Abstract

In this note we produce explicit quasi-isomorphisms computing the cyclic homology of crossed-product algebras.

Résumé


Introduction

A fundamental problem in noncommutative geometry is the explicit computation of the cyclic homology of crossed-product algebras, especially in the context of group actions on manifolds or varieties. By explicit it is meant exhibiting explicit quasi-isomorphisms at the level of chains that give rise to explicit constructions of cyclic cycles. In particular, in the context of group actions on manifolds and varieties, we expect to have close relationships with equivariant cohomology and equivariant characteristic classes. There is a great amount of work on the cyclic homology of crossed-product algebras (see, e.g [1,2,3,6,8,9,10,16]). However, at the exception of the chain map of Connes [6,7] for the homogeneous component of periodic cyclic homology, we do not have explicit quasi-isomorphisms.

The aim of this note is to present the construction of explicit quasi-isomorphisms for the cyclic homology and periodic cyclic homology of crossed-product algebras, including for the localizations at infinite order components. Furthermore, the arguments use only elementary homological algebra. Therefore, this allows us to bypass the difficult homological arguments involved in some previous approaches to the cyclic homology of crossed-product algebras.

The focus of this note is on algebraic crossed-products $\mathcal{A}_\Gamma = \mathcal{A} \rtimes \Gamma$ associated with the action of an arbitrary group $\Gamma$ on a unital algebra $\mathcal{A}$ over a commutative ring $k \supset \mathbb{Q}$. In the sequel [18] we will explain how to apply the results in the contexts of group actions on manifolds and varieties.
1. Cyclic Homology and Triangular S-Modules

We refer to [5,7,14] for background on cyclic homology, including cyclic and bi-cyclic modules, mixed complexes, and S-maps. If \( C = (C_\bullet, b, B) \) is a mixed complex, we let \( C^\bullet = (C_\bullet^\bullet, b + SB) \) be its cyclic complex, where \( C_\bullet^\bullet = C_m \oplus C_{m-2} \oplus \cdots \) and \( S : C_\bullet^\bullet \to C_{\bullet-2}^\bullet \) is the periodicity operator. The homology of the chain complex \( C^\bullet \) is the cyclic homology of \( C \) and is denoted by \( HC_\bullet(C) \). The periodic cyclic homology \( HP_\bullet(C) \) is the homology of the complex \( C^\bullet = (C_\bullet^\bullet, b + B) \), where \( C_\bullet^\bullet = \prod_{i \geq 0} C_{2q+i}, \ i = 0, 1. \)

We refer to [10] for background on paracyclic, bi-paracyclic and cylindrical modules, parachain and cylindrical complexes, and parachain bicomplexes. When \( C \) is a bi-paracyclic module (resp., parachain bicomplex) we denote by \( \text{Diag}(C) \) (resp., \( \text{Tot}(C) \)) its diagonal paracyclic module (resp., total parachain complex). When \( C \) is cylindrical we obtain a cyclic complex (resp., mixed complex) (see [10]).

The \( S \)-modules of Jones-Kassel [12,13] encapsulate various approaches to cyclic homology. More generally, by a \( \text{para-S-module} \) we shall mean the datum of \((C_\bullet, b, S)\), where \( C_m, m \geq 0 \), are \( k \)-modules and \( d : C_\bullet \to C_{\bullet-1} \) and \( S : C_\bullet \to C_{\bullet-2} \) are \( k \)-module maps commuting with each other such that \( d^2 = (1 - T)S \) where \( T : C_\bullet \to C_\bullet \) is some \( k \)-module map commuting with both \( d \) and \( S \). When \( d^2 = 0 \) we obtain an \( S \)-module. For instance, if \( C = (C_\bullet, b, B) \) is a parachain complex, then we can define its cyclic complex of the para-S-module \( C^\bullet = (C_\bullet^\bullet, b + SB, S) \). Notions of para-S-module maps and \( S \)-homotopies of para-S-module maps make sense in the same way as with \( S \)-modules. Therefore, although quasi-isomorphisms do not quite make sense for para-S-modules, \( S \)-homotopy inverse of a para-S-module map and \( S \)-homotopy equivalence of para-S-modules do make sense. This enables us to state a version of the Eilenberg-Zilber theorem for bi-paracyclic modules ([17]). If \( C \) is a bi-paracyclic module, then there is an \( S \)-homotopy equivalence between \( \text{Diag}(C) \) and \( \text{Tot}(C) \). It is given by \( S \)-maps \( \Omega^\bullet : \text{Tot}_\bullet(C)^\bullet \to \text{Diag}_\bullet(C)^\bullet \) and \( AW^\bullet : \text{Diag}_\bullet(C)^\bullet \to \text{Tot}_\bullet(C)^\bullet \) whose zeroth degree components are the shuffle and Alexander-Whitney maps.

A \( \text{horizontal triangular para-S-module} \) is given by the datum of \((C_\bullet, d, b, B, S)\) where \( C_{p,q}, p, q \geq 0 \), are \( k \)-modules, \((C_\bullet, d, S)\) is a para-S-module and \((C_{p,\bullet}, b, B)\) is a parachain complex for all \( p, q \geq 0 \), the horizontal operators \((d, S)\) both commute with each of the vertical differentials \((b, B)\). We say that we have a \( \text{horizontal triangular S-module} \) when \( d^2 + S(bB + Bb) = 0 \). There is a similar definition of vertical triangular para-S-module and \( S \)-module \((C_{\bullet, p}, b, B, d, S)\) where the operators \((d, S)\) acts vertically and the operators \((b, B)\) act horizontally. Triangular (para-)\( S \)-modules provide us with a natural framework for defining the tensor product of (para-)\( S \)-modules with mixed and parachain complexes.

Any triangular para-S-module \( C = (C_\bullet, d, B, B, S) \) gives rise to a \( \text{total para-S-module} \) \( \text{Tot}(C) = (\text{Tot}_\bullet(C), d^\dagger, S) \), where \( \text{Tot}_m(C) = \bigoplus_{p+q=m} C_{p,q} \) and \( d^\dagger = d + (-1)^p(b + SB) \) on \( C_{p,q} \). When \( C \) is a triangular \( S \)-module the condition \( d^2 + S(bB + Bb) = 0 \) ensures us that \( (d^\dagger)^2 = 0 \), and so we actually obtain an \( S \)-module. Furthermore, the filtration by columns of \( \text{Tot}_\bullet(C) \) is a filtration of chain complexes, and so it gives rise to a spectral sequence converging to \( H_\bullet(\text{Tot}(C)) \).

We also observe that if \( C = (C_\bullet, b, B, b) \) is a cylindrical complex, then \( \text{Tot}(C)^\bullet \) is the total \( S \)-module of the horizontal triangular \( S \)-module \( C^\sigma := (C_\bullet^\bullet, b + SB, b, B, S) \), where \( C_{p,q}^\sigma = C_{p,q} \oplus C_{p-2,q} \oplus \cdots \). It is also the total \( S \)-module of the triangular \( S \)-module \( C^\sigma := (C_\bullet^\bullet, B, b + SB, S) \), where \( C_{p,q}^\sigma = C_{p,q} \oplus C_{p,q+2} \oplus \cdots \). As a result, we obtain two spectral sequence converging to \( HC_\bullet(\text{Tot}(C)) \). The spectral sequence of Getzler-Jones [10] is an instance of such a spectral sequence.

2. The Cylindrical Complexes \( C^\Phi(\Gamma, \mathcal{E}) \)

From now on, we assume that \( \Gamma \) is a group acting on a unital algebra \( \mathcal{A} \) over a commutative ring \( k \supset \mathbb{Q} \). The cyclic \( k \Gamma \)-module of \( \Gamma \) is \( C(\Gamma) = (C_\bullet(\Gamma), d, s, t) \), where \( C_m(\Gamma) = k \Gamma^{m+1}, m \geq 0 \), and \( (d, s, t) \)
are given by \( d(\psi_0, \ldots, \psi_m) = (\psi_0, \ldots, \psi_{m-1}) \), \( s(\psi_0, \ldots, \psi_m) = (\psi_m, \psi_0, \ldots, \psi_m) \), and \( t(\psi_0, \ldots, \psi_m) = (\psi_0, \ldots, \psi_{m-1}) \). Its \( b \)-differential is given by \( \partial(\psi_0, \ldots, \psi_m) = \sum_{0 \leq j \leq m} (-1)^j (\psi_0, \ldots, \hat{\psi}_j, \ldots, \psi_m) \). Given any \( k \Gamma \)-module \( \mathcal{M} \), the group homology \( H_*(\Gamma, \mathcal{M}) \) is the homology of the chain complex \((C_*(\Gamma, \mathcal{M}), \partial)\), where \( C_m(\Gamma, \mathcal{M}) = \mathcal{M} \otimes \Gamma^m, m \geq 0 \). The group cohomology \( \overline{H}^* (\Gamma, \mathcal{M}) \) is the cohomology of the dual cochain complex \((C^*(\Gamma, \mathcal{M}), \partial)\), where \( C^m(\Gamma, \mathcal{M}) \) consists of all \( \Gamma \)-equivariant maps \( u : \Gamma^m \rightarrow \mathcal{M} \) and \( \partial u = u \circ \partial \).

Let \( \phi \) be a central element of \( \Gamma \). This gives rise to the paracyclic \( k \Gamma \)-module \( \mathcal{C}^\phi(\Gamma) = (C_*(\Gamma), d, s_\phi, t_\phi) \), where \( d \) is as above and \((s_\phi, t_\phi)\) are given by \( s_\phi(\psi_0, \ldots, \psi_m) = (\phi^{-1} \psi_m, \psi_0, \ldots, \psi_m) \) and \( t_\phi(\psi_0, \ldots, \psi_m) = (\phi^{-1} \psi_m, \psi_0, \ldots, \psi_{m-1}) \). The simplicial module structure of \( \mathcal{C}^\phi(\Gamma) \) agrees with that of \( C(\Gamma) \), and so its \( b \)-differential is the differential \( \partial \) described above. We also note that \( t^m(\psi_0, \ldots, \psi_m) = (\phi^{-1} \psi_0, \ldots, \phi^{-1} \psi_m) \), i.e., \( t^m(\psi) \) is given by the action of \( \phi^{-1} \) on \( C_m(\Gamma) \).

In what follows, by a \( \phi \)-parachain complex we shall mean a parachain complex of \( k \Gamma \)-modules \( \mathcal{C} = (\mathcal{C}_*, b, B) \) such that \( T := 1 - (BB + Bb) \) is given by the action of \( \phi^{-1} \) on \( \mathcal{C}_* \). We also define \( \phi \)-paracyclic \( k \Gamma \)-modules as paracyclic \( k \Gamma \)-modules \( \mathcal{C} = (\mathcal{C}_*, d, s, t) \) such that \( t^{m+1} \) is given by the action of \( \phi^{-1} \) on \( \mathcal{C}_m \) (so that the associated parachain complex is a \( \phi \)-parachain complex). The parachain \( k \Gamma \)-module \( \mathcal{C}^\phi(\Gamma) \) is a \( \phi \)-paracyclic module. Another example of \( \phi \)-paracyclic \( k \Gamma \)-module is the twisted cyclic \( k \Gamma \)-module \( \mathcal{C}^\phi(A) = (C_*(A), d_\phi, s, t_\phi) \), where \( C_m(A) = A^{m-1} \) and \( (d_\phi, s, t_\phi) \) are given by \( d_\phi(a^0 \otimes \cdots \otimes a^m) = [(\phi^{-1} \phi^0 a^0) \otimes \cdots \otimes a^m], s(a^0 \otimes \cdots \otimes a^m) = a^0 \otimes \cdots \otimes a^m \) and \( t_\phi(a^0 \otimes \cdots \otimes a^m) = (\phi^{-1} a^0) \otimes a^0 \otimes \cdots \otimes a^m \). For \( \phi = 1 \) we recover the cyclic module \( C(A) \) of \( A \).

Given (left) \( k \Gamma \)-modules \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) we shall denote by \( \mathcal{M}_1 \otimes_\Gamma \mathcal{M}_2 \) their tensor product over \( \Gamma \), i.e., the quotient of \( \mathcal{M}_1 \otimes \mathcal{M}_2 \) by the action of \( \Gamma \). If \( \mathcal{C} \) and \( \mathcal{C}' \) are parachain complexes of \( k \Gamma \)-modules (resp., paracyclic \( k \Gamma \)-modules), then we can form their tensor product \( \mathcal{C} \otimes \mathcal{C}' \) so as to get a parachain complex of \( k \)-modules (resp., a bi-paracyclic \( k \)-modules). When \( \mathcal{C} \) and \( \mathcal{C}' \) are \( \phi \)-paracyclic modules (resp., \( \phi \)-parachain modules) the tensor product \( \mathcal{C} \otimes \mathcal{C}' \) is cylindrical.

In what follows, when \( \mathcal{C} \) is a \( \phi \)-parachain complex (resp., \( \phi \)-paracyclic \( k \Gamma \)-module) we shall denote by \( \mathcal{C}(\Gamma, \mathcal{C}) \) the cylindrical complex (resp., cylindrical \( k \)-module) \( \mathcal{C}^\phi(\Gamma) \otimes_\Gamma \mathcal{C}^\phi(\Gamma) \). We shall use the notation \( \mathcal{C}(\Gamma, A) \) for \( \mathcal{C} = \mathcal{C}^\phi(A) \). It can be shown that if \( \alpha : \mathcal{C} \rightarrow \mathcal{C}' \) is a quasi-isomorphism of \( \phi \)-parachain complexes, then \( 1 \otimes \alpha : \text{Tot}_* (\mathcal{C}(\Gamma, \mathcal{C})) \rightarrow \text{Tot}_* (\mathcal{C}^\phi(\Gamma, \mathcal{C}')) \) is a quasi-isomorphism of mixed complexes.

3. Splitting along conjugacy classes.

The crossed-product algebra \( \mathcal{A}_\Gamma := A \rtimes \Gamma \) is the unital \( k \)-algebra with generators \( a \in A \) and \( u_\phi, \phi \in \Gamma \) subject to the relations \( a^0 u_{\phi_0} a^1 u_{\phi_1} = a^0 (\phi_0^{-1} a^1) u_{\phi_0 \phi_1} \) for all \( a^0 \in A \) and \( \phi_0, \phi_1 \in \Gamma \). Given any \( \phi \in \Gamma \), we shall denote by \( [\phi] \) its conjugacy class in \( \Gamma \). We then denote by \( \mathcal{A}_\phi(A) \) the cyclic submodule of \( \mathcal{A}(A) \) generated by \( a^0 u_{\phi_0} \otimes \cdots \otimes a^m u_{\phi_m} \), with \( a^0 \in A \) and \( \phi_0, \ldots, \phi_m \in [\phi] \). We then have a direct-sum decomposition of cyclic \( k \)-modules \( \mathcal{C}_*(\mathcal{A}_\phi) = \bigoplus C_*(\mathcal{A})_{[\phi]} \), where the summation goes over all conjugacy classes. This provides us with corresponding decomposition and inclusion at the level of the cyclic and periodic complexes. We shall denote by \( \text{HC}_*(\mathcal{A}_\phi) \) (respectively, \( \text{HP}_*(\mathcal{A}_\phi) \)) the cyclic homology (respectively, periodic cyclic homology) of \( \mathcal{A}(A)_{[\phi]} \). We then have a splitting \( \text{HC}(\mathcal{A}_\phi) = \bigoplus \text{HC}_*(\mathcal{A}_\phi) \), and an inclusion \( \bigoplus \text{HP}_*(\mathcal{A}_\phi) \subseteq \text{HP}(\mathcal{A}_\phi) \), which is onto when \( \Gamma \) has a finite number of conjugacy classes.

Given \( \phi \in \Gamma \), let us denote by \( \phi_0 \) its centralizer in \( \Gamma \). As \( \phi_0 \) is a central element of \( \Gamma_\phi \), we may form the cylindrical complex \( \mathcal{C}^\phi(\Gamma_\phi, A) \) as in Section 2. We have a natural embedding of cyclic \( k \)-modules \( \mu_\phi : \text{Diag}_*(\mathcal{C}(\Gamma_\phi, A)) \rightarrow \mathcal{C}_*(\mathcal{A}_\phi) \) given by \( \mu_\phi ((\psi_0, \ldots, \psi_m) \otimes_{\Gamma_\phi} (a^0 \otimes \cdots \otimes a^m)) = [(\psi_{m-1}^{-1}) a^0] u_{\phi_{m-1}^{-1} \phi_{m-2}^{-1} \cdots \phi_0^{-1}} \otimes (\psi_0 \cdot a^1) u_{\phi_0^{-1} \phi_1 \cdots \phi_{m-1}^{-1}} \otimes \cdots \otimes (\psi_{m-1}^{-1} \cdot a^m) u_{\phi_{m-1}^{-1}}. \) This embedding can be shown to be a quasi-isomorphism. Combining this with the Eilenberg-Zilber theorem for bi-paracyclic modules we then obtain explicit quasi-isomorphisms.

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There are analogous quasi-isomorphisms between the corresponding periodic cyclic complexes. All this reduces the study of the cyclic module of $A_\Gamma$ to that of the mixed complexes $\text{Tot}_*(C^\phi(\Gamma_\phi,A))$, $\phi \in \Gamma$.

\section{The Cyclic Module $C^\phi(A)_{[\phi]}$. Finite Order Case}

Suppose that $\phi$ is an element of $\Gamma$ of finite order $r$. Let $C^\phi(\Gamma_\phi)$ be the mixed complex $(C(\Gamma_\phi), \epsilon, 0)$. Given any $\phi$-invariant mixed complex of $k\Gamma_\phi$-modules $E = (E_\bullet, b, B)$, we shall denote by $C^\phi(\Gamma_\phi, E)$ the mixed bicomplex $C^\phi(\Gamma_\phi) \otimes_{r \phi} E$. More generally, if $E = (E_\bullet, b, B)$ is a $\phi$-parachain complex, then we have a $\phi$-invariant mixed complex $C^\phi = (E^\phi_\bullet, b, B)$, where $E^\phi_\bullet$ is the $\phi$-invariant submodule of $E_\bullet$. Then we may form the mixed bicomplex $C^\phi(\Gamma_\phi, E^\phi)$.

Bearing this in mind, let $\nu_\phi : C^\phi(\Gamma_\phi) \rightarrow C^\phi(\Gamma_\phi)$ be the $k\Gamma_\phi$-module map defined by $\nu_\phi(\psi_0, \ldots, \psi_m) = \frac{1}{r} \sum_{0 \leq i_1 \leq r-1} (\phi^{i_1} \psi_0, \ldots, \phi^{i_1} \psi_m)$, where $\phi$ is the group of permutations of $\{0, \ldots, m\}$. As $\nu_\phi$ and $\epsilon$ both are projections and commute with each other, the composition $\nu_\phi \circ \epsilon$ is a projection as well. It can be checked that the composition $\nu_\phi \circ \epsilon : C^\phi(\Gamma_\phi) \rightarrow C^\phi(\Gamma_\phi)$ is a map of parachain complexes which is both a projection and $\phi$-homotopic to the identity (compare [4, 15]). By elaborating on the perturbation theory of Kassel [13], it can be shown that the associated S-map $\nu_\phi \circ \epsilon : C^\phi(\Gamma_\phi) \rightarrow C^\phi(\Gamma_\phi)$ has an explicit $S$-homotopy inverse $\nu_\phi \circ \epsilon : C^\phi(\Gamma_\phi) \rightarrow C^\phi(\Gamma_\phi)$ which is an $S$-map whose zeroth degree component is $\nu_\phi$. Given any $\phi$-parachain complex $E$, we obtain a parachain bicomplex $(E^\phi \otimes 1 : C^\phi(\Gamma_\phi, E))^\phi \rightarrow C^\phi(\Gamma_\phi, E^\phi)$. A homotopy inverse of the $S$-map $(\nu_\phi \circ \epsilon) \otimes 1 : \text{Tot}_*(C^\phi(\Gamma_\phi, E))^\phi \rightarrow \text{Tot}_*(C^\phi(\Gamma_\phi, E^\phi))^\phi$ is given by $(\nu_\phi \circ \epsilon)^\phi \otimes 1 : \text{Tot}_*(C^\phi(\Gamma_\phi, E))^\phi \rightarrow \text{Tot}_*(C^\phi(\Gamma_\phi, E^\phi))^\phi$. Therefore, we arrive at the following result.

\textbf{Theorem 4.1} Let $\phi \in \Gamma$ have finite order. Suppose we are given a quasi-isomorphism of parachain complex $\alpha : C^\phi_\bullet(A) \rightarrow E_\bullet$, where $E$ is a $\phi$-parachain complex. Then we have quasi-isomorphisms,

$$\text{Tot}_*(C^\phi(\Gamma_\phi, E))^\phi \rightarrow \text{Tot}_*(C^\phi(\Gamma_\phi, A))^\phi \rightarrow \text{Tot}_*(C^\phi(\Gamma_\phi, A))^\phi \nu_\phi \circ \epsilon : C^\phi(\Gamma_\phi, E)^\phi \rightarrow C^\phi(\Gamma_\phi, E^\phi)^\phi.$$
5. The Cyclic Module $C^\phi(A) \Gamma$. Infinite Order

Suppose that $\phi$ is an infinite order element of $\Gamma$. Set $\Gamma_\phi = \Gamma_\phi/\langle \phi \rangle$, where $\langle \phi \rangle$ is the subgroup generated by $\phi$. Composing the natural projection $\pi^3 : C^\phi_3(\Gamma_\phi)^3 \to C^\phi_3(\Gamma_\phi)$ with the antisymmetrization map of $C^\phi_3(\Gamma_\phi)$ defined in the previous section, we get a chain map $\pi^3 : C^\phi_3(\Gamma_\phi)^3 \to C^\phi_3(\Gamma_\phi)$. If $\mathcal{M}$ is a $\phi$-invariant $k\Gamma_\phi$-module, then the action of $\Gamma_\phi$ on $\mathcal{M}$ descends to an action of $\Gamma_\phi$, and so we obtain a chain map $\pi^3 \otimes 1 : C^\phi_3(\Gamma_\phi, \mathcal{M})^3 \to C^\phi_3(\Gamma_\phi, \mathcal{M})$. Using results of Marciniak [15] and Kassel [13], this chain map can be shown to give rise to an $S$-homotopy equivalence. In addition, let $u_\phi \in C^2(\Gamma_\phi, k)$ be a 2-cocycle representing the Euler class $e_\phi \in H^2(\Gamma_\phi, k)$ of the central extension $1 \to \langle \phi \rangle \to \Gamma_\phi \to \Gamma_\phi \to 1$. The cap product with $u_\phi$ then gives rise to a chain map $u_\phi - : H_+(\Gamma_\phi) \to H_{-2}(\Gamma_\phi)$. We then can construct an explicit chain homotopy $h_\phi : C^\phi_3(\Gamma_\phi)^3 \to C^\phi_3(\Gamma_\phi)$ such that $\pi^3 S - (u_\phi - ) S = \partial h_\phi + h_\phi(\partial + B_\phi S)$, where $B_\phi$ is the $B$-differential of $C^\phi_3(\Gamma_\phi)$ (compare [11]).

As $u_\phi -$ is a chain map degree $-2$, we get an $S$-module $C^\phi(\Gamma_\phi) = (\Gamma_\phi, \partial, u_\phi - )$. Given any $\phi$-invariant mixed complex $\mathcal{C} = (\mathcal{C}, b, B)$, we denote by $C^\phi(\Gamma_\phi, \mathcal{C})$ the triangular $S$-module given by the tensor product $C^\phi(\Gamma_\phi) \otimes_{\mathcal{C}} \mathcal{C}$. Its total $S$-module is $\text{Tot}(C^\phi(\Gamma_\phi, \mathcal{C})) = (\text{Tot}_+ C^\phi(\Gamma_\phi, \mathcal{C}))$, $d^\phi, u_\phi -$), where $\text{Tot}_+ C^\phi(\Gamma_\phi, \mathcal{C}) = \bigoplus_{p \in \mathbb{Z}} C^\phi_\mathcal{C}_p(\Gamma_\phi, \mathcal{C})$ and $d^\phi = \partial + (-1)^{p+1}B_\phi - u_\phi -$. We then obtain a chain map $\theta : \text{Tot}_+ (\mathcal{C}^\phi(\Gamma_\phi, \mathcal{C}))^3 \to \text{Tot}_+ (C^\phi(\Gamma_\phi, \mathcal{C}))$ by letting $\theta = \pi^3 \otimes 1 + (-1)^{p+1}(1 \otimes B_\phi) + (B_\phi \otimes 1)$ on $C^\phi_\mathcal{C}_p(\Gamma_\phi, \mathcal{C})$. It can be shown that $\theta$ is a quasi-isomorphism and $\theta S - (u_\phi - ) S = d^\phi (h_\phi + h_\phi(\partial + B_\phi S))$, where we have denoted by $d^\phi$ the differentials of $\text{Tot}(C^\phi(\Gamma_\phi, \mathcal{C}))^3$ and $\text{Tot}(C^\phi(\Gamma_\phi, \mathcal{C}))$.

**Theorem 5.1** Let $\phi \in \Gamma$ have infinite order. Suppose we are given quasi-isomorphism of parachain complexes $\alpha : C^\phi_\mathcal{C}(A) \to \mathcal{C}$, where $\mathcal{C}$ is a $\phi$-invariant mixed complex. Then we have quasi-isomorphisms,

$$\text{Tot}_+ \alpha (C^\phi(\Gamma_\phi, \mathcal{C})) \xleftarrow{\text{Tot}_+ \alpha (C^\phi(\Gamma_\phi, \mathcal{C}))} \xrightarrow{s^\phi, \phi} \text{Tot}_+ \alpha (C^\phi(\Gamma_\phi, \mathcal{C})) \xrightarrow{\text{Tot}_+ \alpha (C^\phi(\Gamma_\phi, \mathcal{C}))} \xrightarrow{\text{Tot}_+ \alpha (C^\phi(\Gamma_\phi, \mathcal{C}))}$$

This gives an isomorphism $HC_\mathcal{C}(\mathcal{C}_\mathcal{C}(A)) \xrightarrow{\alpha} H_\mathcal{C}(\text{Tot}(C^\phi(\Gamma_\phi, \mathcal{C})))$, under which the periodicity operator of $HC_\mathcal{C}(\mathcal{C}_\mathcal{C}(A))$ is given by the cap product $e_\phi - : H_\mathcal{C}(\text{Tot}(C^\phi(\Gamma_\phi, \mathcal{C}))) \to H_{-2}(\text{Tot}(C^\phi(\Gamma_\phi, \mathcal{C})))$.

Actions satisfying the assumptions of Theorem 5.1 naturally appear in the context of group actions on manifolds (cf. [18]). In general, we have the following result.

**Theorem 5.2** Let $\phi \in \Gamma$ have infinite order. Suppose we are given a quasi-isomorphism of parachain complexes $\alpha : C^\phi_\mathcal{C}(A) \to \mathcal{C}$, where $\mathcal{C} = (\mathcal{C}, b, B)$ is a $\phi$-parachain complex. Then we have a spectral sequence,

$$E_2^2_{p,q} = H_p(\phi, H_q(\mathcal{C})) \Longrightarrow HC_{p+q}(\mathcal{C}_\mathcal{C}(A))$$

where $H_\mathcal{C}(\mathcal{C})$ is the homology of $(\mathcal{C}, b)$. If $b = 0$, then $E_2^2_{p,q} = H_p(\phi, H_q(\mathcal{C}))$ and the $E^2$-differential is given by $(-1)^{p+q}B_\phi(u_\phi - ) : H_p(\phi, H_q(\mathcal{C})) \to H_{p-2}(\phi, H_{q+1}(\mathcal{C}))$.

**Remark 5.3** When $\mathcal{C} = C^\phi(A)$ and $\alpha = \text{id}$, the above spectral sequence specializes to the spectral sequence of Feigin-Tsygan [9].

Let $\delta : C^\phi_3(\Gamma_\phi) \to \text{Diag}_+ (C^\phi_3(\Gamma_\phi, k) \otimes C^\phi_3(\Gamma_\phi))$ be the paracyclic $k$-module map given by $\delta(\psi_0, \ldots, \psi_m) = [(\psi_0, \ldots, \psi_m)]_{\otimes \phi} \otimes (\psi_0, \ldots, \psi_m)$. Combining it with the bi-paracyclic Alexander-Whitney map we obtain a para-$S$-module map $\text{AW}^3 \phi : C^\phi_3(\Gamma_\phi) \to \text{Tot}_+ (C^\phi_3(\Gamma_\phi, k) \otimes C^\phi_3(\Gamma_\phi))^3$. Let $C^\phi_3(\Gamma_\phi, k)$ be the mixed complex $(C_\mathcal{C}_3(\Gamma_\phi), \partial, 0)$, and let us form the vertical triangular para-$S$-module $C^\phi_3(\Gamma_\phi, k) \otimes \mathcal{C}_3(\Gamma_\phi)$. We have a para-$S$-module map $(\pi^3) \otimes 1 : \text{Tot}_+ (C^\phi_3(\Gamma_\phi, k) \otimes C^\phi_3(\Gamma_\phi))^3 \to \text{Tot}_+ (C^\phi_3(\Gamma_\phi, k) \otimes C^\phi_3(\Gamma_\phi))^3$. We thus
obtain a para-$S$-module map $\Delta^2 := ((\varepsilon \pi) \otimes 1) AW^\delta \circ \delta : C_0^\phi(\Gamma_\phi)^2 \rightarrow \text{Tot}_\bullet(C_0^\phi(\Gamma_\phi,k) \otimes C_0^\phi(\Gamma_\phi)^2)$. This gives rise to a bilinear differential graded map,

\[(2) \quad \triangleright : C_0^\phi(\Gamma_\phi,k) \times \text{Tot}_\bullet(C_0^\phi(\Gamma_\phi,A))^2 \rightarrow \text{Tot}_\bullet(C_0^\phi(\Gamma_\phi,A))^2).\]

More precisely, given any cochain $u \in C_p^\phi(\Gamma_\phi,k)$, $p \geq 0$, and chains $\eta \in C_\bullet^\phi(\Gamma_\phi)^2$ and $\xi \in C_\bullet(A)$, we have $u \triangleright (\eta \otimes_k \xi) = [(u \otimes P_\phi \pi_1) \Delta^2 \eta] \otimes_{\Gamma_\phi} \xi$.

**Theorem 5.4** Let $\phi \in \Gamma$ have infinite order. The quasi-isomorphisms (1) and the bilinear map (2) give rise to an associative action of the cohomology ring $H^\bullet(\Gamma_\phi,k)$ on $HC_\bullet(A_{\Gamma})[\phi]$. Moreover, the periodicity operator of $HC_\bullet(A_{\Gamma})[\phi]$ is given by the action of the Euler class $e_\phi$. In particular, $HP_\bullet(A_{\Gamma})[\phi] = 0$ whenever $e_\phi$ is nilpotent in $H^\bullet(\Gamma_\phi,k)$.

**Remark 5.5** The result that $HC_\bullet(A_{\Gamma})[\phi]$ is a module over $H^\bullet(\Gamma_\phi,k)$ and the action of $e_\phi$ gives the periodicity is due to Nistor [16]. The improvement with respect to [16] is twofold. First, we are able to bypass the difficult homological algebra arguments of [16]. Second, we have an explicit description of the action at the level of chains.

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**References**


