Coactions of Hopf $C^*$-algebras on Cuntz-Pimsner algebras

Kim, Dong-woon

Seoul National University

August 9, 2014
Motivation and Aim

C*-correspondence \((X, A) \rightsquigarrow \text{Cuntz-Pimsner algebra } \mathcal{O}_X = C^*\langle X, A \rangle\).

Related to group actions and coactions on \(\mathcal{O}_X\) we have:

- \(G^{(\gamma, \alpha)} \times (X, A) \rightsquigarrow \begin{cases} G \beta \curvearrowright \mathcal{O}_X & \text{(Hao and Ng, 2008)} \\ (X \rtimes_{\gamma, r} G, A \rtimes_{\alpha, r} G) & \text{(Echterhoff et al., 2005)} \end{cases}\)

  If \(G\) is amenable, \(\mathcal{O}_X \rtimes_{\beta} G \cong \mathcal{O}_{X \rtimes_{\gamma} G}\) (Hao and Ng, 2008).

- The same is true for a group coaction on \((X, A)\) under a mild condition (Kaliszewski et al., 2013).

We aim to unify the above two works into a single framework of Hopf C*-algebra and show that essentially the same result can be obtained if \(G\) is replaced by a Hopf C*-algebra \(S\).
A \textit{C*-correspondence} is a (right) Hilbert module $X$ over a C*-algebra $A$ equipped with a homomorphism $\varphi_A : A \rightarrow \mathcal{L}(X)$, called the \textit{left action}. We write $(X, A)$ to refer to a C*-correspondence.

A \textit{representation} of $(X, A)$ on a C*-algebra $B$ is a pair

$$(\psi, \pi) : (X, A) \rightarrow B$$

of a linear map $\psi : X \rightarrow B$ and a homomorphism $\pi : A \rightarrow B$ such that

(i) $(\psi, \pi)$ preserves the left action: $\psi(\varphi_A(a)\xi) = \pi(a)\psi(\xi)$,

(ii) $(\psi, \pi)$ preserves the inner product: $\pi(\langle \xi, \eta \rangle_A) = \psi(\xi)^*\psi(\eta)$.

(It is automatic that $\psi(\xi \cdot a) = \psi(\xi)\pi(a)$.)
The ideal $J_X$; universal covariant representation

Every representation $(\psi, \pi) : (X, A) \to B$ determines a homomorphism $\psi^{(1)} : \mathcal{K}(X) \to B$ such that $\psi^{(1)}(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*$. 

For $(X, A)$, $J_X$ denotes the largest ideal of $A$ which is mapped injectively into $\mathcal{K}(X)$ by $\varphi_A$:

$$J_X := \{ a \in A : \varphi_A(a) \in \mathcal{K}(X), \ a \ker \varphi_A = 0 \}.$$ 

A representation $(\psi, \pi) : (X, A) \to B$ is said to be covariant if $\psi^{(1)} \circ \varphi_A = \pi$ on $J_X$ (Katsura, 2004).

We denote by $(k_X, k_A)$ the universal covariant representation of $(X, A)$. 
Cuntz-Pimsner algebra

The Cuntz-Pimsner algebra $\mathcal{O}_X$ is the $C^*$-algebra generated by $k_X(X)$ and $k_A(A)$ (Pimsner, 1997; Katsura, 2004). Hence by universality, we have the commutative diagram for a covariant representation $(\psi, \pi) : (X, A) \rightarrow B$:

$$
\begin{array}{c}
(k_X, k_A) \\
| \\
| \exists! \psi \times \pi \\
| \\
(X, A) \downarrow (\psi, \pi) \\
\end{array}
\rightarrow
\begin{array}{c}
\mathcal{O}_X \\
| \\
| \\
| \\
B.
\end{array}
$$

Basic examples:

- $\varphi \in Aut(A) \leadsto (X, A) = (A(\varphi), A)$, where $A(\varphi) = A$ as Hilbert $A$-modules and $\varphi_A(a)$ is the left multiplication by $\varphi(a)$. Then $\mathcal{O}_X = A \rtimes_\varphi \mathbb{Z}$.

- If $(X, A) = (\mathbb{C}^n, \mathbb{C})$ with $\varphi_A(1) = 1$ ($n \geq 2$), then $\mathcal{O}_X = \mathcal{O}_n$. 
Multiplier correspondence

From now on, \((X, A)\) is assumed to be nondegenerate, i.e., \(\varphi_A : A \to \mathcal{L}(X)\) is nondegenerate.

For \((X, A)\), let \(M(X) := \mathcal{L}_A(A, X)\). The multiplier correspondence is the \(\text{C}^*\)-correspondence \((M(X), M(A))\) such that

\[
m \cdot a := ma, \quad \langle m, n \rangle_{M(A)} := m^* n, \quad \varphi_{M(A)} := \overline{\varphi_A}
\]

for \(m, n \in M(X)\). Note that \(X \subseteq M(X)\) via \(X \cong \mathcal{K}_A(A, X)\).

The strict topology on \(M(X)\) is the locally convex topology generated by the family of seminorms

\[
m \mapsto \| Tm \| + \| m \cdot a \|
\]

on \(M(X)\) \((T \in \mathcal{K}(X), \ a \in A)\).
Tensor product

The tensor product of $C^*$-algebras is always minimal.

For a $C^*$-correspondence $(X, A)$ and a $C^*$-algebra $B$,

- The tensor product $X \otimes B$ of the Hilbert modules $X$ and $B$ is exterior:
  \[
  (\xi \otimes b) \cdot (a \otimes b') = \xi \cdot a \otimes bb' \]
  \[
  \langle \xi \otimes b, \xi' \otimes b' \rangle_{A \otimes B} = \langle \xi, \xi' \rangle_A \otimes b^* b',
  \]

- $\varphi_{A \otimes B} := \varphi_A \otimes \text{id}_B : A \otimes B \to \mathcal{L}_A(X) \otimes \mathcal{L}_B(B) \subseteq \mathcal{L}_{A \otimes B}(X \otimes B)$.

Then $(X \otimes B, A \otimes B)$ becomes a $C^*$-correspondence.
Hopf $C^*$-algebra; reduced Hopf $C^*$-algebra

By a *Hopf $C^*$-algebra* we mean a pair $(S, \Delta)$ of a $C^*$-algebra $S$ and a nondegenerate homomorphism $\Delta : S \to M(S \otimes S)$ satisfying

(i) $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$;

(ii) $\Delta(S)(1 \otimes S) = S \otimes S = \Delta(S)(S \otimes 1)$.

A unitary $V \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ which is

(i) *multiplicative*: $V_{12} V_{13} V_{23} = V_{23} V_{12}$,

(ii) *regular*: $\{ \text{id} \otimes \omega(\sum V) : \omega \in \mathcal{L}(\mathcal{H})_* \} = \mathcal{K}(\mathcal{H})$

determines a pair of Hopf $C^*$-algebras

$$S_V := \{ \omega \otimes \text{id}(V) : \omega \in \mathcal{L}(\mathcal{H})_* \}, \quad \Delta_V(s) = V(s \otimes 1)V^*,$$

$$\hat{S}_V := \{ \text{id} \otimes \omega(V) : \omega \in \mathcal{L}(\mathcal{H})_* \}, \quad \hat{\Delta}_V(x) = V^*(1 \otimes x)V,$$

called the *reduced* and *dual reduced* Hopf $C^*$-algebras, respectively. They are nondegenerate subalgebras of $\mathcal{L}(\mathcal{H})$ (Baaj and Skankalasis, 1993).
Commutative and cocommutative Hopf $C^*$-algebras

The followings are motivated examples of Hopf $C^*$-algebras:

- $(C_0(G), \Delta_G)$ with $\Delta_G(f)(r, s) = f(rs)$;
- $(C^*(G), \delta_G)$ with $\delta_G(r) = r \otimes r$, and similarly for $(C^*_r(G), \delta_G)$.

One can verify that

$$(C_0(G), \Delta_G) \cong (S_{\widehat{W}_G}, \Delta_{\widehat{W}_G}), \quad (C^*_r(G), \delta_G) = (\widehat{S}_{\widehat{W}_G}, \widehat{\Delta}_{\widehat{W}_G}),$$

for $\widehat{W}_G \in \mathcal{L}(G \times G)$ given by $\widehat{W}_G(h)(r, s) = h(sr, s)$. 
A coaction of a Hopf C*-algebra \((S, \Delta)\) on \((X, A)\) is a pair

\[(\sigma, \delta) : (X, A) \to (M(X \otimes S), M(A \otimes S))\]

of a linear map \(\sigma\) and a homomorphism \(\delta\) satisfying

\[
\sigma(\varphi_A(a)\xi) = \varphi_{M(A \otimes S)}(\delta(a)) \sigma(\xi)
\]

\[
\delta(\langle \xi, \eta \rangle_A) = \langle \sigma(\xi), \sigma(\eta) \rangle_{M(A \otimes S)}
\]

such that

(i) \(\varphi_{M(A \otimes S)}(1_{M(A)} \otimes S) \sigma(X) = X \otimes S; \quad (1_{M(A)} \otimes S) \delta(A) = A \otimes S;\)

(ii) \((\sigma \otimes \text{id}_S) \circ \sigma = (\text{id}_X \otimes \Delta) \circ \sigma; \quad (\delta \otimes \text{id}_S) \circ \delta = (\text{id}_X \otimes \Delta) \circ \delta.\)

If \(S = C^*(G)\) and \(\delta\) is injective, \((\sigma, \delta)\) is by definition a nondegenerate coaction of \(G\) on \((X, A)\) (Echterhoff et al., 2005).
The $C^*$-correspondence $(M(X \otimes C_0(G)), M(A \otimes C_0(G)))$

Recall that $M(A \otimes C_0(G)) = C_b(G, M(A)_s)$ (Akemann et al., 1973).

**Theorem (K.)**

Let $(X, A)$ be a (nondegenerate) $C^*$-correspondence and $G$ be a locally compact Hausdorff space. Then

$$(M(X \otimes C_0(G)), M(A \otimes C_0(G))) = (C_b(G, M(X)_s), C_b(G, M(A)_s)).$$
Coaction of $C_0(G)$ on $(X, A)$

Let $Aut(X, A)$ be the group of isomorphisms on $(X, A)$. An action of a LCG $G$ on $(X, A)$ is a homomorphism

$$(\gamma, \alpha) : G \rightarrow Aut(X, A), \quad r \mapsto (\gamma_r, \alpha_r)$$

such that for each $\xi \in X$ and $a \in A$, both maps

$$G \ni r \mapsto \gamma_r(\xi) \in X, \quad G \ni r \mapsto \alpha_r(a) \in A$$

are continuous (Echterhoff et al., 2005).

**Theorem (K.)**

There exists a one-to-one correspondence

$$\{\text{actions of } G \text{ on } (X, A)\} \ni (\gamma, \alpha) \leftrightarrow (\sigma^\gamma, \delta^\alpha) \in \{\text{coactions } (X, A) \rightarrow (M(X \otimes C_0(G)), M(A \otimes C_0(G)))\}$$

such that $\sigma^\gamma(\xi)(r) = \gamma_r(\xi)$ and similarly for $\delta^\alpha$. 
Weak $\delta$-invariancy

For $(X, A)$, recall that $J_X$ is the following ideal of $A$:

$$J_X := \{ a \in A : \varphi_A(a) \in \mathcal{K}(X), \ a \ker \varphi_A = 0 \}.$$

**Definition (K., cf. Kaliszewski et al., for group coaction)**

For a coaction $(\sigma, \delta)$ of $S$ on $(X, A)$, $J_X$ is said to be *weakly $\delta$-invariant* if

$$\delta(J_X)(1_{M(A)} \otimes S) \subseteq J_X \otimes S.$$

If $S = C_0(G)$, this is automatic.
Induced coaction $\zeta$ of $S$ on $\mathcal{O}_X$

Recall that $\mathcal{O}_X = C^*\langle k_X(X), k_A(A) \rangle$.

**Theorem (K., cf. Kaliszewski et al., for group coaction)**

Let $(\sigma, \delta)$ be a coaction of $S$ on $(X, A)$ such that $J_X$ is weakly $\delta$-invariant. Then there exists a unique coaction $\zeta$ of $S$ on $\mathcal{O}_X$ such that the diagram

\[
\begin{array}{ccc}
(X, A) & \xrightarrow{((k_X \otimes \text{id}_S) \circ \sigma, (k_A \otimes \text{id}_S) \circ \delta)} & M(\mathcal{O}_X \otimes S) \\
& \downarrow & \downarrow \\
& \exists! \zeta = ((k_X \otimes \text{id}_S) \circ \sigma) \times ((k_A \otimes \text{id}_S) \circ \delta) & \end{array}
\]

commutes.

If $S = C_0(G)$, then we can recover Hao and Ng’s Theorem:

$G \overset{(\gamma, \alpha)}{\curvearrowright} (X, A) \leadsto G \overset{\beta}{\curvearrowright} \mathcal{O}_X$ with

\[
\begin{array}{ccc}
(X, A) & \xrightarrow{(k_X \circ \gamma_r, k_A \circ \alpha_r)} & \mathcal{O}_X \\
& \downarrow & \downarrow \\
& \exists! \beta_r = (k_X \circ \gamma_r) \times (k_A \circ \alpha_r) & \end{array}
\]
Reduced crossed product correspondence

From now on, we consider only $S = S_V (\subseteq \mathcal{L}(\mathcal{H}))$. Let $\iota : S \hookrightarrow \mathcal{L}(\mathcal{H})$ be the inclusion map. For a coaction $(\sigma, \delta)$, let $(\sigma_\iota, \delta_\iota)$ be the following composition:

$$(X, A) \xrightarrow{(\sigma_\iota, \delta_\iota)} (M(X \otimes \mathcal{K}(\mathcal{H})), M(A \otimes \mathcal{K}(\mathcal{H}))).$$

Let $1 \otimes \hat{S} \subseteq M(A \otimes \mathcal{K}(\mathcal{H}))$ be the canonical embedding of $\hat{S}$. Define

- $X \rtimes_\sigma \hat{S} := \sigma_\iota(X) \cdot (1 \otimes \hat{S}) \subseteq M(X \otimes \mathcal{K}(\mathcal{H}))$,
- $A \rtimes_\delta \hat{S} := \delta_\iota(A)(1 \otimes \hat{S}) \subseteq M(A \otimes \mathcal{K}(\mathcal{H}))$ (Baaj and Skandalis, 1993).

**Theorem (K.)**

*With respect to the operation of $(M(X \otimes \mathcal{K}(\mathcal{H})), M(A \otimes \mathcal{K}(\mathcal{H})))$, $(X \rtimes_\sigma \hat{S}, A \rtimes_\delta \hat{S})$ is a (nondegenerate) $C^*$-correspondence.*
The representation $(\Psi_\sigma, \Pi_\delta)$

- $(\psi, \pi) : (X, A) \to B$ is called a representation if 
  $\psi(\varphi_A(a)\xi) = \pi(a)\psi(\xi)$ and $\pi(\langle \xi, \eta \rangle_A) = \psi(\xi)^*\psi(\eta)$.

- $(\psi, \pi)$ is said to be covariant if $\psi^{(1)} \circ \varphi_A = \pi$ on $J_X$.

- $(k_X, k_A)$ denotes the universal covariant representation of $(X, A)$.

- $(\sigma, \delta)$ on $(X, A)$ indeces $\zeta$ on $O_X$ under the assumption of weak $\delta$-invariancy of $J_X$.

Proposition (K.)

Let $(\sigma, \delta)$ be a coaction of $S$ on $(X, A)$ such that $J_X$ is weakly $\delta$-invariant. Then there exists a representation

$$(\Psi_\sigma, \Pi_\delta) : (X \rtimes_\sigma \hat{S}, A \rtimes_\delta \hat{S}) \to O_X \rtimes_\zeta \hat{S}$$

such that $\Psi_\sigma(\sigma_\nu(\xi) \cdot (1_{M(A)} \otimes x)) = \zeta_\nu(k_X(\xi))(1_{M(O_X)} \otimes x)$ and similarly for $\Pi_\delta$. 
Main result

Theorem (K.)

Let \((\sigma, \delta)\) be a coaction of \(S\) on \((X, A)\) such that \(J_X\) is weakly \(\delta\)-invariant. Then

(i) If \((\Psi_\sigma, \Pi_\delta)\) is covariant, then

\[ \Psi_\sigma \times \Pi_\delta : \mathcal{O}_X \rtimes_\zeta \hat{S} \cong \mathcal{O}_X \rtimes_\sigma \hat{S}. \]

(ii) The followings are equivalent.

1. \((\Psi_\sigma, \Pi_\delta)\) is covariant;
2. \(J_X \rtimes_\sigma \hat{S}_V (A \otimes \mathcal{K}(\mathcal{H})) \subseteq J_X \otimes \mathcal{K}(\mathcal{H})\);
3. \(J_X \rtimes_\sigma \hat{S}_V (\ker \varphi_A \otimes \mathcal{K}(\mathcal{H})) = 0\).

In particular, if either \(J_X = A\) or \(\varphi_A\) is injective, then \((\Psi_\sigma, \Pi_\delta)\) is covariant.

(iii) Suppose that \(S\) is defined by an amenable regular multiplicative unitary (in the sense of Baaj and Skandalis). If \(A\) is nuclear (or exact) then the same is true for \(\mathcal{O}_X \rtimes_\zeta \hat{S}\).
Application to group actions

Considering the multiplicative unitary $\hat{W}_G$ we can extend Hao and Ng’s theorem:

**Corollary (K., cf. HN, 2008 for amenable $G$)**

Let $(\gamma, \alpha)$ be an action of a LCG $G$ on $(X, A)$ and $\beta$ be the induced action of $G$ on $O_X$. If either $J_X = A$ or $\varphi_A$ is injective, then

$$O_X \rtimes_{\beta, r} G \cong O_X \rtimes_{\gamma, r} G.$$
Examples

Let $\delta$ be a coaction of $S$ on a $C^*$-algebra $A$ and $\varphi \in Aut(A)$ such that

$$\delta \circ \varphi = (\varphi \otimes \text{id}_S) \circ \delta.$$ 

Consider $(X, A) := (A(\varphi), A)$. Then $(\delta, \delta)$ defines a coaction of $S$ on $(X, A)$, and induces a coaction $\zeta$ of $S$ on the $C^*$-algebra $\mathcal{O}_X = A \rtimes \varphi \mathbb{Z}$. Applying the main theorem we have $\mathcal{O}_X \rtimes \zeta \hat{S} = \mathcal{O}_{X \rtimes \delta \hat{S}}$, or equivalently

$$(A \rtimes \varphi \mathbb{Z}) \rtimes \zeta \hat{S} = (A \rtimes \delta \hat{S}) \rtimes \varphi \rtimes \text{id} \mathbb{Z},$$

where $\varphi \rtimes \text{id} \in Aut(A \rtimes \delta \hat{S})$ is the automorphism given by

$$(\varphi \rtimes \text{id})(\delta_\ell(a)(1 \otimes x)) = \delta_\ell(\varphi a)(1 \otimes x) \quad (a \in A \times \in \hat{S}).$$
Examples

The quantum permutation group $\mathbb{G} = A_{aut}(X_n)$ is the quantum automorphism group acting on $n$-point set $X_n$ (Wang, 1998). As a $C^*$-algebra, $C(\mathbb{G})$ is a universal $C^*$-algebra generated by $n^2$ projections $u_{ij}$ satisfying $\sum_i u_{ij} = 1 = \sum_j u_{ij}$. It is a compact quantum group.

Consider the $C^*$-correspondence $(\mathbb{C}^n, \mathbb{C})$ ($n \geq 2$). Define

$$\sigma(e_j) = \sum_i e_i \otimes u_{ij}, \quad \delta(1) = 1.$$ 

Then $(\sigma, \delta)$ is a coaction of $C(\mathbb{G})$ on $(\mathbb{C}^n, \mathbb{C})$ and induces a coaction $\zeta$ of $C(\mathbb{G})$ on $\mathcal{O}_n$. We see that $\mathcal{O}_n \rtimes_\zeta \mathbb{G} \cong \mathcal{O}_{\mathbb{C}^n \rtimes_\sigma \mathbb{G}}.$