Lecture 4: Ito's Stochastic Calculus and SDE

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Preliminaries

- What is Calculus ?

Integral, Differentiation.

- Differentiation

- Integral

Riemann integral, Lebesgue integral.

- Ordinary differential equations (ODE):

$$\begin{aligned} \dot{X}(t) &= f(X(t)), \quad t > 0, \quad \dot{X} := \frac{dX}{dt}, \\ X(0) &= X_0, \end{aligned}$$

where $f : R^n \rightarrow R^n$ is a Lipschitz continuous function.

or equivalently the IVP can be rewritten as an integral equation:

$$X(t) = X_0 + \int_0^t f(X(s))ds, \quad t \geq 0.$$

We add (white) noise which is responsible for random fluctuations

$$\begin{aligned} \dot{X}(t) &= f(X(t)) + \sigma\xi(t), \quad t > 0, \quad \dot{X} := \frac{dX}{dt}, \\ X(0) &= X_0, \end{aligned}$$

where $\xi = \xi(t)$ is a white noise satisfying

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t), \xi(s) \rangle = \delta(t - s).$$

We formally set

$$\xi(t) = \frac{dW(t)}{dt}.$$

Then the stochastically perturbed ODE becomes

$$\begin{aligned} & \dot{X}(t) = f(X(t)) + \sigma\xi(t) \\ \iff \quad & \frac{dX(t)}{dt} = f(X(t)) + \sigma\frac{dW(t)}{dt} \\ \iff \quad & dX(t) = f(X(t))dt + \sigma dW(t). \end{aligned}$$

- Example 1:

$$\dot{x} = ax, \quad x(0) = x_0.$$

- Example 2:

$$\dot{x} = x^2, \quad x(0) = x_0.$$

- Heat equation (or diffusion equation)

The fundamental solution $K = K(x,t)$ is defined to be the solution of the following IVP:

$$\begin{aligned} u_t &= \sigma^2 u_{xx}, \quad x \in R,\ t > 0, \\ u(x,0) &= \delta(x). \end{aligned}$$

$$K(x,t) = \frac{1}{\sqrt{4\pi\sigma^2 t}} \exp\Big(-\frac{x^2}{4\sigma^2 t}\Big).$$

The solution to the IVP for heat equation:

$$
\begin{aligned}
u_t &= \sigma u_{xx}, \quad x \in R, \ t > 0, \\
u(x,0) &= u_0(x)
\end{aligned}
$$

is given by

$$u(x,t) = K(x,t) * u_0.$$

Motivation

- Stochastic Differential Equations (SDE)

$$\begin{aligned} dX &= b(X,t)dt + B(X,t)dW, \\ X(0) &= X_0. \end{aligned}$$

which means

$$X(t) = X_0 + \int_0^t b(X(s),s)ds + \int_0^t B(X(s),s)dW(s), \quad t \geq 0.$$

Need to define Ito's integral (1949):

$$\int_0^T GdW, \quad \text{or} \quad \int_0^T G(t,\omega)dW(t,\omega), \quad G: \text{adapted process.}$$

If W is differentiable, (which is not true), we can define

$$\int_0^T GdW = \int_0^T GW'dt.$$

Of course, B.M W is not differentiable in probability 1.

Construction of Ito's integral

- General guideline:

Step 1: Construction of Ito's Integral for simple adapted process.

Step 2: Construction of Ito's Integral for general L^2-adapted process.

- Definition (Simple adopted process)

$\Delta = \Delta(t)$ is a simple process if and only if for some partition $\mathcal{P} = \{t_0 = 0 < t_1 < \cdots < t_n\}$ of $[0, T]$, $\Delta(t)$ is constant in t on each subinterval $[t_j, t_{j+1})$.

Question: How to define $I(t) := \int_0^t \Delta(s) dW(s)$?

Hueristic interpretation

• $W(t)$: the price per share of an asset at time t.

• $t_0, t_1, \cdots, t_n$: trading dates in the asset.

• $\Delta(t_0), \Delta(t_1), \cdots, \Delta(t_n)$: the number of shares take in the asset at each trading date and held to the next trading date.

Then the gain $I(t)$ from trading at each time t is given by

$$
\begin{aligned}
I(t) &= \Delta(0)[W(t) - W(t_0)] = \Delta(0)W(t), \quad 0 \le t \le t_1, \\
I(t) &= \Delta(0)W(t_1) + \Delta(t_1)[W(t) - W(t_1)], \quad t_1 \le t \le t_2, \\
I(t) &= \Delta(0)W(t_1) + \Delta(t_1)[W(t_2) - W(t_1)] + \Delta(t_2)[W(t) - W(t_2)], \quad t_2 \le t \le t_3, \\
I(t) &= \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)], \quad t_k \le t \le t_{k+1}.
\end{aligned}
\tag{2}
$$

- Theorem

Ito's integral is a martingale.

Proof.

• Theorem (Ito's isometry)

$$E[I^2(t)] = E\int_0^t \Delta^2(s)ds.$$

Proof.

- Theorem (Quadratic Variation)

$$[I, I](t) := \int_0^t \Delta^2(s)ds = t.$$

Proof.

- Construction of $I(t)$ for L^2-process $\Delta(t)$:

$$E\int_0^T \Delta^2(t)dt < \infty.$$

Step 1: Choose a sequence $\Delta_n(t)$ of simple processes such that

$$\lim_{n\to\infty} E\int_0^T |\Delta_n(t) - \Delta(t)|^2 dt = 0.$$

Step 2: For each adopted simple process Δ_n, we define an Ito's integral I_n:

$$I_n := \int_0^T \Delta_n(t)dW(t).$$

Step 3: Define an Ito's integral $I(T)$ as a limit of I_n, i.e.,

$$\int_0^T \Delta(t)dW(t) := \lim_{n \to \infty} \int_0^T \Delta_n(t)dW(t).$$

- Theorem. Ito's integral $I(t) = \int_0^t \Delta(s)dW(s)$ satisfies

1. (Continuity): The sample paths of $I(t)$ are continuous.

2. (Adaptivity): For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.

3. (Linearity): For every constants λ, μ,

$$\int_0^t (\lambda\Delta_1(s)+\mu\Delta_2(s))dW(s) = \lambda \int_0^t \Delta_1(s)dW(s)+\mu \int_0^t \Delta_2(s)dW(s).$$

1. (Martingale): $I(t)$ is a martingale.

2. (Ito's isometry):

$$E(I^2(t)) = E\int_0^t \Delta^2(s)ds.$$

3. (Quadratic variation):

$$[I, I](t) = \int_0^t \Delta^2(s)ds.$$

Ito-Doeblin Formula

Question: We want to differentiate $f(W(t))$, f is a differentiable function and $W(s)$ is a B.M.

- Heuristic explanation.

For one-dimensional case $n = 1$, consider a SDE:

$$\begin{aligned} dX(t) &= A(t)dt + B(t)dW, \\ X(0) &= X_0. \end{aligned}$$

Let $f : R \to R$ and define

$$Y(t) = f(X(t)).$$

- (Wrong answer). If f is differentiable,

$$\begin{aligned}\frac{d}{dt}Y(t) &= f'(X(t))X'(t), \quad \text{or} \\ dY(t) &= f'(X(t))X'(t)dt \\ &= f'(X(t))A(t)dt + f'(X(t)B(t)dW(t).\end{aligned}$$

• Right approach: We use a heuristic principle "$dW = (dt)^{\frac{1}{2}}$" and "Taylor expansion" to find

$$\begin{aligned} dY(t) &= df(X(t)) \\ &= f'(X(t)dX(t) + \frac{1}{2}f''(X(t))dX(t)^2 + \frac{1}{6}f'''(X(t))dX(t)^3 + \cdots \\ &= f'(X(t))\Big(A(t)dt + B(t)dW(t)\Big) \\ &+ \frac{1}{2}f''(X(t))\Big(A(t)dt + B(t)dW(t)\Big)^2 + \cdots \end{aligned}$$

Note that

$$\Big(A(t)dt + B(t)dW(t)\Big)^2$$
$$= A(t)^2 dt^2 + 2A(t)B(t)dtdW(t) + B(t)^2 dW(t)^2$$

This is the Ito-Doeblin's formula in differential form. Integrating this, we also obtain a mathematically meaningful form:

$$
\begin{aligned}
Y(t) - Y(0) &= \underbrace{\int_0^t f'(X(s))B(s)dW(s)}_{\text{Ito's integral}} \\
&+ \underbrace{\int_0^t \Big(f'(X(s))A(s) + \frac{1}{2}B(s)^2 f''(X(s)) \Big) ds}_{\text{Lebesgue Integral}} .
\end{aligned}
$$

- (Higher dimensions)

$$\begin{aligned} dX(t) &= A(t)dt + \sigma dW(t), \quad t \geq 0, \\ X(0) &= x^0, \end{aligned}$$

where $X(t) = (x_1(t), \cdots, x_n(t))^T$.

For $f : R^n \times [0, \infty) \to R$ and $Y(t) = f(X(t), t)$, we have

$$\begin{aligned} dY(t) &= df(X(t), t) \\ &= f_t(X(t), t)dt + \sum_{i=1}^{n} f_{x_i}(X(t), t)dx_i(t) \end{aligned}$$

$$+ \quad \frac{1}{2}\sum_{i,j} f_{x_i x_j}(X(t),t)dx_i(t)dx_j(t),$$

$$dW_i = (dt)^{\frac{1}{2}}, \quad dW_i dW_j = \begin{cases} dt, & i = j, \\ 0, & i \neq j. \end{cases}$$

Hence we have

$$
\begin{aligned}
dY(t) &= f_t(X(t),t)dt + \sum_{i=1}^{n} f_{x_i}(X(t),t)dx_i(t) \\
&+ \frac{1}{2}\sum_{i,j} f_{x_i x_j}(X(t),t)dx_i(t)dx_j(t) \\
&= f_t(X(t),t)dt + \sum_{i=1}^{n} f_{x_i}(X(t),t)(A_i(t)dt + \sigma dW_i(t)) \\
&+ \frac{\sigma^2}{2}\sum_{i} f_{x_i x_i}(X(t),t)dt \\
&= f_t(X(t),t)dt + \nabla_x f(X(t),t)\cdot(A(t)dt + \sigma dW(t)) \\
&+ \frac{\sigma^2}{2}\Delta f(X(t),t)dt.
\end{aligned}
$$

- **Theorem**.

Let $f = f(t,x)$ be a $C^{1,2}_{t,x}$-function, and let $W(t)$ be a B.M. Then for every $T > 0$, we have

$$
\begin{aligned}
f(T, W(T)) &= f(0, W(0)) + \int_0^T f_t(t, W(t))dt \\
&+ \int_0^T f_x(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt.
\end{aligned}
$$

Remark.

$$\int_0^t W(s)dW(s) = ?$$

If W is differentiable, then we might expect

$$\int_0^t W(s)dW(s) = \int_0^t W(s)W'(s)ds = \int_0^t \left(\frac{1}{2}W(s)^2\right)' ds = \frac{1}{2}W(t)^2.$$

Of course, this is not true. We now apply Ito and Doeblin's formula for $f(x) = \frac{1}{2}x^2$ to find

$$\begin{aligned} \frac{1}{2}W^2(T) &= f(W(T)) - f(W(0)) \\ &= \int_0^T f'(W(t))dW(t) + \frac{1}{2}\int_0^t f''(W(t))dt \\ &= \int_0^T W(t)dW(t) + \frac{1}{2}T. \end{aligned}$$

Hence

$$\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{T}{2}.$$

Ito's process

• Definition. Let $W(t)$, $t > 0$ be a Brownian motion, and let $\mathcal{F}(t)$ be an associated filtration. An Ito's process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Delta(s)dW(s) + \int_0^t \Theta(s)ds,$$

where $X(0)$ is nonrandom, and $\Delta(s)$ and $\Theta(s)$ are adapted processes.

• Theorem (Quadratic variation).

$$[X, X](t) = \int_0^t \Delta^2(s)ds.$$

Formal hueristic proof. Rewrite Ito's process in differential form

$$dX(t) = \Delta(t)dW(t) + \Theta(t)dt.$$

Then we use $dW(t)dW(t) = dt, dW(t)dt = dtdt = 0$ to get

$$\begin{aligned} dX(t)dX(t) &= \Delta^2(t)dW(t)dW(t) + 2\Delta(t)\Theta(t)dW(t)dt + \Theta^2(t)dtdt \\ &= \Delta^2(t)dt. \end{aligned}$$

Integral with respect to Ito process

• Definition. Let $X(t), t \geq 0$ be an Ito process, and let $\Gamma(t), t \geq 0$ be an adopted process. Define the integral with respect to Ito's process

$$\int_0^t \Gamma(s) dX(s) = \int_0^t \Gamma(s)\Delta(s) dW(s) + \int_0^t \Gamma(s)\Theta(s) ds.$$

• Theorem.(Ito-Doeblin formula for an Ito's process)

Let $X(t), t \geq 0$ be an Ito process, and let f be a $C^{1,2}_{t,x}$-function. Then for any $T \geq 0$, we have

$$
\begin{aligned}
f(T, X(T)) &= f(0, X(0)) + \int_0^T f_t(t, X(t))dt \\
&+ \int_0^T f_x(t, X(t))dX(t) + \frac{1}{2}\int_0^T f_{xx}(t, X(t))d[X, X](t) \\
&= f(0, X(0)) + \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))\Delta(t)dW(t) \\
&+ \int_0^T f_x(t, X(t))\Theta(t)dt + \frac{1}{2}\int_0^T f_{xx}(t, X(t))\Delta^2(t)dt.
\end{aligned}
$$

Examples for Ito's processes

1. Geometric Brownian Motion.

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t), \qquad \alpha, \sigma\text{: constants}$$

Apply Ito's formula to $\ln S(t)$, i.e.,

$$\begin{aligned} d\ln S(t) &= \frac{dS(t)}{S(t)} - \frac{1}{2S(t)^2}\sigma^2 S^2(t)dt \\ &= \left(\alpha - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t). \end{aligned}$$

We integrate the above equality from 0 to t to get

$$S(t) = S(0)e^{(\alpha - \frac{\sigma^2}{2})t + \sigma W(t)}.$$

2. Generalized geometric Brownian Motion.

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t).$$

As before, we apply Ito's formula to $\ln S(t)$ to find

$$d\ln S(t) = \left(\alpha(t) - \frac{1}{2}\sigma(t)^2\right)dt + \sigma(t)dW(t).$$

Direct integration yields

$$S(t) = S(0)e^{\int_0^t (\alpha(s) - \frac{\sigma(s)^2}{2})ds + \int_0^t \sigma(s)dW(s)}.$$

3. Vasicek interest rate model.

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t).$$

Here α, β and σ are positive constants.

We apply Ito's formula to $e^{\beta t}R(t)$ to get

$$\begin{aligned} d(e^{\beta t}R(t)) &= \beta e^{\beta t}R(t)dt + e^{\beta t}dR(t) \\ &= \alpha e^{\beta t}dt + \sigma e^{\beta t}dW(t). \end{aligned}$$

We now integrate the above relation from 0 to t and find

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma \int_0^t e^{-\beta(t-s)}dW(s).$$

4. Cox-Ingersoll-Ross (CIR) interest rate model.

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t).$$

We apply Ito's formula to $e^{\beta t}R(t)$ to find

$$d(e^{\beta t}R(t)) = \alpha e^{\beta t}dt + \sigma e^{\beta t}\sqrt{R(t)}dW(t).$$

We integrate the above relation to get

$$e^{\beta t}R(t) = R(0) + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma\int_0^t e^{\beta u}\sqrt{R(t)}dW(u).$$

Black-Scholes-Merton equation

- Derivation of B-S-M equation

Please see the separate note.

Connection between SDE and PDE

• Definition. A stochastic differential equation (in short SDE) is an equation of the form

$$\begin{aligned} dX(s) &= \beta(s, X(s))ds + \gamma(s, X(s))dW(s), \quad s \geq t, \\ X(t) &= X_0. \end{aligned}$$

where

$$\beta(s, x): \text{ drift}, \qquad \gamma(s, x): \text{ diffusion}.$$

or equivalently,

$$X(T) = x + \int_t^T \beta(s, X(s))ds + \int_t^T \gamma(s, X(s))dW(s).$$

Consider one-dimensional linear SDE:

$$dX(s) = (a(s) + b(s)X(s))ds + (\gamma(s) + \sigma(s)X(s))dW(s),$$

where a, b, γ, σ are nonrandom function of time s.

• Examples 1. Geometric Brownian motion.

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t).$$

2. Hull-White interest rate model.

$$dR(t) = (a(t) - b(t)R(t))dt + \sigma(t)d\tilde{W}(t).$$

Markov property

Consider SDE:

$$\begin{aligned} dX(s) &= \beta(s, X(s))ds + \gamma(s, X(s))dW(s), \quad s \geq t, \\ X(t) &= X_0. \end{aligned}$$

Let $0 \leq t \leq T$ be given, and let $h(y)$ be a Borel-measurable function. We denote by

$$g(t, x) := E^{t,x}h(X(T)),$$

where $X(T)$ is the solution of SDE with initial data $X(t) = x$.

• Theorem. Let $X(s), s \geq 0$ be a solution to the stochastic differential equation with initial condition given at time 0. Then for $0 \leq t \leq T$,

$$E[h(X(T))|\mathcal{F}(t)] = g(t, X(t)).$$

• Corollary.

Solutions to SDE are Markov process.

Feynman-Kac's formula

- Theorem. Consider the stochastic differential equation

$$dX(s) = \beta(s, X(s))ds + \gamma(s, X(s))dW(s).$$

Let $h(y)$ be a Boreal-measurable function. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

$$g(t, x) = E^{t,x} h(X(T)).$$

Then $g(t, x)$ satisfies the following PDE of parabolic type:

$$g_t(t, x) + \beta(t, x) g_x(t, x) + \frac{1}{2}\gamma^2(t, x) g_{xx}(t, x) = 0,$$

with the terminal condition:

$$g(T, x) = h(x), \quad \text{for all} \quad x.$$

• Lemma. Let $X = X(s)$ be a solution to the SDE:

$$dX(s) = \beta(s, X(s))ds + \gamma(s, X(s))dW(s),$$

with initial condition given at time 0. Let $h(y)$ be a Borel-measurable function. Fix $T > 0$, and let $g = g(t, x)$ be given as before. Then stochastic process

$$g(t, X(t)), \quad 0 \leq t \leq T, \qquad \text{is a martingale.}$$

Outline of proof of Feynman-Kac's formula: Let $X(t)$ be the solution to the SDE starting at time zero. Since $g(t, X(t))$ is a martingale, the net dt in the differential $g(t, X(t))$ must be zero.

$$
\begin{aligned}
dg(t, X(t)) &= g_t dt + g_x dX + \frac{1}{2} g_{xx} dX dX \\
&= g_t dt + \beta g_x dt + \gamma g_x dW + \frac{1}{2}\gamma^2 g_{xx} dt \\
&= \left[g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx} \right] dt + \gamma g_x dW.
\end{aligned}
$$

Hence we have

$$
g_t(t, X(t)) + \beta(t, X(t)) g_x(t, X(t)) + \frac{1}{2}\gamma^2(t, X(t)) g_{xx}(t, X(t)) = 0,
$$

along every path of X. Therefore, we have

$$
g_t(t, x) + \beta(t, x) g_x(t, x) + \frac{1}{2}\gamma^2(t, x) g_{xx}(t, x) = 0.
$$