G_{ab}^K$. The partition function is $(1 - q^K)^{1/2}$.

There is an arithmetic subalgebra of $(A_K, \sigma_t)$. In particular, the phase transition does not occur in the local case.
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Connection between local and global ones

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\( K_p \): localization of \( K \) at \( p \)
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We have “\( \mathbb{R} \)-equivariant” \((A_K, A_{K_p})\)-correspondence \( E_p \)

\( \mathbb{R} \)-equivariance means that there is one-parameter group of isometries \( U_t \) on \( E_p \) such that

- \( U_t a\xi = \sigma_t(a) U_t \xi \)
- \( \langle U_t \xi, U_t \eta \rangle = \sigma_t(\langle \xi, \eta \rangle) \)

for any \( a \in A_K, \xi, \eta \in E_p \) and \( t \in \mathbb{R} \).
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$Y_K = \hat{O}_K \times \hat{O}_K^* G^a{}_b^b \subset X_K = \mathbb{A}_K \times \hat{O}_K^* G^a{}_b^b$
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$Y_K = \hat{\mathcal{O}}_K \times \hat{\mathcal{O}}_K^* \times G_{K}^{ab} \subset X_K = \mathbb{A}_K f \times \hat{\mathcal{O}}_K^* \times G_{K}^{ab}$

$Y_p = \mathcal{O}_p \times \mathcal{O}_p^* \times G_{K}^{ab} \subset X_p = K_p \times \mathcal{O}_p^* \times G_{K}^{ab}$
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$A_K = C(Y_K) \times I_K$, $A_p = C(Y_p) \times \mathbb{N}$
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\[ Y_K = \hat{O}_K \times \hat{O}_K^* \cdot G_{K}^{ab} \subset X_K = A_{K,f} \times \hat{O}_K^* \cdot G_{K}^{ab} \]
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\[ A_K = C(Y_K) \rtimes I_K, \quad A_p = C(Y_p) \rtimes \mathbb{N} \]
\[ \tilde{A}_K = C_0(X_K) \rtimes J_K, \quad \tilde{A}_p = C_0(X_p) \rtimes \mathbb{Z} \]
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Construction of $E_p$

$$Y_K = \hat{O}_K \times \hat{O}_K^* G_K^{ab} \subset X_K = \mathbb{A}_K, f \times \hat{O}_K^* G_K^{ab}$$

$$Y_p = \mathcal{O}_p \times \mathcal{O}_p^* G_K^{ab} \subset X_p = K_p \times \mathcal{O}_p^* G_K^{ab}$$

$$A_K = C(Y_K) \rtimes I_K, A_p = C(Y_p) \rtimes \mathbb{N}$$

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$$1_{Y_K} \tilde{A}_K 1_{Y_K} = A_K, 1_{Y_p} \tilde{A}_p 1_{Y_p} = A_p$$

We have $X_p \hookrightarrow X_K$ by $[\rho, g] \mapsto [(\rho, 1), g|_K]$
Connection between local and global ones

Let \( E_p = C(JK) \otimes C(Z) \sim A_p \) (right Hilbert \( A_p \)-module), where \( Z \) is identified with the subgroup of \( JK \) generated by \( p \).

\( A_K \) acts on \( E_p \) by \((fu)(ub \otimes g) = uab \otimes ((ab)^{-1} \cdot f) |_{Xp} \) for \( f \in C(XK) \), \( g \in C_0(Xp) \) and \( a, b \in JK \).

Define \( E_p = \bigoplus K \sim E_p \).

One-parameter group of isometries on \( E_p \) is defined by \( U_t(ua \otimes f) = N(a)\cdot it\cdot ua \otimes f \).
Let $\tilde{E}_p = C^*(J_K) \otimes_{C^*(\mathbb{Z})} \tilde{A}_p$ (right Hilbert $\tilde{A}_p$-module), where $\mathbb{Z}$ is identified with the subgroup of $J_K$ generated by $p$. 
Let $\tilde{E}_p = C^*(J_K) \otimes_{C^*(\mathbb{Z})} \tilde{A}_p$ (right Hilbert $\tilde{A}_p$-module), where $\mathbb{Z}$ is identified with the subgroup of $J_K$ generated by $p$. $\tilde{A}_K$ acts on $E_p$ by

$$(fu_a)(u_b \otimes g) = u_{ab} \otimes ((ab)^{-1}.f)|_{X_p} g$$

for $f \in C(X_K), g \in C_0(X_p)$ and $a, b \in J_K$.
Let $\tilde{E}_p = C^*(J_K) \otimes_{C^*(\mathbb{Z})} \tilde{A}_p$ (right Hilbert $\tilde{A}_p$-module), where $\mathbb{Z}$ is identified with the subgroup of $J_K$ generated by $p$. $\tilde{A}_K$ acts on $E_p$ by

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for $f \in C(X_K)$, $g \in C_0(X_p)$ and $a, b \in J_K$. Define $E_p = 1_{\gamma_K} \tilde{E}_p 1_{\gamma_p}$.
Let $\tilde{E}_p = C^*(J_K) \otimes_{C^*(\mathbb{Z})} \tilde{A}_p$ (right Hilbert $\tilde{A}_p$-module), where $\mathbb{Z}$ is identified with the subgroup of $J_K$ generated by $p$. $\tilde{A}_K$ acts on $E_p$ by

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One-parameter group of isometries on $E_p$ is defined by

$U_t(u_a \otimes f) = N(a)^{it}u_a \otimes f$. 
Thank you for the attention!