The Game of Hex
and
the Brouwer Fixed-Point Theorem

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Good News :)  

Takes less than 1 hour and Main Idea is attractive  
(Written in English for 2 Post-Docs)
Bad News

Because It’s content is far from what we have learned in this course,

Need to get some Background Information
Hex Theorem

= 

Brouwer Fixed-Point Theorem
Contents

1. Hex?

2. Brower Fixed-Point Theorem?

3. =?
What is Hex?

2 Player Board game

- Marking hexagon with X or O alternatively

- Marking a connected set of tiles meeting the boundary regions X~X' (or O~O')
What is Hex?

Who is the winner?

• Neither player has won

• But the X-player has a sure win in 3 moves

• Red curve is called X-path
What is Hex?

Is there always a winner?

• Yes! It can never end in a draw
Is there always winner?

If every tile of the Hex board is marked either x or o, then there is either

1) an x-path connecting regions X and X’

2) or an o-path connecting regions O and O’
Hex Theorem

PROOF?

• Rigorous proof is not provided here

SO?

• funny analogy gives you some insight
Hex Theorem

ANALOGY

• X is trying to build a dam by putting down stones

• O is compared to flowing water
Hex Theorem

In the End?
There could be ONLY TWO cases

- X can succeed in blocking water (X WIN)
- Or NOT (O WIN)

BUT NOT BOTH
Brouwer Fixed-Point Theorem

Basic Theorem in Topology

Let $f$ be a continuous mapping from the unit square $I^2$ into itself.

Then there exists $x \in I^2$ such that $f(x) = x$
Sorry!
Wait For a Moment!
Graphical Representation

5 X 5 Hex Board

N, S, E, W : Set of Vertices $z = (z_1, z_2)$ of Bk
$z_2 = k, z_2 = 0, z_1 = k, z_1 = 0$
Revision of Hex Theorem

If every tile of the Hex board is marked either $x$ or $o$, then there is either

1) an $x$-path connecting regions $X$ and $X'$
2) or an $o$-path connecting regions $O$ and $O'$

Let $B_k$ be covered by two sets $H$ and $V$. Then either

1) $H$ contains a connected set meeting $E$ and $W$
2) $V$ contains a connected set meeting $N$ and $S$
Two Theorems are equivalent!

Let $B_k$ be covered by two sets $H$ and $V$. Then either

1) $H$ contains a connected set meeting $E$ and $W$

2) $V$ contains a connected set meeting $N$ and $S$

Let $f$ be a continuous mapping from the unit square $I^2$ into itself.

Then there exists $x \in I^2$ such that $f(x) = x$
“Hex” Implies “Brouwer”

Sorry!
Wait For a Moment!
"Hex" Implies "Brouwer"

Four subsets $H^+$, $H^-$, $V^+$, $V^-$ of $B_k$

\[
H^+ = \{ z \mid f_1(z/k) - z_1/k > \varepsilon \}
\]
\[
H^- = \{ z \mid z_1/k - f_1(z/k) > \varepsilon \}
\]
\[
V^+ = \{ z \mid f_2(z/k) - z_2/k > \varepsilon \}
\]
\[
V^- = \{ z \mid z_2/k - f_2(z/k) > \varepsilon \}
\]

\[
H = H^+ \cup H^-, \quad V = V^+ \cup V^-
\]

Claim: $H$ and $V$ Can't Cover $B_k$
“Hex” Implies “Brouwer”

Let \( B_k \) be covered by two sets \( H \) and \( V \). Then either

1) \( H \) contains a connected set meeting \( E \) and \( W \)
2) \( V \) contains a connected set Meeting \( N \) and \( S \)

If two conditions below are satisfied

1) \( H \) contains no connected set meeting \( E \) and \( W \)
2) \( V \) contains no connected set Meeting \( N \) and \( S \)

Two Sets \( H \) and \( V \) cannot covered \( B_k \)

\[ H = H^+ \cup H^-, \quad V = V^+ \cup V^- \]
Each point of $I^2$ is uniquely expressible as convex combination of some set of vertices, all of which are adjacent.

$f : B_k \to \mathbb{R}$ can be extended to $f' : l_k^2 \to \mathbb{R}$

$$x = \lambda_1 z^1 + \lambda_2 z^2 + \lambda_3 z^3 \quad \hat{f}(x) = \lambda_1 f(z^1) + \lambda_2 f(z^2) + \lambda_3 f(z^3).$$
“Brouwer” Implies “Hex”

Partitioning Bk by H and V

1) \( W' = \) Vertices connected to \( W \) by H path
   \( E' = H - W' \)

2) \( S' = \) Vertices connected to \( S \) by V path
   \( N' = V - S' \)

- Disjoint
  - Not Contiguous

Proof is by contradiction
(We assumes that there is no H path from \( E \) to \( W \) and no V path from \( N \) to \( S \))

(\( \rightarrow \) There is NO FIXED POINT in some function \( f \))
Define mapping \( f : B_k \rightarrow B_k \)

\[
\begin{align*}
f(z) &= z + e^1 \text{ for } z \in \hat{W} \\
&= z - e^1 \text{ for } z \in \hat{E} \\
&= z + e^2 \text{ for } z \in \hat{S} \\
&= z - e^2 \text{ for } z \in \hat{N}.
\end{align*}
\]

Need to Check \( f \) is in \( B_k \)

Extend to all of \( I_k^2 \) and we can see that \( f \) has NO FIXED POINT

\( W' = \text{Vertices connected to } W \text{ by } H \text{ path} \)

\( E' = H - W' \)

(We assumes that there is no \( H \) path from \( E \) to \( W \) and no \( V \) path from \( N \) to \( S \))
Define mapping \( f : B_k \rightarrow B_k \)

\[
\begin{align*}
  f(z) &= z + e^1 \text{ for } z \in \hat{W} \\
        &= z - e^1 \text{ for } z \in \hat{E} \\
        &= z + e^2 \text{ for } z \in \hat{S} \\
        &= z - e^2 \text{ for } z \in \hat{N}.
\end{align*}
\]

Extend to all of \( I_k^2 \) and We can see that \( f \) has **NO FIXED POINT**

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**Lemma 1.** Let \( z^1, z^2, z^3 \) be vertices of any triangle \( \triangle \) in \( R^2 \) and let \( \hat{\rho} \) be the simplicial extension of the mapping \( \rho \) defined by \( \rho(z^i) = z^i + v^i \) where \( v^1, v^2, v^3 \) are given vectors. Then \( f \) has a fixed point if and only if \( 0 \) lies in the convex hull of \( v^1, v^2, v^3 \).

- Let \( x = \lambda_1 z^1 + \lambda_2 z^2 + \lambda_3 z^3 \). Then \( \hat{\rho}(x) = \hat{\lambda}_1 (z^1 + v^1) + \hat{\lambda}_2 (z^2 + v^2) + \hat{\lambda}_3 (z^3 + v^3) \) and \( x \) is fixed if and only if \( \lambda_1 v^1 + \lambda_2 v^2 + \lambda_3 v^3 = 0 \).
Thank You!