

ON THE CONE MULTIPLIER IN $\mathbb{R}^3$

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ABSTRACT. We prove the sharp L^3 bounds for the cone multiplier in $\mathbb{R}^3$ and the associated square function, which is known as Mockenhaupt's square function.

1. INTRODUCTION

We consider the cone multiplier operator of order α which is defined by

$$\widehat{\mathcal{C}^\alpha f}(\xi, \tau) = (\tau^2 - |\xi|^2)_+^\alpha \phi(\tau) \widehat{f}(\xi, \tau), \quad (\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R},$$

where ϕ is a smooth function supported in $[1/2, 2]$. It has been conjectured (see [23]) that for $1 \leq p \leq \infty$

$$\|\mathcal{C}^\alpha f\|_{L^p} \leq C \|f\|_p, \text{ if and only if } \alpha > \alpha(p) = \max\left(\left|1 - \frac{2}{p}\right| - \frac{1}{2}, 0\right).$$

There has been a lot of work devoted to this problem [2, 5, 13, 18, 19, 21] (also see [9, 12, 11] for results in higher dimensions). The sharp bounds for $p > 74$ were first obtained by Wolff [21], and by a refinement of his argument the range was further extended by Garrigós and Seeger [6] and Garrigós, Schlag and Seeger [5]. The conjecture is now known to be true for $p > 20$ and at the critical space, L^4 , the inequality is true for $\alpha > 1/9$ ([5]).

Recently, Bourgain and Guth [3] made new progress on (spherical) restriction and Bochner-Riesz problems by an approach based on the multilinear restriction estimate due to Bennett, Carbery and Tao [1]. By adapting the argument in [3], we prove

Theorem 1.1. *If $\alpha > 0$, $\|\mathcal{C}^\alpha f\|_3 \leq C \|f\|_3$.*

As it was shown by Fefferman [7], the estimate is sharp in that the condition $\alpha > 0$ cannot be removed. Hence by interpolation with the obvious L^2 estimate and duality the conjecture is verified for $3 \leq p \leq \frac{3}{2}$.

By scaling and rotation, for L^p bounds of $\mathcal{C}^\alpha$ we may assume that $\widehat{f}$ is supported in a small neighborhood of $(1, 0, 1)$. By a linear change of variables $(\xi_2, \xi_1 - \tau, \xi_1 + \tau) \rightarrow (\eta, \tau, \rho)$ we modify the operator $\mathcal{C}^\alpha$ so that

$$\widehat{\mathcal{C}^\alpha f}(\eta, \tau, \rho) = (\tau - \eta^2/\rho)_+^\alpha \phi(\eta, \tau, \rho) \widehat{f}(\eta, \tau, \rho), \quad (\eta, \tau, \rho) \in \mathbb{R}^2 \times \mathbb{R}.$$

Here ϕ is a smooth function supported in a small neighborhood of $2e_3 = (0, 0, 2)$. Obviously the L^p boundedness properties of the two operators $\mathcal{C}^\alpha$ and $\mathcal{C}^\alpha$ are equivalent. The second form is especially convenient to perform nonisotropic rescalings adapted to the cone.

Now we take $\psi \in C_c^\infty[1/2, 4]$ and define an operator $\mathcal{C}_\delta$ by

$$\widehat{\mathcal{C}_\delta f}(\eta, \tau, \rho) = \psi\left(\frac{\tau - \eta^2/\rho}{\delta}\right) \widehat{f}(\eta, \tau, \rho)$$

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for f of which the Fourier transform is supported in the set $\{(\eta, \tau, \rho) : \rho \in [2^{-2}, 2^2], |\eta/\rho| \leq 2^2\}$. Let $\phi \in C_c^\infty[-1, 1]$ and $0 < \delta \leq 1$. For $\nu \in \sqrt{\delta}\mathbb{Z} \cap [-1, 1]$, define a projection operator by

$$(1) \quad \widehat{S^\nu f} = \phi\left(\frac{\nu - \eta/\rho}{\sqrt{\delta}}\right) \widehat{f}.$$

In what follows we obtain a sharp square function estimate.

Theorem 1.2. *Suppose that $\widehat{f}$ is supported in $B(2e_3, 1)$. For $\epsilon > 0$,*

$$(2) \quad \left\| \sum_{\nu} C_{\delta} S^{\nu} f \right\|_3 \leq C_{\epsilon} \delta^{-\epsilon} \left\| \left(\sum_{\nu} |C_{\delta} S^{\nu} f|^2 \right)^{\frac{1}{2}} \right\|_3.$$

The critical L^4 estimate

$$(3) \quad \left\| \sum_{\nu} C_{\delta} S^{\nu} f \right\|_4 \leq C_{\gamma} \delta^{-\gamma} \left\| \left(\sum_{\nu} |C_{\delta} S^{\nu} f|^2 \right)^{\frac{1}{2}} \right\|_4$$

has been of interest (see [2, 6, 13, 19, 22]) and it is conjectured that (3) holds for $\gamma > 0$. In [5] (3) was shown for $\gamma > 1/9$. This may be improved further by making use of (2) and the argument in [5, 6] but we do not pursue it here.

As it was shown in [14] (also see [19]), the square function estimates (2) and (3) can be used to show the local smoothing estimate for the wave equation:

$$(4) \quad \|e^{it\sqrt{-\Delta}} f\|_{L_{x,t}^p(I \times \mathbb{R}^2)} \leq C \|f\|_{L_{\beta}^p}$$

where I is a compact interval and L_{β}^p is the L^p -Sobolev space of order β . It was conjectured by Sogge [16] that (4) holds for $p \geq 2$ if $\beta > \alpha(p)$. By the works [5, 6, 21], this is now verified for $p > 20$. Also see [10, 15, 17] for related results and recent development in higher dimensions. Combining (2) with the Nikodym type maximal estimate in [14, Lemma 1.4]¹⁾ we obtain

Corollary 1.3. *Let $p = 3$. Then (4) holds for all $\beta > 0$.*

This can be interpolated with known results to extend the range for (4).

Finally we make a few remarks on the paper and the notation. In section 2 we obtain estimates based on the multilinear restriction estimate and in section 3 we prove the theorems. The constant C may vary line to line and in addition to $\widehat{}$ we also use $\mathcal{F}$ to denote the Fourier transform.

2. TRILINEAR ESTIMATES

In this section we state various trilinear estimates which are deduced from the multilinear restriction estimate in [1].

Transversality of conic sectors. Let us consider a subset Γ of the cone, which is given by

$$\Gamma = \{(\eta, \tau, \rho) : \tau = \eta^2/\rho, \rho \in [3/2, 5/2], |\eta/\rho| \leq 3\}.$$

Let us define a map $\theta : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\theta(\eta, \tau, \rho) = \eta/\rho.$$

¹⁾In fact, we need an L^3 estimate for the maximal function, but it follows from interpolation between L^2 and the trivial L^∞ estimate.

One may identify $\theta = \eta/\rho$ as an angular variable of (η, τ, ρ) . Then we may write

$$\Gamma = \{\rho(\theta, \theta^2, 1) : \rho \in [3/2, 5/2], |\theta| \leq 3\}.$$

The normal vector to Γ at (η, τ, ρ) is parallel to

$$(2\eta/\rho, -1, -\eta^2/\rho^2) = (2\theta, -1, -\theta^2).$$

Consider three points (η_1, τ_1, ρ_1) , (η_2, τ_2, ρ_2) , and $(\eta_3, \tau_3, \rho_3) \in \Gamma$ with angular positions $\theta_1, \theta_2, \theta_3$, $(\theta_i = \eta_i/\rho_i)$, respectively. Since

$$\det \begin{pmatrix} 2\theta_1 & -1 & -\theta_1^2 \\ 2\theta_2 & -1 & -\theta_2^2 \\ 2\theta_3 & -1 & -\theta_3^2 \end{pmatrix} = 2(\theta_1 - \theta_3)(\theta_1 - \theta_2)(\theta_2 - \theta_3),$$

we see that three conical sectors are transversal as long as their angular variables are separated. Hence it is possible to make use of the multilinear (trilinear) restriction estimate [1, Theorem 1.16] provided that the supports of the three functions are angularly separated.

Let us denote by $d\sigma$ the induced Lebesgue measure in Γ .

Theorem 2.1. *Let Γ_1, Γ_2 , and Γ_3 be subsets of Γ and $\epsilon_o > 0$. Suppose that $\theta(\Gamma_1)$, $\theta(\Gamma_2)$ and $\theta(\Gamma_3)$ are mutually separated by a distance $\gtrsim \epsilon_o > 0$. Let $R \gg \epsilon_o^{-1}$. Then, for $\epsilon > 0$, there is a constant $C_\epsilon = C_\epsilon(\epsilon_o)$ such that*

$$\left\| \prod_{i=1}^3 \widehat{g_i d\sigma} \right\|_{L^1(B_R)} \leq C_\epsilon R^\epsilon \prod_{i=1}^3 \|g_i\|_2$$

whenever g_i is supported in Γ_i , $i = 1, 2, 3$.

An equivalent statement can be given as follows (see Lemma 2.2 in [1]. The implication from Theorem 2.1 to (5) is a trilinear version of Stein's argument in [8].):

Suppose that $\widehat{F_i}$ is supported in $\Gamma_i + O(R^{-1})$, $i = 1, 2, 3$. Then, for $\epsilon > 0$ there is a constant $C_\epsilon = C_\epsilon(\epsilon_o)$ such that

$$(5) \quad \left\| \prod_{i=1}^3 F_i \right\|_{L^1(B_R)} \leq C_\epsilon R^{-\frac{3}{2}} R^\epsilon \prod_{i=1}^3 \|F_i\|_2.$$

Equivalence can be shown without difficulty by Plancherel's theorem together with a slicing argument (see [20] for the details).

For $0 < \lambda$, let us set

$$\mathcal{A}(\lambda) = \left\{ (\eta, \tau, \rho) : \rho \in \left[\frac{1}{2}, \frac{7}{2} \right], |\eta/\rho| \leq \lambda \right\}.$$

If $\widehat{f}$ is supported in $\mathcal{A}(3)$, we may assume that the convolution kernel of C_δ is rapidly decaying outside of a ball radius $\gg \delta^{-1}$ because $\mathcal{F}(C_\delta f)$ is supported in $\Gamma + O(\delta)$. By the standard localization argument it is sufficient to consider the L^p norm over a ball of radius δ^{-1} provided that $\widehat{f}$ is supported in $\mathcal{A}(3)$. By making use of (5) with $R = \delta^{-1}$ it is easy to see

Lemma 2.2. *Let $1 \gg \epsilon_o \gg \delta > 0$. Suppose that $\widehat{f_1}, \widehat{f_2}$, and $\widehat{f_3}$ are supported in $\mathcal{A}(3)$. If $\theta(\text{supp } \widehat{f_1})$, $\theta(\text{supp } \widehat{f_2})$, and $\theta(\text{supp } \widehat{f_3})$ are mutually separated by a distance $\gtrsim \epsilon_o$, then for $\epsilon > 0$ there is a constant $C_\epsilon = C_\epsilon(\epsilon_o)$ such that*

$$\left\| \prod_{i=1}^3 C_\delta f_i \right\|_{L^1} \leq C_\epsilon \delta^{\frac{3}{2}-\epsilon} \prod_{i=1}^3 \|f_i\|_2.$$

Square function. In this subsection we assume that f_1, f_2 , and f_3 satisfy the assumption in Lemma 2.2. Let us set $R = \delta^{-1}$. Let ψ be a Schwartz function such that $\psi \geq 1$ on $B(0, 1)$ with its Fourier transform supported in $B(0, 1)$. Set

$$\psi_z = \psi\left(\frac{\cdot - z}{\sqrt{R}}\right).$$

Let $z_0 \in \mathbb{R}^3$. Making use of Lemma 2.2 (or (5)) and orthogonality,

$$(6) \quad \left\| \prod_{i=1}^3 \left(\sum_{\nu} C_{\delta} S^{\nu} f_i \right) \right\|_{L^1(B(z_0, R^{\frac{1}{2}}))} \leq C_{\epsilon} R^{-\frac{3}{4} + \epsilon} R^{\frac{9}{2}(\frac{1}{2} - \frac{1}{3})} \prod_{i=1}^3 \left\| |\psi_{z_0}| \left(\sum_{\nu} |C_{\delta} S^{\nu} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^3}.$$

In fact, the left hand side is bounded by $\left\| \prod_{i=1}^3 \psi_{z_0}^2 \left(\sum_{\nu} C_{\delta} S^{\nu} f_i \right) \right\|_{L^1}$. Since $\mathcal{F}(\psi_{z_0}^2 \left(\sum_{\nu} C_{\delta} S^{\nu} f_i \right))$ is supported in $\Gamma + O(R^{-1/2})$, by Lemma 2.2 or by (5), it follows that

$$\left\| \prod_{i=1}^3 \psi_{z_0}^2 \left(\sum_{\nu} C_{\delta} S^{\nu} f_i \right) \right\|_{L^1} \leq C_{\epsilon} R^{-\frac{3}{4} + \epsilon} \prod_{i=1}^3 \left\| \psi_{z_0}^2 \left(\sum_{\nu} C_{\delta} S^{\nu} f_i \right) \right\|_2.$$

Note that the supports of $\mathcal{F}(\psi_{z_0}^2 C_{\delta} S^{\nu} f_i)$ are essentially disjoint. Hence we see that the right hand side is bounded by

$$C R^{-\frac{3}{4} + \epsilon} \prod_{i=1}^3 \left\| |\psi_{z_0}|^2 \left(\sum_{\nu} |C_{\delta} S^{\nu} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \leq C_{\epsilon} R^{-\frac{3}{4} + \epsilon} R^{\frac{9}{2}(\frac{1}{2} - \frac{1}{3})} \prod_{i=1}^3 \left\| |\psi_{z_0}| \left(\sum_{\nu} |C_{\delta} S^{\nu} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^3}.$$

The last inequality follows from Hölder's inequality. Thus we get the desired bound (6).

Let $\{z_0\} \subset \mathbb{R}^3$ be a collection of points separated by $\sim R^{\frac{1}{2}}$ such that $\bigcup_{z_0} B(z_0, R^{\frac{1}{2}}) = \mathbb{R}^3$. By taking summation along the balls $\{B(z_0, R^{\frac{1}{2}})\}$ which are boundedly overlapping (of which centers z_0 are separated by $R^{\frac{1}{2}}$) and by using (6), we see

$$\begin{aligned} \left\| \prod_{i=1}^3 \left(\sum_{\nu} C_{\delta} S^{\nu} f_i \right) \right\|_{L^1} &\leq \sum_{z_0} \left\| \prod_{i=1}^3 \left(\sum_{\nu} C_{\delta} S^{\nu} f_i \right) \right\|_{L^1(B(z_0, R^{\frac{1}{2}}))} \\ &\leq C_{\epsilon} R^{-\frac{3}{4} + \epsilon} R^{\frac{9}{2}(\frac{1}{2} - \frac{1}{3})} \sum_{z_0} \prod_{i=1}^3 \left\| |\psi_{z_0}| \left(\sum_{\nu} |C_{\delta} S^{\nu} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^3} \\ &\leq C_{\epsilon} R^{\epsilon} \prod_{i=1}^3 \left(\sum_{z_0} \left\| |\psi_{z_0}| \left(\sum_{\nu} |C_{\delta} S^{\nu} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^3}^3 \right)^{\frac{1}{3}}. \end{aligned}$$

By the rapid decay of ψ_{z_0} outside of $B(z_0, CR^{\frac{1}{2}})$ it follows that

$$\prod_{i=1}^3 \left\| |\psi_{z_0}| \left(\sum_{\nu} |C_{\delta} S^{\nu} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^3} \leq C \prod_{i=1}^3 \left\| \left(\sum_{\nu} |C_{\delta} S^{\nu} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^3}^3.$$

Therefore we obtain the following.

Proposition 2.3. *Under the same assumption as in Lemma 2.2, for $\epsilon > 0$ there is a constant $C_{\epsilon} = C_{\epsilon}(\epsilon_0)$ such that*

$$\left\| \prod_{i=1}^3 \left(\sum_{\nu} C_{\delta} S^{\nu} f_i \right) \right\|_{L^1} \leq C_{\epsilon} \delta^{-\epsilon} \prod_{i=1}^3 \left\| \left(\sum_{\nu} |C_{\delta} S^{\nu} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^3}.$$

Before closing this section we recall the following estimate for the square function.

Lemma 2.4. *Let $2 \leq p \leq 4$. Then for $\epsilon > 0$, $\left\| (\sum_{\nu} |C_{\delta} S^{\nu} f|^2)^{\frac{1}{2}} \right\|_{L^p} \leq C \delta^{-\epsilon} \|f\|_p$.*

When $p = 2$ the above is clear from Plancherel's theorem. Then by interpolation it is sufficient to consider $p = 4$. The estimate $\left\| (\sum_{\nu} |C_{\delta} S^{\nu} f|^2)^{\frac{1}{2}} \right\|_{L^4} \leq C \delta^{-\epsilon} \|f\|_4$ can be obtained by making use of the maximal estimate due to Córdoba [4] and a duality argument (for example see [13]).

3. PROOFS OF THEOREM 1.1 AND 1.2: PROOF OF (2)

In order to prove Theorem 1.1, it is sufficient to show the estimate

$$\|C_{\delta} f\|_3 \leq C \delta^{-\epsilon} \|f\|_3.$$

This follows from the estimate (2) and Lemma 2.4. So for the proof of the theorems we only need to show (2).

We now prove (2) by using trilinear estimate in Proposition 2.3. To do this, we need to decompose the function f in such a way that the trilinear estimate can be effectively used. It is important to decompose the operator C_{δ} into two parts so that the one is bounded by a sum of trilinear operators with transversality meanwhile the other part is controlled by sum of operators which have relatively small supports at the Fourier transform side. Unlike the argument in [3] we don't need to use the lower dimensional restriction estimates. So the decomposition here is simpler than the one in [3].

The rest of the paper is devoted to the proof of (2).

Decomposition. We assume that $\widehat{f}$ is supported in $\mathcal{A}(2)$. Let $0 < \delta \ll 1$ and $S^{\nu} f$ be given by (1) so that $\sum_{\nu \in \sqrt{\delta} \mathbb{Z}} S^{\nu} f = f$. Let K_1, K_2 be large fixed numbers such that $1 \ll K_1 \ll K_2 \ll \delta^{-\epsilon}$.

Let us set

$$\{\mathfrak{J}^1\} = \left\{ \frac{k}{K_1} \in [-1, 1] : k \in \mathbb{Z} \right\}, \quad \{\mathfrak{J}^2\} = \left\{ \frac{k}{K_2} \in [-1, 1] : k \in \mathbb{Z} \right\}.$$

First we group $S^{\nu} f$ into functions $f_{\mathfrak{J}^2}$ by setting

$$(7) \quad f_{\mathfrak{J}^2} = \sum_{\nu \in (\mathfrak{J}^2 - (2K_2)^{-1}, \mathfrak{J}^2 + (2K_2)^{-1})} S^{\nu} f.$$

So, it follows that

$$(8) \quad \begin{aligned} \text{supp } \widehat{f_{\mathfrak{J}^2}} &\subset (\mathfrak{J}^2 - \frac{51}{100K_2}, \mathfrak{J}^2 + \frac{51}{100K_2}), \\ \sum_{\mathfrak{J}^2} f_{\mathfrak{J}^2} &= \sum_{\nu} S^{\nu} f = f. \end{aligned}$$

Similarly we also group $f_{\mathfrak{J}^2}$ into functions $f_{\mathfrak{J}^1}$ by setting

$$(9) \quad f_{\mathfrak{J}^1} = \sum_{\mathfrak{J}^2 \in (\mathfrak{J}^1 - (2K_1)^{-1}, \mathfrak{J}^1 + (2K_1)^{-1})} f_{\mathfrak{J}^2}.$$

Hence we have

$$(10) \quad \begin{aligned} \text{supp } \widehat{f_{\mathfrak{J}^1}} &\subset (\mathfrak{J}^1 - \frac{51}{100K_1}, \mathfrak{J}^1 + \frac{51}{100K_1}), \\ \sum_{\mathfrak{J}^1} f_{\mathfrak{J}^1} &= \sum_{\mathfrak{J}^2} f_{\mathfrak{J}^2} = f. \end{aligned}$$

We now fix $x \in \mathbb{R}^3$. Then, there are two possibilities;

$$(11) \quad |C_\delta f(x)| \leq 16 \max_{\mathfrak{J}_1^1} |C_\delta f_{\mathfrak{J}_1^1}(x)|,$$

$$(12) \quad |C_\delta f(x)| \geq 16 \max_{\mathfrak{J}_1^1} |C_\delta f_{\mathfrak{J}_1^1}(x)|.$$

For the second case (12) we claim that there are $\mathfrak{J}_1^1, \mathfrak{J}_2^1$, such that

$$|C_\delta f(x)| \leq CK_1 |C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)|^{\frac{1}{2}}$$

and $|\mathfrak{J}_1^1 - \mathfrak{J}_2^1| \geq \frac{4}{K_1}$. In fact, let us denote by $\mathfrak{J}_*^1$ the cube $\mathfrak{J}_1^1$ such that $|C_\delta f_{\mathfrak{J}_*^1}(x)| = \max_{\mathfrak{J}_1^1} |C_\delta f_{\mathfrak{J}_1^1}(x)|$. Since $|C_\delta f(x)| \leq \sum_{\mathfrak{J}_1^1} |C_\delta f_{\mathfrak{J}_1^1}(x)|$,

$$\begin{aligned} |C_\delta f(x)| &\leq \sum_{|\mathfrak{J}_*^1 - \mathfrak{J}_1^1| \geq \frac{4}{K_1}} |C_\delta f_{\mathfrak{J}_1^1}(x)| + 8 |C_\delta f_{\mathfrak{J}_*^1}(x)| \\ &\leq \sum_{|\mathfrak{J}_*^1 - \mathfrak{J}_1^1| \geq \frac{4}{K_1}} |C_\delta f_{\mathfrak{J}_1^1}(x)| + \frac{1}{2} |C_\delta f(x)|. \end{aligned}$$

So, there is a $\mathfrak{J}_1^1$ such that $|\mathfrak{J}_*^1 - \mathfrak{J}_1^1| \geq \frac{4}{K_1}$ and $|C_\delta f(x)| \leq 2K_1 |C_\delta f_{\mathfrak{J}_1^1}(x)|$. By taking $\mathfrak{J}_1^1 = \mathfrak{J}_*^1$ and $\mathfrak{J}_2^1 = \mathfrak{J}_1^1$, the claim follows. Hence, combining this with the case (12) we see that

$$(13) \quad |C_\delta f(x)| \leq C \max_{\mathfrak{J}_1^1} |C_\delta f_{\mathfrak{J}_1^1}(x)| + CK_1 \max_{\mathfrak{J}_1^1, \mathfrak{J}_2^1: |\mathfrak{J}_1^1 - \mathfrak{J}_2^1| \geq \frac{4}{K_1}} |C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)|^{\frac{1}{2}}.$$

We intend to decompose the second term in the right hand side. Let $\mathfrak{J}_1^1$ and $\mathfrak{J}_2^1$ be separated by distance $\geq \frac{4}{K_1}$. Using (10) we write

$$(14) \quad f_{\mathfrak{J}_1^1} = \sum_{\mathfrak{J}_1^2} f_{\mathfrak{J}_1^2}, \quad f_{\mathfrak{J}_2^1} = \sum_{\mathfrak{J}_2^2} f_{\mathfrak{J}_2^2}.$$

Here $\mathfrak{J}_1^2, \mathfrak{J}_2^2 \in \{\mathfrak{J}^2\}$ and $\mathfrak{J}_i^2 \in (\mathfrak{J}_i^1 - (2K_1)^{-1}, \mathfrak{J}_i^1 + (2K_1)^{-1}]$ for $i = 1, 2$. Note that $|\mathfrak{J}_1^2 - \mathfrak{J}_2^2| \geq \frac{2}{K_1}$ by the second condition of (10) since $|\mathfrak{J}_1^1 - \mathfrak{J}_2^1| \geq \frac{4}{K_1}$. Let us denote by $\mathfrak{J}_{1*}^2$ and $\mathfrak{J}_{2*}^2$ those indices such that

$$|C_\delta f_{\mathfrak{J}_{1*}^2}(x)| = \max_{\mathfrak{J}_1^2} |C_\delta f_{\mathfrak{J}_1^2}(x)|, \quad |C_\delta f_{\mathfrak{J}_{2*}^2}(x)| = \max_{\mathfrak{J}_2^2} |C_\delta f_{\mathfrak{J}_2^2}(x)|,$$

where max are respectively taken over the indices $\mathfrak{J}_1^2, \mathfrak{J}_2^2$ which are appearing the first and the second summations in (14). We consider the cases

$$(15) \quad |C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)| \leq 2^5 \max_{\mathfrak{J}_1^2, \mathfrak{J}_2^2} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)|,$$

$$(16) \quad |C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)| \geq 2^5 \max_{\mathfrak{J}_1^2, \mathfrak{J}_2^2} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)|,$$

separately. For the second case (16) we claim that if $|\mathfrak{J}_1^1 - \mathfrak{J}_2^1| \geq \frac{4}{K_1}$,

$$(17) \quad \begin{aligned} |C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)| &\leq CK_2^{-50} \max_{\mathfrak{J}^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\}} |C_\delta f_{\mathfrak{J}^2}(x)|^2 \\ &\quad + CK_2^{45} \max_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\} \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{2}{K_2}}} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x) C_\delta f_{\mathfrak{J}_3^2}(x)|^{\frac{2}{3}}. \end{aligned}$$

Proof of (17). To begin with, from (14) we write

$$|C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)| \leq \sum_{\mathfrak{J}_1^2, \mathfrak{J}_2^2} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)|.$$

In the summation those terms such that $|C_\delta f_{\mathfrak{J}_1^2}(x)| \leq K_2^{-100} |C_\delta f_{\mathfrak{J}_{2*}^2}(x)|$, or $|C_\delta f_{\mathfrak{J}_2^2}(x)| \leq K_2^{-100} |C_\delta f_{\mathfrak{J}_{1*}^2}(x)|$ are bounded by $K_2^{-50} C \max_{\mathfrak{J}^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\}} |C_\delta f_{\mathfrak{J}^2}(x)|^2$. Hence

$$(18) \quad |C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)| \leq K_2^{-50} \max_{\mathfrak{J}^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\}} |C_\delta f_{\mathfrak{J}^2}(x)|^2 + \sum_{\mathfrak{J}_1^2, \mathfrak{J}_2^2}^{\circ} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)|.$$

Here, in the summation $\sum_{\mathfrak{J}_1^2, \mathfrak{J}_2^2}^{\circ}$ we are assuming

$$(19) \quad |C_\delta f_{\mathfrak{J}_1^2}(x)| > K_2^{-100} |C_\delta f_{\mathfrak{J}_{2*}^2}(x)|, \text{ and } |C_\delta f_{\mathfrak{J}_2^2}(x)| > K_2^{-100} |C_\delta f_{\mathfrak{J}_{1*}^2}(x)|.$$

We break $\sum_{\mathfrak{J}_1^2, \mathfrak{J}_2^2}^{\circ}$ so that

$$(20) \quad \sum_{\mathfrak{J}_1^2, \mathfrak{J}_2^2}^{\circ} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)| \leq \sum_{|\mathfrak{J}_1^2 - \mathfrak{J}_{1*}^2| \geq \frac{4}{K_2}, \mathfrak{J}_2^2}^{\circ} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)| \\ + \sum_{\mathfrak{J}_1^2, |\mathfrak{J}_2^2 - \mathfrak{J}_{2*}^2| \geq \frac{4}{K_2}}^{\circ} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)| + \sum_{|\mathfrak{J}_1^2 - \mathfrak{J}_{1*}^2| \leq \frac{4}{K_2}, |\mathfrak{J}_2^2 - \mathfrak{J}_{2*}^2| \leq \frac{4}{K_2}}^{\circ} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)|.$$

Now let us consider the first and the second summations in the right hand side. Using (19), for $\mathfrak{J}_1^2$ and $\mathfrak{J}_2^2$ appearing in the first sum (in the right hand side of (20)) we have

$$|C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)| \leq |C_\delta f_{\mathfrak{J}_1^2}(x)|^{\frac{2}{3}} |C_\delta f_{\mathfrak{J}_1^2}(x)|^{\frac{1}{3}} |C_\delta f_{\mathfrak{J}_{2*}^2}(x)|^{\frac{2}{3}} |C_\delta f_{\mathfrak{J}_{2*}^2}(x)|^{\frac{1}{3}} \\ \leq K_2^{40} |C_\delta f_{\mathfrak{J}_1^2}(x)|^{\frac{2}{3}} |C_\delta f_{\mathfrak{J}_{2*}^2}(x)|^{\frac{2}{3}} |C_\delta f_{\mathfrak{J}_{1*}^2}(x)|^{\frac{2}{3}}.$$

Now note that $|\mathfrak{J}_{1*}^2 - \mathfrak{J}_{2*}^2| \geq \frac{4}{K_1}$, $|\mathfrak{J}_1^2 - \mathfrak{J}_{2*}^2| \geq \frac{4}{K_1}$ and $|\mathfrak{J}_1^2 - \mathfrak{J}_{1*}^2| \geq \frac{4}{K_2}$. So, it follows that

$$(21) \quad \sum_{|\mathfrak{J}_1^2 - \mathfrak{J}_{1*}^2| \geq \frac{4}{K_2}, \mathfrak{J}_2^2}^{\circ} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)| \\ \leq K_2^{45} \max_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\} \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{4}{K_2}}} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x) C_\delta f_{\mathfrak{J}_3^2}(x)|^{\frac{2}{3}}.$$

And similarly for $\mathfrak{J}_1^2$ and $\mathfrak{J}_2^2$ appearing the second sum (in the right hand side of (20)) we have

$$|C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)| \leq K_2^{40} |C_\delta f_{\mathfrak{J}_{1*}^2}(x)|^{\frac{2}{3}} |C_\delta f_{\mathfrak{J}_{2*}^2}(x)|^{\frac{2}{3}} |C_\delta f_{\mathfrak{J}_2^2}(x)|^{\frac{2}{3}}.$$

Since $|\mathfrak{J}_{1*}^2 - \mathfrak{J}_{2*}^2| \geq \frac{4}{K_1}$, $|\mathfrak{J}_{1*}^2 - \mathfrak{J}_2^2| \geq \frac{4}{K_1}$ and $|\mathfrak{J}_2^2 - \mathfrak{J}_{2*}^2| \geq \frac{4}{K_2}$, we also have

$$(22) \quad \sum_{\mathfrak{J}_1^2, |\mathfrak{J}_2^2 - \mathfrak{J}_{2*}^2| \geq \frac{4}{K_2}}^{\circ} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)| \\ \leq K_2^{45} \max_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\} \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{4}{K_2}}} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x) C_\delta f_{\mathfrak{J}_3^2}(x)|^{\frac{2}{3}}.$$

Also, since we are assuming (16),

$$\sum_{|\mathfrak{J}_1^2 - \mathfrak{J}_{1*}^2| \leq \frac{4}{K_2}, |\mathfrak{J}_2^2 - \mathfrak{J}_{2*}^2| \leq \frac{4}{K_2}}^{\circ} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)| \leq \frac{1}{2} |C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)|.$$

By combining this with (21), (22), and using (20), it follows that

$$\begin{aligned} \sum_{\mathfrak{J}_1^2, \mathfrak{J}_2^2}^{\circ} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)| &\leq 2K_2^{45} \max_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\} \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{4}{K_2}}} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x) C_\delta f_{\mathfrak{J}_3^2}(x)|^{\frac{2}{3}} \\ &\quad + \frac{1}{2} |C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)|. \end{aligned}$$

Therefore, by this and (18) and we get (17). $\square$

Also considering the case (15) together with (17), we have

$$\begin{aligned} |C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)| &\leq 2K_2^{-50} \max_{\mathfrak{J}^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\}} |C_\delta f_{\mathfrak{J}^2}(x)|^2 + 2^5 \max_{\mathfrak{J}_1^2, \mathfrak{J}_2^2} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)| \\ &\quad + K_2^{50} \max_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\} \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{2}{K_2}}} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x) C_\delta f_{\mathfrak{J}_3^2}(x)|^{\frac{2}{3}}. \end{aligned}$$

Combining this with (13), for any $x \in \mathbb{R}^3$ we get

$$\begin{aligned} (23) \quad |C_\delta f(x)| &\leq C \max_{\mathfrak{J}_1^1} |C_\delta f_{\mathfrak{J}_1^1}(x)| + CK_1 \max_{\mathfrak{J}_2^2} |C_\delta f_{\mathfrak{J}_2^2}(x)| \\ &\quad + K_2^{50} \max_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{2}{K_2}}} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x) C_\delta f_{\mathfrak{J}_3^2}(x)|^{\frac{1}{3}}. \end{aligned}$$

Bounds for the square function. For $0 < \delta \leq 1$, we define $\mathfrak{S}(\delta)$ to be the best constant for which

$$(24) \quad \left\| \sum_{\nu} C_\delta S^\nu f \right\|_3 \leq \mathfrak{S}(\delta) \left\| \left(\sum_{\nu} |C_\delta S^\nu f|^2 \right)^{\frac{1}{2}} \right\|_3$$

holds whenever $\widehat{f}$ is supported in $\mathcal{A}(2)$. For the proof of Theorem 1.2, it is sufficient to show that $\mathfrak{S}(\delta) \leq C\delta^{-\epsilon}$ for any $\epsilon > 0$.

As it was observed in [3], smallness of support at Fourier side can be cooperated with rescaling to give better bounds. It is also true for the square function. More precisely we have the following lemma.

Lemma 3.1. *Let $0 < \delta \leq \gamma^2 \leq 1$. Suppose that $\widehat{f}$ is supported in $\mathcal{A}(2)$ and the diameter of $\theta(\text{supp } \widehat{f}) \leq \gamma$, then*

$$\left\| \sum_{\nu} C_\delta S^\nu f \right\|_3 \leq \mathfrak{S}(\delta/\gamma^2) \left\| \left(\sum_{\nu} |C_\delta S^\nu f|^2 \right)^{\frac{1}{2}} \right\|_3.$$

This can be shown by rescaling and making use of Lemma 3.2 below. Let $0 < \gamma \leq 1$ and $\theta_\circ \in [-3, 3]$ and set

$$T_{\theta_\circ, \gamma} = \begin{pmatrix} \gamma & 0 & \theta_\circ \\ 2\gamma\theta_\circ & \gamma^2 & \theta_\circ^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We also denote by $T_{\theta_\circ, \gamma}^*$ the adjoint of $T_{\theta_\circ, \gamma}$.

Lemma 3.2. *Let $0 < \delta \leq \gamma^2 \leq 1$ and $\theta_o \in [-2, 2]$. Suppose that $\widehat{f}$ is supported in $\mathcal{A}(2)$ and $\theta(\text{supp } \widehat{f}) \subset [\theta_o - \gamma/2, \theta_o + \gamma/2]$. Then*

$$(25) \quad C_\delta f(x) = C_{\delta/\gamma^2}(f \circ T_{\theta_o, \gamma}^{*-1})(T_{\theta_o, \gamma}^* x)$$

and $\mathcal{F}(f \circ T_{\theta_o, \gamma}^{*-1})$ is supported in $\mathcal{A}(1)$.

Proof. To begin with we carry out the changes of variables at Fourier side, $\eta \rightarrow \eta + \theta_o \rho$, $\tau \rightarrow \tau + \theta_o^2 \rho + 2\theta_o \eta$, $\eta \rightarrow \gamma \eta$, $\tau \rightarrow \gamma^2 \tau$ which change successively $\tau - \eta^2/\rho$ to

$$\tau - \theta_o^2 \rho - 2\theta_o \eta - \eta^2/\rho \Rightarrow \tau - \eta^2/\rho \Rightarrow \gamma^2(\tau - \eta^2/\rho).$$

Composing all the linear changes of variables in order, we see

$$(\eta, \tau, \rho) \rightarrow T_{\theta_o, \gamma}(\eta, \tau, \rho) = (\gamma\eta + \theta_o \rho, 2\gamma\theta_o \eta + \gamma^2 \tau + \theta_o^2 \rho, \rho)$$

and also it follows that

$$(26) \quad \widehat{C_\delta f}(T_{\theta_o, \gamma}(\eta, \tau, \rho)) = \psi\left(\gamma^2 \frac{(\tau - \eta^2/\rho)}{\delta}\right) \widehat{f}(T_{\theta_o, \gamma}(\eta, \tau, \rho)).$$

By inversion we have $C_\delta f(T_{\theta_o, \gamma}^{*-1} x) = C_{\delta/\gamma^2}(f \circ T_{\theta_o, \gamma}^*)(x)$. Hence we get the desired (25). The last statement is obvious from the change of variables because $\mathcal{F}(f \circ T_{\theta_o, \gamma}^{*-1}) = |\det T_{\theta_o, \gamma}| \widehat{f}(T_{\theta_o, \gamma}(\eta, \tau, \rho))$. $\square$

For $\theta \in \mathbb{R}$ and $\nu \in \sqrt{\delta}\mathbb{Z}$, let us set

$$\mathcal{F}(S^{\nu, \theta} f) = \phi\left(\frac{\nu + \theta - \eta/\rho}{\sqrt{\delta}}\right) \widehat{f}(\eta, \tau, \rho).$$

For the proof of Lemma 3.1 we make the observation that (24) implies

$$(27) \quad \left\| \sum_{\nu} C_\delta S^{\nu, \theta} f \right\|_3 \leq \mathfrak{S}(\delta) \left\| \left(\sum_{\nu} |C_\delta S^{\nu, \theta} f|^2 \right)^{\frac{1}{2}} \right\|_3$$

for any $\theta \in \mathbb{R}$ whenever $\widehat{f}$ is supported in $\mathcal{A}(1)$. It can be shown by making use of (25) (in fact with $T_{\theta, 1}$). Indeed, by shifting the indices ν , we may assume that $|\theta| < \sqrt{\delta}$. Since $\mathcal{F}(S^{\nu, \theta} f)(T_{\theta, 1}(\eta, \tau, \rho)) = \phi\left(\frac{\nu - \eta/\rho}{\sqrt{\delta}}\right) \widehat{f}(T_{\theta, 1}(\eta, \tau, \rho))$, by 25 (or (26) equivalently) $C_\delta S^{\nu, \theta} f(x) = C_\delta S^\nu(f \circ T_{\theta, 1}^{*-1})(T_{\theta, 1}^* x)$. So, it follows that

$$\left\| \sum_{\nu} C_\delta S^{\nu, \theta} f \right\|_3 = |\det T_{\theta, 1}^*|^{-\frac{1}{3}} \left\| \sum_{\nu} C_\delta S^\nu(f \circ T_{\theta, 1}^{*-1}) \right\|_3.$$

Note that $\mathcal{F}(f \circ T_{\theta, 1}^{*-1})$ is supported in $\mathcal{A}(2)$ because $\widehat{f}$ is supported in $\mathcal{A}(1)$ and $|\theta| < \sqrt{\delta}$. By the definition of $\mathfrak{S}$ we have

$$\left\| \sum_{\nu} C_\delta S^{\nu, \theta} f \right\|_3 \leq \mathfrak{S}(\delta) |\det T_{\theta, 1}^*|^{-\frac{1}{3}} \left\| \left(\sum_{\nu} |C_\delta S^\nu(f \circ T_{\theta, 1}^{*-1})|^2 \right)^{\frac{1}{2}} \right\|_3.$$

Now, by (25) and reversing the change variables we get (27).

Proof of Lemma 3.1. We may assume that $\theta(\text{supp } \widehat{f}) \subset [\theta_o - \gamma/2, \theta_o + \gamma/2]$ for some $\theta_o \in [-3, 3]$. Now note that

$$\mathcal{F}(S^\nu f)(T_{\theta_o, \gamma}(\eta, \tau, \rho)) = \phi\left(\frac{\gamma^{-1}\nu - \gamma^{-1}\theta_o - \eta/\rho}{\sqrt{\delta/\gamma^2}}\right) \widehat{f}(T_{\theta_o, \gamma}(\eta, \tau, \rho)).$$

As before, by 25 (or (26) equivalently) we see

$$C_\delta S^\nu f(x) = C_{\delta/\gamma^2}[S^{\gamma^{-1}\nu, -\gamma^{-1}\theta_o}](f \circ T_{\theta_o, \gamma}^*)(T_{\theta_o, \gamma}^{*-1} x).$$

So it follows that

$$\begin{aligned} \left\| \sum_{\nu} C_{\delta} S^{\nu} f \right\|_3 &= |\det T_{\theta_o, \gamma}|^{\frac{1}{3}} \left\| \sum_{\nu \in \sqrt{\delta} \mathbb{Z}} C_{\delta/\gamma^2} [S^{\gamma^{-1}\nu, -\gamma^{-1}\theta}] (f \circ T_{\theta_o, \gamma}^*) \right\|_3 \\ &= |\det T_{\theta_o, \gamma}|^{\frac{1}{3}} \left\| \sum_{k \in \sqrt{\delta/\gamma^2} \mathbb{Z}} C_{\delta/\gamma^2} [S^{k, -\gamma^{-1}\theta}] (f \circ T_{\theta_o, \gamma}^*) \right\|_3. \end{aligned}$$

Note that $\mathcal{F}(f \circ T_{\theta_o, \gamma}^*)$ is supported in $\mathcal{A}(1)$. Hence, by (24), (27), and replacing δ with δ/γ^2 we see that

$$\begin{aligned} \left\| \sum_{\nu} C_{\delta} S^{\nu} f \right\|_3 &\leq \mathfrak{S}(\delta/\gamma^2) |\det T_{\theta_o, \gamma}|^{\frac{1}{3}} \left\| \left(\sum_{k \in \sqrt{\delta/\gamma^2} \mathbb{Z}} |C_{\delta/\gamma^2} [S^{k, -\gamma^{-1}\theta}] (f \circ T_{\theta_o, \gamma}^*)|^2 \right)^{\frac{1}{2}} \right\|_3 \\ &= \mathfrak{S}(\delta/\gamma^2) \left\| \left(\sum_{\nu} |C_{\delta/\gamma^2} [S^{\gamma^{-1}\nu, -\gamma^{-1}\theta}] (f \circ T_{\theta_o, \gamma}^*) (T_{\theta_o, \gamma}^{*-1} \cdot)|^2 \right)^{\frac{1}{2}} \right\|_3 \\ &= \mathfrak{S}(\delta/\gamma^2) \left\| \left(\sum_{\nu} |C_{\delta} S^{\nu} f|^2 \right)^{\frac{1}{2}} \right\|_3. \end{aligned}$$

So this proves Lemma 3.1. $\square$

Proof of Theorem 1.2. Let $0 < \delta \leq 1$. For the proof we need to show $\mathfrak{S}(\delta) \leq C\delta^{-\epsilon}$ for $\epsilon > 0$. Clearly, $\mathfrak{S}(\delta) \leq C\delta^{-1/4}$. By interpolation between known results the exponent can be replaced by a smaller one but it is not relevant here. Hence, we may assume $1 \ll \delta^{-1}$ because the bound is trivial otherwise.

Let $1 \ll K_1 \ll K_2 \ll \delta^{-\epsilon}$. Then, using (23) and the embedding $\ell^3 \subset \ell^\infty$, and raising 3rd power we have for $x \in \mathbb{R}^3$

$$\begin{aligned} \left| \sum_k C_{\delta} S^{\nu} f(x) \right|^3 &\leq C \sum_{\mathfrak{J}^1} |C_{\delta} f_{\mathfrak{J}^1}(x)|^3 + CK_1^3 \sum_{\mathfrak{J}^2} |C_{\delta} f_{\mathfrak{J}^2}(x)|^3 \\ &\quad + CK_2^{150} \sum_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{2}{K_2}}} |C_{\delta} f_{\mathfrak{J}_1^2}(x) C_{\delta} f_{\mathfrak{J}_2^2}(x) C_{\delta} f_{\mathfrak{J}_3^2}(x)|. \end{aligned}$$

By integrating both sides, it follows that

$$\begin{aligned} (28) \quad \|C_{\delta} f\|_3^3 &\leq C \sum_{\mathfrak{J}^1} \|C_{\delta} f_{\mathfrak{J}^1}\|_3^3 + CK_1^3 \sum_{\mathfrak{J}^2} \|C_{\delta} f_{\mathfrak{J}^2}\|_3^3 \\ &\quad + CK_2^{150} \sum_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{2}{K_2}}} \|C_{\delta} f_{\mathfrak{J}_1^2} C_{\delta} f_{\mathfrak{J}_2^2} C_{\delta} f_{\mathfrak{J}_3^2}\|_1. \end{aligned}$$

Now we use Proposition 2.3 to handle the last term in the right hand side. Since there are at most CK_2^3 triples $(\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2)$, by Proposition 2.3, with $\epsilon_o = \frac{1}{K_2}$, we see

$$\sum_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{2}{K_2}}} \int |C_{\delta} f_{\mathfrak{J}_1^2}(x) C_{\delta} f_{\mathfrak{J}_2^2}(x) C_{\delta} f_{\mathfrak{J}_3^2}(x)| dx \leq C_{\epsilon} \left(\frac{1}{K_2} \right) \delta^{-\epsilon} \prod_{i=1}^3 \left\| \left(\sum_{\nu} |C_{\delta} f^{\nu}|^2 \right)^{\frac{1}{2}} \right\|_3.$$

Now, recalling (7)–(10), for simplicity we write

$$f_{\mathfrak{J}^1} = \sum_{\nu \sim \mathfrak{J}^1} S^{\nu} f, \quad f_{\mathfrak{J}^2} = \sum_{\nu \sim \mathfrak{J}^2} S^{\nu} f$$

so that $f = \sum_{\mathfrak{J}^1} \sum_{\nu \sim \mathfrak{J}^1} S^\nu f$ and $f = \sum_{\mathfrak{J}^2} \sum_{\nu \sim \mathfrak{J}^2} S^\nu f$. Note that the diameter of $\theta(f_{\mathfrak{J}^2}) \leq \frac{51}{50K_2}$ and set $c = 50^2/51^2$. By Lemma 3.1

$$\begin{aligned} \sum_{\mathfrak{J}^2} \|C_\delta f_{\mathfrak{J}^2}\|_3^3 &= \sum_{\mathfrak{J}^2} \left\| \sum_{\nu \sim \mathfrak{J}^2} C_\delta S^\nu f \right\|_3^3 \leq [\mathfrak{S}(c\delta K_2^2)]^3 \sum_{\mathfrak{J}^2} \left\| \left(\sum_{\nu \sim \mathfrak{J}^2} |C_\delta S^\nu f|^2 \right)^{\frac{1}{2}} \right\|_3^3 \\ &\leq [\mathfrak{S}(c\delta K_2^2)]^3 \left\| \left(\sum_{\mathfrak{J}^2} \sum_{\nu \sim \mathfrak{J}^2} |C_\delta S^\nu f|^2 \right)^{\frac{1}{2}} \right\|_3^3 \\ &\leq [\mathfrak{S}(c\delta K_2^2)]^3 \left\| \left(\sum_{\nu} |C_\delta S^\nu f|^2 \right)^{\frac{1}{2}} \right\|_3^3. \end{aligned}$$

Similarly, for the first term in the right hand side of (28) we have

$$\sum_{\mathfrak{J}^1} \|C_\delta f_{\mathfrak{J}^1}\|_3^3 \leq [\mathfrak{S}(c\delta K_1^2)]^3 \left\| \left(\sum_{\nu} |C_\delta S^\nu f|^2 \right)^{\frac{1}{2}} \right\|_3^3.$$

Now combining all these estimates, we get

$$\left\| \sum_{\nu} C_\delta S^\nu f \right\|_3 \leq C \left(\mathfrak{S}(c\delta K_1^2) + K_1 \mathfrak{S}(c\delta K_2^2) + K_2^{50} C_\epsilon \left(\frac{1}{K_2} \right) \delta^{-\epsilon} \right) \left\| \left(\sum_{\nu} |C_\delta S^\nu f|^2 \right)^{\frac{1}{2}} \right\|_3.$$

Taking sup along f of which the Fourier transform is supported in $\mathcal{A}(2)$, we see for $0 < \delta \leq 1$

$$(29) \quad \mathfrak{S}(\delta) \leq C \left(\mathfrak{S}(c\delta K_1^2) + K_1 \mathfrak{S}(c\delta K_2^2) + K_2^{50} C_\epsilon \left(\frac{1}{K_2} \right) \delta^{-\epsilon} \right).$$

Now let us define

$$\overline{\mathfrak{S}}_\beta(\delta) = \sup_{\delta \leq \delta' \leq 1} (\delta')^\beta \mathfrak{S}(\delta').$$

And let $\delta \leq \delta_o \leq 1$. Then by (29) it follows that

$$\begin{aligned} \delta_o^\beta \mathfrak{S}(\delta_o) &\leq C \left(\delta_o^\beta \mathfrak{S}(c\delta_o K_1^2) + K_1 \delta_o^\beta \mathfrak{S}(c\delta_o K_2^2) + K_2^{50} C_\epsilon \left(\frac{1}{K_2} \right) \delta_o^\beta \delta_o^{-\epsilon} \right) \\ &\leq C \left(K_1^{-2\beta} (c\delta_o K_1^2)^\beta \mathfrak{S}(c\delta_o K_1^2) + K_1 K_2^{-2\beta} (c\delta_o K_2^2)^\beta \mathfrak{S}(c\delta_o K_2^2) + K_2^{50} C_\epsilon \left(\frac{1}{K_2} \right) \delta_o^{\beta-\epsilon} \right) \end{aligned}$$

Since $1 \leq cK_2^2$, cK_1^2 and $\delta \leq \delta_o$, for $\beta < \epsilon$,

$$\delta_o^\beta \mathfrak{S}(\delta_o) \leq C \left(K_1^{-2\beta} \overline{\mathfrak{S}}_\beta(\delta) + K_1 K_2^{-2\beta} \overline{\mathfrak{S}}_\beta(\delta) + K_2^{50} C_\epsilon \left(\frac{1}{K_2} \right) \delta^{\beta-\epsilon} \right).$$

Taking sup along $\delta_o \geq \delta$, we have

$$\overline{\mathfrak{S}}_\beta(\delta) \leq C \left(K_1^{-2\beta} \overline{\mathfrak{S}}_\beta(\delta) + K_1 K_2^{-2\beta} \overline{\mathfrak{S}}_\beta(\delta) + K_2^{50} C_\epsilon \left(\frac{1}{K_2} \right) \delta^{\beta-\epsilon} \right).$$

Finally, we choose $CK_1^{-2\beta} < \frac{1}{4}$ and $CK_1 K_2^{-2\beta} < \frac{1}{4}$, to get $\overline{\mathfrak{S}}_\beta(\delta) \leq CK_2^{50} C_\epsilon \left(\frac{1}{K_2} \right) \delta^{\beta-\epsilon}$ provided that $\epsilon > \beta > 0$. Hence

$$\mathfrak{S}(\delta) \leq CK_2^{50} C_\epsilon \left(\frac{1}{K_2} \right) \delta^{-\epsilon}.$$

Therefore we get (2) for $\epsilon > 0$. This completes the proof.

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