Which subnormal Toeplitz operators are either normal or analytic?

Raúl E. Curto, In Sung Hwang, and Woo Young Lee

Abstract

We study subnormal Toeplitz operators on the vector-valued Hardy space of the unit circle, along with an appropriate reformulation of P.R. Halmos’s Problem 5: Which subnormal block Toeplitz operators are either normal or analytic? We extend and prove Abrahamse’s Theorem to the case of matrix-valued symbols; that is, we show that every subnormal block Toeplitz operator with bounded type symbol (i.e., a quotient of two bounded analytic functions), whose analytic and co-analytic parts have the “left coprime factorization,” is normal or analytic. We also prove that the left coprime factorization condition is essential. Finally, we examine a well known conjecture, of whether every subnormal Toeplitz operator with finite rank self-commutator is normal or analytic.

1 Introduction

Toeplitz operators arise naturally in several fields of mathematics and in a variety of problems in physics (in particular, in the field of quantum mechanics). On the other hand, the theory of subnormal operators is an extensive and highly developed area, which has made important contributions to a number of problems in functional analysis, operator theory, and mathematical physics. Thus, it becomes of central significance to describe in detail subnormality for Toeplitz operators. This paper focuses on subnormality for block Toeplitz operators and more precisely, the case of block Toeplitz operators with bounded type symbols. Our main result is an appropriate generalization of Abrahamse’s Theorem to the case of matrix-valued symbols; that is, we show that every subnormal block Toeplitz operator with bounded type symbol (i.e., a quotient of two bounded analytic functions), whose analytic and co-analytic parts have the “left coprime factorization,” is normal or analytic.

Naturally, this research is closely related to the study of subnormal operators with finite rank self-commutator, a class that has been extensively researched by many authors. However, until now a complete description of that class has proved elusive. Recently, D. Yakubovich [Ya] has shown that if $S$ is a pure subnormal operator with finite rank self-commutator and admits a normal extension with no nonzero eigenvectors, then $S$ is unitarily equivalent to a block Toeplitz operator with analytic rational normal matrix symbol. A corollary of our main result illustrates, in a certain sense, the case of subnormal Toeplitz operators with finite rank self-commutator.

MSC(2010): Primary 47B20, 47B35, 46J15; Secondary 30H10, 47A57

Keywords: Block Toeplitz operators; subnormal; hyponormal; bounded type functions

The work of the first named author was partially supported by NSF Grant DMS-0801168. The work of the second named author was supported by NRF grant funded by the Korea government(MEST)(2009-0075890). The work of the third author was supported by the Basic Science Research Program through the NRF grant funded by the Korea government(MEST)(2010-0001983).
To describe our results in more detail, we first need to review a few essential facts about (block) Toeplitz operators, and for that we will use [Do1], [Do2], [GGK], and [Ni]. Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be hyponormal if its self-commutator $[T^* T - T T^*]$ is positive (semi-definite), and subnormal if there exists a normal operator $N$ on some Hilbert space $K \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant under $N$ and $N|_\mathcal{H} = T$. Let $\mathbb{T} \equiv \partial \mathbb{D}$ be the unit circle in the complex plane. Let $L^2 \equiv L^2(\mathbb{T})$ be the set of all square-integrable measurable functions on $\mathbb{T}$ and let $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$, that is, $H^\infty$ is the set of bounded analytic functions on $\mathbb{D}$. Given $\phi \in L^\infty$, the Toeplitz operator $T_\phi$ and the Hankel operator $H_\phi$ are defined by
\[ T_\phi g := P(\phi g) \quad \text{and} \quad H_\phi(g) := J \rho(\phi g) \quad (g \in H^2), \]
where $P$ and $P^\perp$ denote the orthogonal projections that map from $L^2$ onto $H^2$ and $(H^2)^\perp$, respectively, and where $J$ denotes the unitary operator on $L^2$ defined by $J(f)(z) = \overline{f(\overline{z})}$.

In the early 1960’s, normal Toeplitz operators were characterized by a property of their symbols $\phi \in L^\infty$ and the hyponormality of $T_\phi$ was understood much later, in 1988, via Cowen’s theorem [Co3].

**Cowen’s theorem.** ([Co3], [NT]) For each $\phi \in L^\infty$, let
\[ \mathcal{E}(\phi) \equiv \{ k \in H^\infty : ||k||_\infty \leq 1 \ \text{and} \ \phi - k \overline{\phi} \in H^\infty \}.
\]
Then $T_\phi$ is hyponormal if and only if $\mathcal{E}(\phi)$ is nonempty.

The elegant and useful theorem of C. Cowen has been used in the works [CuL1], [CuL2], [FL], [Gu1], [Gu2], [GS], [HKL1], [HKL2], [HL1], [HL2], [HL3], [Le], [NT] and [Zhu], which have been devoted to the study of hyponormality for Toeplitz operators on $H^2$. When one studies hyponormality (also, normality and subnormality) of the Toeplitz operator $T_\phi$ one may, without loss of generality, assume that $\phi(0) = 0$; this is because hyponormality is invariant under translation by scalars. We now recall that a function $\phi \in L^\infty$ is said to be of bounded type (or in the Nevanlinna class) if there are analytic functions $\psi_1, \psi_2 \in H^\infty(\mathbb{D})$ such that
\[ \phi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}. \]

It is well known [Ab, Lemma 3] that if $\phi \notin H^\infty$ then $\phi$ is of bounded type $\iff$ $\ker H_\phi \neq \{0\}$. If $\phi \in L^\infty$, we write
\[ \phi_+ \equiv P \phi \in H^2 \quad \text{and} \quad \phi_- \equiv \overline{P^\perp \phi} \in zH^2.\]
Assume now that both $\phi$ and $\overline{\phi}$ are of bounded type. Since $T_\phi H_\phi = H_\phi T_\phi$ for all $\psi \in L^\infty$, it follows from Beurling’s Theorem that $\ker H^-_\overline{\phi} = \theta_0 H^2$ and $\ker H^+_{\overline{\phi}} = \theta_+ H^2$ for some inner functions $\theta_0, \theta_+$. We thus have $b := \overline{\phi} \theta_0 \in H^2$, and hence we can write
\[ \phi_- = \theta_0 \overline{\phi} \quad \text{and similarly } \phi_+ = \theta_+ \overline{\psi} \quad \text{for some } \psi \in H^2. \]

In particular, if $T_\phi$ is hyponormal and $\phi \notin H^\infty$, and since
\[ [T_\phi^*, T_\phi] = H^+_{\overline{\phi}} H^-_{\phi} - H^+_{\phi} H^-_{\overline{\phi}}, \]

it follows that $||H^-_{\overline{\phi}} f|| \geq ||H^-_{\overline{\phi}} f||$ for all $f \in H^2$, and hence
\[ \theta_+ H^2 = \ker H^-_{\overline{\phi}} \subseteq \ker H^-_{\overline{\phi}} = \theta_0 H^2. \]
which implies that \( \theta_0 \) divides \( \theta_+ \), i.e., \( \theta_+ = \theta_0 \theta_1 \) for some inner function \( \theta_1 \). We write, for an inner function \( \theta \),

\[
\mathcal{H}_\theta := H^2 \ominus \theta H^2.
\]

Note that if \( f = \theta \pi \in L^2 \), then \( f \in H^2 \) if and only if \( \pi \in \mathcal{H}_{\theta} \); in particular, if \( f(0) = 0 \) then \( a \in \mathcal{H}_{\theta} \). Thus, if \( \phi = \phi_+ + \phi_- \in L^\infty \) is such that \( \phi \) and \( \phi \) are of bounded type such that \( \phi_+(0) = 0 \) and \( T_\phi \) is hyponormal, then we can write

\[
\phi_+ = \theta_0 \theta_1 \tilde{a} \quad \text{and} \quad \phi_- = \theta_0 \tilde{b}, \quad \text{where} \quad a \in \mathcal{H}_{\theta_0}, \theta_1 \text{ and } b \in \mathcal{H}_{\theta_0}.
\]

By Kronecker’s Lemma [Ni, p. 183], if \( f \in H^\infty \) then \( \overline{f} \) is a rational function if and only if \( \text{rank} H_f < \infty \), which implies that

\[
\overline{f} \text{ is rational } \iff f = \theta \overline{b} \text{ with a finite Blaschke product } \theta.
\]

On the other hand, M. Abrahamse [Ab, Lemma 6] also showed that if \( T_\phi \) is hyponormal, if \( \phi \notin H^\infty \), and if \( \phi \) or \( \overline{\phi} \) is of bounded type then both \( \phi \) and \( \overline{\phi} \) are of bounded type.

We now introduce the notion of block Toeplitz operators. For a Banach space \( X \), let \( L^2_X \equiv L^2_X(\mathbb{T}) \) be the Hilbert space of \( X \)-valued norm square-integrable measurable functions on \( \mathbb{T} \) and let \( H^2_X \equiv H^2_X(\mathbb{T}) \) be the corresponding Hardy space. We observe that \( L^2_X = L^2 \otimes \mathbb{C}^n \) and \( H^2_X = H^2 \otimes \mathbb{C}^n \). If \( \Phi \) is a matrix-valued function in \( L^\infty_M \equiv L^\infty_M(\mathbb{T}) \) \( (= \mathbb{L}^\infty \otimes M_n) \) then \( T_\Phi : H^2_X \to H^2_X \) denotes the block Toeplitz operator with symbol \( \Phi \) defined by

\[
T_\Phi F := P_n(\Phi F) \quad \text{for } F \in H^2_X,
\]

where \( P_n \) is the orthogonal projection of \( L^2_X \) onto \( H^2_X \). A block Hankel operator with symbol \( \Phi \in L^\infty_M \) is the operator \( H_\Phi : H^2_X \to H^2_X \) defined by

\[
H_\Phi F := J_n P_n^\perp (\Phi F) \quad \text{for } F \in H^2_X,
\]

where \( J_n \) denotes the unitary operator from \( (H^2_X) \) to \( H^2_X \) given by \( J_n(F)(z) := 1_n F(z) \) for \( F \in H^2_X \), and where \( I_n \) is the \( n \times n \) identity matrix. If we set \( H^2_X := H^2 \otimes \cdots \otimes H^2 \) then we see that

\[
T_\Phi = \begin{bmatrix} T_{\phi_{11}} & \cdots & T_{\phi_{1n}} \\ \vdots & \ddots & \vdots \\ T_{\phi_{n1}} & \cdots & T_{\phi_{nn}} \end{bmatrix} \quad \text{and} \quad H_\Phi = \begin{bmatrix} H_{\phi_{11}} & \cdots & H_{\phi_{1n}} \\ \vdots & \ddots & \vdots \\ H_{\phi_{n1}} & \cdots & H_{\phi_{nn}} \end{bmatrix},
\]

where

\[
\Phi = \begin{bmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & \ddots & \vdots \\ \phi_{n1} & \cdots & \phi_{nn} \end{bmatrix} \in L^\infty_M.
\]

For \( \Phi \in L^\infty_M \), write

\[
\overline{\Phi}(z) := \Phi^*(z).
\]

A matrix-valued function \( \Theta \in H^\infty_{M_{n \times m}} \) \((= \mathbb{L}^\infty \otimes M_{n \times m}) \) is called inner if \( \Theta(z)^* \Theta(z) = I_m \) for almost all \( z \in \mathbb{T} \). The following basic relations can be easily derived:

\[
\begin{align*}
T_\Phi^* & = T_{\Phi^*}, \quad H_\Phi^* = H_\Phi \quad (\Phi \in L^\infty_M); \\
T_\Phi \Theta - T_\Phi T_\Theta & = H_\Phi^* H_\Phi \quad (\Phi, \Psi \in L^\infty_M); \\
H_\Phi T_\Theta & = H_\Phi \Psi, \quad H_\Psi = T_\Phi^* H_\Phi \quad (\Phi \in L^\infty_M, \Psi \in H^\infty_M); \\
H_\Phi^* H_\Psi - H^*_\Phi H_\Theta & = H^*_\Phi H_\Theta - H^*_\Phi H_\Theta \quad (\Theta \in H^\infty_M \text{ is inner, } \Phi \in L^\infty_M).
\end{align*}
\]

For a matrix-valued function \( \Phi = [\phi_{ij}] \in L^\infty_M \), we say that \( \Phi \) is of bounded type if each entry \( \phi_{ij} \) is of bounded type and that \( \Phi \) is rational if each entry \( \phi_{ij} \) is a rational function. The shift operator \( S \) on \( H^2_X \) is defined by

\[
S := \sum_{j=1}^n \bigoplus T_j.
\]

3
The following fundamental result known as the Beurling-Lax-Halmos Theorem is useful in the sequel.

**The Beurling-Lax-Halmos Theorem.** A subspace $M$ of $H^n_{2C}$ is invariant under the shift operator $S$ on $H^n_{2C}$ if and only if $M = \Theta H^n_{2C}$, where $\Theta$ is an inner matrix function in $H^n_{M \times M}$ ($m \leq n$).

In view of (4), the kernel of a block Hankel operator $H_\Phi$ is an invariant subspace of the shift operator on $H^n_{2C}$. Thus if $\ker H_\Phi \neq \{0\}$ then by the Beurling-Lax-Halmos Theorem,

$$\ker H_\Phi = \Theta H^n_{2C}$$

for some inner matrix function $\Theta$. But we don’t guarantee that $\Theta$ is a square matrix. In fact, as we will refer in the sequel, $\Theta$ is square if and only if $\Phi$ is of bounded type. Recently, Gu, Hendricks and Rutherford [GHR] considered the hyponormality of block Toeplitz operators and characterized the hyponormality of block Toeplitz operators in terms of their symbols. In particular, they showed that if $T_\Phi$ is a hyponormal block Toeplitz operator on $H^n_{2C}$, then $\Phi$ is normal, i.e., $\Phi^* \Phi = \Phi \Phi^*$. Their characterization for hyponormality of block Toeplitz operators resembles the Cowen’s theorem except for an additional condition — the normality condition of the symbol.

**Hyponormality of Block Toeplitz Operators** (Gu-Hendricks-Rutherford [GHR]) For each $\Phi \in L_{M_n}^\infty$, let

$$E(\Phi) := \left\{ K \in H_{M_n}^\infty : ||K||_\infty \leq 1 \text{ and } \Phi - K \Phi^* \in H_{M_n}^\infty \right\}.$$ 

Then $T_\Phi$ is hyponormal if and only if $\Phi$ is normal and $E(\Phi)$ is nonempty.

For a matrix-valued function $\Phi \in H^2_{M \times r}$, we say that $\Delta \in H^2_{M \times m}$ is a left inner divisor of $\Phi$ if $\Delta$ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M \times r}$ ($m \leq n$). We also say that two matrix functions $\Phi \in H^2_{M \times r}$ and $\Psi \in H^2_{M \times m}$ are left coprime if the only common left inner divisor of both $\Phi$ and $\Psi$ is a unitary constant and that $\Phi \in H^2_{M \times r}$ and $\Psi \in H^2_{M \times m}$ are right coprime if $\Phi$ and $\Psi$ are left coprime. Two matrix functions $\Phi$ and $\Psi$ in $H^2_{M_n}$ are said to be coprime if they are both left and right coprime. We remark that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then any left inner divisor $\Delta$ of $\Phi$ is square, i.e., $\Delta \in H^2_{M_n}$. If $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then we say that $\Delta \in H^2_{M_n}$ is a right inner divisor of $\Phi$ if $\Delta$ is a left inner divisor of $\Phi$.

The following lemma will be useful in the sequel.

**Lemma 1.1.** ([GHR]) For $\Phi \in L_{M_n}^\infty$, the following statements are equivalent:

(i) $\Phi$ is of bounded type;
(ii) $\ker H_\Phi = \Theta H^2_{C^n}$, for some square inner matrix function $\Theta$;
(iii) $\Phi = A \Theta^*$, where $A \in H_{M_n}^\infty$ and $A$ and $\Theta$ are right coprime.

For $\Phi \in L_{M_n}^\infty$ we write

$$\Phi_+ := P_n \Phi \in H^2_{M_n} \text{ and } \Phi_- := (P_n^\perp \Phi)^* \in H^2_{M_n}.$$ 

Thus we can write $\Phi = \Phi^* + \Phi_+$. For an inner matrix function $\Theta \in H_{M_n}^\infty$, write

$$H_\Theta := (\Theta H^2_{C^n})^\perp \equiv H^2_{C^n} \ominus \Theta H^2_{C^n}.$$ 

Suppose $\Phi = [\phi_{ij}] \in L_{M_n}^\infty$ is such that $\Phi^*$ is of bounded type. Then we may write $\phi_{ij} = \theta_{ij} \overline{b}_{ij}$, where $\theta_{ij}$ is an inner function and $b_{ij}$ are coprime. Thus if $\theta$ is the least common multiple
of $\theta_{ij}$’s (i.e., the $\theta_{ij}$ divide $\theta$ and if they divide an inner function $\theta'$ then $\theta$ in turn divides $\theta'$), then we can write
\[ \Phi = [\phi_{ij}] = [\theta_{ij}^* \tilde{b}_{ij}] = [\theta a_{ij}] = \Theta A^* \quad (\Theta = \Theta_1 I_n, \ A \in H^2_{M_n}). \tag{5} \]
We note that the representation (5) is “minimal,” in the sense that if $\omega I_n$ ($\omega$ is inner) is a common inner divisor of $\Theta$ and $A$, then $\omega$ is constant. Let $\Phi \equiv \Phi_+ + \Phi_- \in L^\infty_{M_n}$ be such that $\Phi$ and $\Phi^*$ are of bounded type. Then in view of (5) we can write
\[ \Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*, \]
where $\Theta_i = \theta_i I_n$ with an inner function $\theta_i$ for $i = 1, 2$ and $A, B \in H^2_{M_n}$. In particular, if $\Phi \in L^\infty_{M_n}$ is rational then the $\theta_i$ are chosen as finite Blaschke products as we observed in (2).

We would remark that, in (5), by contrast with scalar-valued functions, $\Theta$ and $A$ need not be (right) coprime: indeed, if $\Phi := \left[ \begin{smallmatrix} z & \bar{z} \\ 0 & z \end{smallmatrix} \right]$ then we can write
\[ \Phi = \Theta A^* = \left[ \begin{array}{cc} z & 0 \\ 0 & z \end{array} \right] \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right], \]
but $\Theta := \left[ \begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix} \right]$ and $A := \left[ \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right]$ are not right coprime because $\frac{1}{\sqrt{2}} \left[ \begin{smallmatrix} z & -z \\ 1 & -1 \end{smallmatrix} \right]$ is a common right inner factor, i.e.,
\[ \Theta = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \cdot \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} z & -z \\ 1 & 1 \end{array} \right] \quad \text{and} \quad A = \sqrt{2} \left[ \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right] \cdot \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} z & -z \\ 1 & 1 \end{array} \right]. \tag{6} \]

In this paper we consider the subnormality of block Toeplitz operators and in particular, the block version of Halmos’s Problem 5: Which subnormal block Toeplitz operators are either normal or analytic? In 1976, M. Abrahamse showed that if $\Phi = \overline{\varphi} + f \in L^\infty$ ($f, g \in H^2$) is such that $\overline{\varphi}$ or $\overline{\varphi}$ is of bounded type, if $T_{\overline{\varphi}}$ is hyponormal, and if $\ker [T_{\overline{\varphi}}, T_g]$ is invariant under $T_{\overline{\varphi}}$ then $T_{\overline{\varphi}}$ is normal or analytic. The purpose of this paper is to establish an extension of Abrahamse’s theorem for block Toeplitz operators. In Section 2 we make a brief sketch on Halmos’s Problem 5 and Abrahamse’s theorem. Section 3 is devoted to the proof of the main result. In Section 4 we consider the scalar Toeplitz operators with finite rank self-commutators.

**Acknowledgment.** The authors are deeply indebted to the referee for many helpful comments that helped improve the presentation and mathematical content of the paper.

## 2 Halmos’s Problem 5 and Abrahamse’s theorem

In 1970, P.R. Halmos posed the following problem, listed as Problem 5 in his lectures “Ten problems in Hilbert space” [Hal1], [Hal2]:

> Is every subnormal Toeplitz operator either normal or analytic?

A Toeplitz operator $T_\phi$ is called analytic if $\phi \in H^\infty$. Any analytic Toeplitz operator is easily seen to be subnormal: indeed, $T_{\phi} h = P(\phi h) = \phi h = M_\phi h$ for $h \in H^2$, where $M_\phi$ is the normal operator of multiplication by $\phi$ on $L^2$. The question is natural because the two classes, the normal and analytic Toeplitz operators, are fairly well understood and are subnormal. Halmos’s Problem 5 has been partially answered in the affirmative by many authors (cf. [Ab], [AIW], [CuL1], [CuL2], [NT], and etc). In 1984, Halmos’s Problem 5 was answered in the negative by C. Cowen and J. Long [CoL]: they found an analytic function $\psi$ for which $T_{\psi^{\alpha \beta}}$ (0 < $\alpha$ < 1) is subnormal - in fact, this Toeplitz operator is unitarily equivalent to a subnormal weighted shift $W_\beta$ with weight sequence $\beta \equiv \{ \beta_n \}$, where $\beta_n = (1-\alpha^{2n+2})^{\frac{1}{2}}$ for $n = 0, 1, 2, \ldots$. Unfortunately, Cowen and Long’s construction does not provide an intrinsic connection between subnormality and the theory of Toeplitz operators.
Until now researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. On the other hand, surprisingly, as C. Cowen notes in [Co1] and [Co2], some analytic Toeplitz operators are unitarily equivalent to non-analytic Toeplitz operators; i.e., the analyticity of Toeplitz operators is not invariant under unitary equivalence. In this sense, we might ask whether Cowen and Long’s non-analytic subnormal Toeplitz operator is unitarily equivalent to an analytic Toeplitz operator. To this end, we have:

**Observation.** Cowen and Long’s non-analytic subnormal Toeplitz operator $T_\phi$ is not unitarily equivalent to any analytic Toeplitz operator.

**Proof.** Assume to the contrary that $T_\phi$ is unitarily equivalent to an analytic Toeplitz operator $T_f$. Then by the above remark, $T_f$ is unitarily equivalent to the subnormal weighted shift $W_\beta$ with weight sequence $\beta \equiv \{\beta_n\}$, where $\beta_n = (1 - \alpha^{2n+2})^{1/2}$ $(0 < \alpha < 1)$ for $n = 0, 1, 2, \ldots$; i.e., there exists a unitary operator $V$ such that

$$V^* T_f V = W_\beta.$$  

Thus if $\{e_n\}$ is the canonical orthonormal basis for $\ell^2$ then

$$V^* T_f V e_j = W_\beta e_j = \beta_j e_{j+1} \text{ for } j = 0, 1, 2, \ldots.$$  

We thus have

$$(V^* T_{|f|^2} V) e_j = W_\beta^* W_\beta e_j = \beta_j^2 e_j,$$  

and hence,

$$T_{|f|^2 - \beta_j^2} (V e_j) = 0 \text{ for } j = 0, 1, 2, \ldots.$$  

Fix $j \geq 0$ and observe that $Ve_j \in \ker (T_{|f|^2 - \beta_j^2})$. By Coburn’s Theorem, if $|f|^2 - \beta_j^2$ is nonzero then either $T_{|f|^2 - \beta_j^2}$ or $T_{|f|^2 - \beta_j^2}^*$ is one-one. It follows that $|f|^2 = \beta_j^2$ for $j = 0, 1, 2, \ldots$. This readily implies that $\beta_0 = \beta_1 = \beta_2 = \cdots$, a contradiction.  

Consequently, even if we interpret “is” in Halmos Problem 5 as “is up to unitary equivalence,” the answer to Halmos Problem 5 is still negative.

We would like to reformulate Halmos’s Problem 5 as follows:

**Halmos’s Problem 5 reformulated.** Which Toeplitz operators are subnormal?

The most interesting partial answer to Halmos’s Problem 5 was given by M. Abrahamse [Ab]. M. Abrahamse gave a general sufficient condition for the answer to Halmos’s Problem 5 to be affirmative.

Abrahamse’s theorem can be then stated as:

**Abrahamse’s Theorem** ([Ab, Theorem]). Let $\phi = \overline{\gamma} + f \in L^\infty$ $(f, g \in H^2)$ be such that $\phi$ or $\overline{\phi}$ is of bounded type. If $T_\phi$ is hyponormal and $\ker [T_\phi^*, T_\phi]$ is invariant under $T_\phi$ then $T_\phi$ is normal or analytic.

Consequently, if $\phi = \overline{\gamma} + f \in L^\infty$ $(f, g \in H^2)$ is such that $\phi$ or $\overline{\phi}$ is of bounded type, then every subnormal Toeplitz operator must be normal or analytic.

We say that a block Toeplitz operator $T_\Phi$ is analytic if $\Phi \in H^\infty_{M_{n \times n}}$. Evidently, any analytic block Toeplitz operator with a normal symbol is subnormal because the multiplication operator $M_{\Phi}$ is a normal extension of $T_\Phi$. As a first inquiry in the above reformulation of Halmos’s Problem 5 the following question can be raised:

Is Abrahamse’s Theorem valid for block Toeplitz operators?

In this paper we answer this question in the affirmative (Theorem 3.5).  

6
3 Abrahamse’s Theorem for matrix-valued symbols

Recall the representation (5), and for $\Psi \in L^\infty_{M_n}$ such that $\Psi^*$ is of bounded type, write $\Psi = \Theta_2 B^* = B^* \Theta_2$. Let $\Omega$ be the greatest common left inner divisor of $B$ and $\Theta_2$. Then $B = \Omega B_\ell$ and $\Theta_2 = \Omega \Theta_2$ for some $B_\ell \in H^2_{M_n}$ and some inner matrix $\Theta_2$. Therefore we can write

$$\Psi = B^*_\ell \Omega_2,$$

where $B_\ell$ and $\Omega_2$ are left coprime:

in this case, $B^*_\ell \Omega_2$ is called a left coprime factorization of $\Psi$. Similarly,

$$\Psi = \Delta_2 B^*_r,$$

where $B_r$ and $\Delta_2$ are right coprime:

in this case, $\Delta_2 B^*_r$ is called a right coprime factorization of $\Psi$.

To prove our main result (Theorem 3.5), we need several auxiliary lemmas.

We begin with:

**Lemma 3.1.** Suppose $\Phi = \Phi^* + \Phi_+ \in L^\infty_{M_n}$ is such that $\Phi$ and $\Phi^*$ are of bounded type of the form

$$\Phi_+ = A^* \Theta_1 \quad \text{and} \quad \Phi_- = B^* \Theta_2,$$

where $\Theta_i := \theta_i I_n$ with an inner function $\theta_i$ ($i = 1, 2$). If $T_\Phi$ is hyponormal, then $\Theta_2$ is a right inner divisor of $\Theta_1$.

**Proof.** Suppose $T_\Phi$ is hyponormal. Then there exists a matrix function $K \in H^\infty_{M_n}$ such that $\Phi^* - K \Phi^* \in H^2_{M_n}$. Thus $B \Theta^*_2 - K A \Theta^*_1 = F$ for some $F \in H^2_{M_n}$, which implies that $B \Theta^*_2 \Theta_1 \in H^2_{M_n}$.

Now we write $\Phi_- = \begin{bmatrix} f_{ij} \end{bmatrix}_{n \times n}$. Since $\Phi$ is of bounded type we can write $f_{ij} = \theta_{ij} \tau_{ij}$, where $\theta_{ij}$ is an inner function, $c_{ij}$ is in $H^2$, and $\theta_{ij}$ and $c_{ij}$ are coprime. Write $B = [b_{ij}]_{n \times n}$. We thus have

$$f_{ij} = \theta_{ij} c_{ij} = \theta_{2} \tau_{ij}$$

for each $i, j = 1, \ldots, n$.

which implies that $b_{ij} = \overline{\theta}_{ij} \theta_2 c_{ij}$. But since $B \Theta^*_2 \Theta_1 = [\theta_{ij} \overline{\theta}_{i} b_{ij}] \in H^2_{M_n}$, we have $\theta_{ij} \overline{\theta}_{ij} c_{ij} \in H^2$. Since $\theta_{ij}$ and $c_{ij}$ are coprime for each $i, j = 1, \ldots, n$, it follows that $\overline{\theta}_{ij} \theta_1 \in H^2$, which implies that $\overline{\theta}_2 \theta_1 \in H^2$ and therefore, $\Theta_2$ divides $\Theta_1$, i.e., $\Theta_1 = \Theta_0 \Theta_2$ for some inner matrix function $\Theta_0$. \(\square\)

In the sequel, when we consider the symbol $\Phi = \Phi^* + \Phi_+ \in L^\infty_{M_n}$, which is such that $\Phi$ and $\Phi^*$ are of bounded type and for which $T_\Phi$ is hyponormal, we will, in view of Lemma 3.1, assume that

$$\Phi_+ = A^* \Omega_1 \Omega_2 \quad \text{and} \quad \Phi_- = B^*_r \Omega_2$$

(left coprime factorization),

where $\Omega_1 \Omega_2 = \Theta = \theta I_n$. We also note that $\Omega_2 \Omega_1 = \Theta_2$, indeed, if $\Omega_1 \Omega_2 = \Theta = \theta I_n$, then $(\overline{\theta}_1 \Omega_1) \Omega_2 = I_n$, so that $\Omega_1 (\overline{\theta}_1 \Omega_2) = I_n$, which implies that $\overline{\theta}_1 \Omega_1 = I_n$, and hence $\Omega_2 \Omega_1 = \theta I_n = \Theta$.

**Lemma 3.2.** Suppose $\Phi = \Phi^* + \Phi_+ \in L^\infty_{M_n}$ is such that $\Phi$ and $\Phi^*$ are of bounded type of the form

$$\Phi_+ = \Delta_1 A^*_r \quad \text{(right coprime factorization)} \quad \text{and} \quad \Phi_- = \Delta_2 B^*_r \quad \text{(right coprime factorization)}.$$

If $T_\Phi$ is hyponormal, then $\Delta_2$ is a left inner divisor of $\Delta_1$, i.e., $\Delta_1 = \Delta_2 \Delta_0$ for some $\Delta_0$.

**Proof.** Suppose $T_\Phi$ is hyponormal. Then there exists $K \in H^\infty_{M_n}$ such that $\Phi - K \Phi^* \in H^\infty_{M_n}$. Thus $H_\Phi = H_{K \Phi^*} = T^*_K H_{\Phi^*}$, which implies that $\ker H_{\Phi^*} \subset \ker H_{\Phi}$, so that by Lemma 1.1, $\Delta_1 H^2_{M_n} \subset \Delta_2 H^2_{M_n}$. It follows (cf. [FF, Corollary IX.2.2]) that $\Delta_2$ is a left inner divisor of $\Delta_1$. \(\square\)
On the other hand, the condition “(left/right) coprime factorization” is not so easy to check in general. For example, consider a simple case: \( \Phi_{-} := \begin{bmatrix} z & 1 \\ 0 & z \end{bmatrix} \). One is tempted to write

\[ \Phi_{-} := \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^*. \]

But \( \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \) and \( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) are not right coprime as we have seen in the Introduction. On the other hand, observe that

\[ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \equiv \Delta B^*, \]

where

\[ \Delta := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ -1 & z \end{bmatrix} \] is inner and \( B := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 2z \\ 0 & 2z \end{bmatrix} \).

Again, \( \Delta \) and \( B \) are not right coprime because \( \ker H_{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} = H_{\mathbb{C}^2}^2 \). Thus we might choose

\[ \Phi_{-} = (zI_2 \Delta) \cdot B^* \quad \text{or} \quad \Phi_{-} = \Delta \cdot (zI_2 B)^*. \]

A straightforward calculation show that \( \ker H_{\Phi_{-}} = \Delta H_{\mathbb{C}^2}^2 \). Hence the latter of the above factorizations is the desired factorization: i.e., \( \Delta \) and \( zI_2 \) are right coprime.

However, if \( \Theta \) is given in a form \( \Theta = \theta I_n \) with a finite Blaschke product \( \theta \), then we can obtain a more tractable criterion on the coprime-ness of \( \Theta \) and \( B \in H_{n}^2 \). To see this, recall that an \( n \times n \) matrix-valued function \( D \) is called a finite Blaschke-Potapov product if \( D \) is of the form

\[ D(z) = \nu \prod_{m=1}^{M} \left( b_m(z)P_m + (I - P_m) \right), \]

where \( \nu \) is an \( n \times n \) unitary constant matrix, \( b_m \) is a Blaschke factor, which is of the form

\[ b_m(z) := \frac{z - \alpha_m}{1 - \overline{\alpha_m} z} \quad (\alpha_m \in \mathbb{D}), \]

and \( P_m \) is an orthogonal projection in \( \mathbb{C}^n \). In particular, a scalar-valued function \( D \) reduces to a finite Blaschke product \( D(z) = \nu \prod_{m=1}^{M} b_m(z) \), where \( \nu = e^{i\omega} \). It was known [Po] that an \( n \times n \) matrix-valued function \( D \) is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product.

We write \( \mathcal{Z}(\theta) \) for the set of zeros of an inner function \( \theta \). We then have:

**Lemma 3.3.** Let \( B \in H_{n}^2 \) and \( \Theta := \theta I_n \) with a finite Blaschke product \( \theta \). Then the following statements are equivalent:

(a) \( B(\alpha) \) is invertible for each \( \alpha \in \mathcal{Z}(\theta) \);

(b) \( B \) and \( \Theta \) are right coprime;

(c) \( B \) and \( \Theta \) are left coprime.

**Proof.** We first write

\[ \Theta(z) = e^{i\xi} \prod_{i=1}^{N} \left( \frac{z - \alpha_i}{1 - \overline{\alpha_i} z} \right)^{m_i} \left( \sum_{i=1}^{N} m_i := d \right). \]

(a) \( \iff \) (b): Suppose \( B(\alpha) \) is invertible for each \( \alpha \in \mathcal{Z}(\theta) \). Assume to the contrary that \( B \) and \( \Theta \) are not right coprime. Thus there exists a finite Blaschke-Potapov product \( D \) of the form

\[ D(z) = \nu \prod_{m=1}^{M} \left( b_m(z)P_m + (I - P_m) \right). \]
satisfying that $B = B_1D$ and $\Theta = \Theta_0D$ for some inner function $\Theta_0$.

Thus if $\alpha \in \mathcal{Z}(b_{m_0})$ for some $1 \leq m_0 \leq M$, then $\Theta(\alpha) = \Theta_0(\alpha)D(\alpha)$ is not invertible. But since $\Theta = \theta I_n$, it follows that $\Theta(\alpha) = 0$ and hence $\alpha \in \mathcal{Z}(\theta)$. Moreover,

$$\det B(\alpha) = \det B_1(\alpha) \det D(\alpha) = \det(\nu) \det B_1(\alpha) \prod_{m=1}^M \det \left( b_m(\alpha)P_m + (I - P_m) \right) = 0,$$

giving a contradiction. Therefore $B$ and $\Theta$ are right coprime.

For the converse we assume that $B(\alpha_{i_0})$ is not invertible for some $i_0$. Then the following matrix is not invertible:

$$B := \begin{bmatrix}
    B_0 & 0 & 0 & 0 & \cdots & 0 \\
    B_1 & B_0 & 0 & 0 & \cdots & 0 \\
    B_2 & B_1 & B_0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    B_{m_{i_0}-2} & B_{m_{i_0}-3} & \cdots & \cdots & \cdots & B_0 \\
    B_{m_{i_0}-1} & B_{m_{i_0}-2} & \cdots & B_2 & B_1 & B_0
\end{bmatrix} \begin{bmatrix}
    (B_j)_{(\alpha_{i_0})} \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{bmatrix} = \begin{bmatrix}
    B_0 & 0 & 0 & 0 & \cdots & 0 \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{bmatrix}.$$

Thus there exists a nonzero $n \times m_{i_0}$ matrix $\mathbf{G} = \left[ \mathbf{G}_1 \mathbf{G}_2 \cdots \mathbf{G}_{m_{i_0}-1} \right]$ such that $B \mathbf{G} = 0$. We now want to show that there exists $\mathbf{h} = (h_1 \ h_2 \ \cdots \ h_n)^t \in H^2_{C^\alpha}$ satisfying the following property:

$$\mathbf{h}^{(j)}(\alpha_i) = \begin{cases}
    \mathbf{G}_j & (i = i_0) \\
    0 & (i \neq i_0)
\end{cases} \quad (7)$$

This is exactly the classical Hermite-Fejér interpolation problem (cf. [FF]), so that we use an argument of a solution for the interpolation of this type. Thus we can construct a function (in fact, a polynomial) $\mathbf{h}(z) \equiv P(z)$ satisfying (7) (see [FF, p.299]). Then $P(z)$ belongs to $\ker H_{B\Theta^*}$. Since

$$\mathbf{G} = \left[ \mathbf{G}_0 \mathbf{G}_1 \cdots \mathbf{G}_{m_{i_0}-1} \right] \neq 0,$$

it follows that $P(z) \notin \Theta H^2_{C^\alpha}$. Therefore we have $\ker H_{B\Theta^*} \neq \Theta H^2_{C^\alpha}$, which implies that $B$ and $\Theta$ are not right coprime.

(b) ⇔ (c): Suppose $B$ and $\Theta$ are right coprime. If $B$ and $\Theta$ are not left coprime, there exists a nonconstant inner matrix $\Delta \in H^2_{M_{n}}$ such that $B = \Delta B_1$ and $\Theta = \Delta \Omega$. We thus have that for each $i = 1, 2, \cdots, N$

$$\begin{bmatrix}
    \Delta_{i,0} & 0 & 0 & 0 & \cdots & 0 \\
    \Delta_{i,1} & \Delta_{i,0} & 0 & 0 & \cdots & 0 \\
    \Delta_{i,2} & \Delta_{i,1} & \Delta_{i,0} & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    \Delta_{i,m_i-2} & \Delta_{i,m_i-3} & \cdots & \cdots & \cdots & \Delta_{i,0} \\
    \Delta_{i,m_i-1} & \Delta_{i,m_i-2} & \cdots & \Delta_{i,2} & \Delta_{i,1} & \Delta_{i,0}
\end{bmatrix} = 0,$$

where

$$\Delta_{i,j} := \frac{\Delta^{(j)}(\alpha_i)}{j!} \quad \text{and} \quad \Omega_{i,j} := \frac{\Omega^{(j)}(\alpha_i)}{j!}.$$

But since $B(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$, we have that $\Delta_{i,0}$ is invertible for each $i = 1, 2, \cdots, N$. Thus

$$\Omega_{i,j} = 0 \quad (i = 1, 2, \cdots, N, \ j = 0, 1, 2, \cdots, m_i - 1),$$

which implies that $\Omega = \Theta \Omega_1$ for some $\Omega_1 \in H^2_{M_n}$. Thus $\Theta = \Delta \Omega = \Delta \Theta \Omega_1$, so that $I = \Delta \Omega_1$ and hence $\Delta^* = \Omega_1$, which implies that $\Delta$ is a constant matrix, a contradiction. Thus $B$ and $\Theta$ are left coprime. The converse follows from the same argument. This completes the proof. ☐
Lemma 3.4. Let $\theta_0$ be a nonconstant inner function. Then $\mathcal{H}_{\theta_0}$ contains an outer function that is invertible in $H^\infty$.

Proof. If $\theta_0$ has at least one Blaschke factor, say $\frac{\omega - \alpha}{1 - \overline{\alpha} \omega}$ ($|\alpha| < 1$), then $1 - \frac{1}{1 - \overline{\alpha} \omega}$ is an outer function and $\frac{1}{1 - \overline{\alpha} \omega} \in \mathcal{H}_{\theta_0}$ because $\frac{1}{1 - \overline{\alpha} \omega}$ is the reproducing kernel for $\alpha$, so that for any $f \in H^2$, 
\[
\left< \theta_0 f, \frac{1}{1 - \overline{\alpha} \omega} \right> = \theta_0(\alpha)f(\alpha) = 0.
\]

Now suppose $\theta_0$ is a nonconstant singular inner function of the form 
\[
\theta_0(z) := \exp \left( -\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(\theta) \right),
\]
where $\mu$ is a finite positive Borel measure on $\mathbb{T}$ which is singular with respect to Lebesgue measure. We put 
\[
\omega(z) := \exp \left( -\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \frac{d\mu(\theta)}{2} \right).
\]
Then $\omega^2 = \theta_0$. If $\alpha := \overline{\omega}(0)$ then evidently, $0 < |\alpha| < 1$ since $\omega$ is not constant. Note that $\overline{\omega} \left( \omega - \frac{1}{\alpha} \right) = \overline{\omega} - \frac{1}{\alpha} \overline{\omega_0} \in (H^2)^+$, since $\overline{\omega} - \frac{1}{\alpha} \overline{\omega_0}(0) = \alpha - \frac{1}{\alpha} \alpha^2 = 0$. We thus have $\omega - \frac{1}{\alpha} \in \mathcal{H}_{\theta_0}$. Also a straightforward calculation shows that $\frac{1}{\omega - \frac{1}{\alpha}}$ is bounded and analytic in $D$, which says that $\omega - \frac{1}{\alpha}$ is invertible in $H^\infty$. Hence $\omega - \frac{1}{\alpha}$ is an outer function in $\mathcal{H}_{\theta_0}$. This completes the proof. $\square$

Before proving the main result, we recall the inner-outer factorization of vector-valued functions. If $D$ and $E$ are Hilbert spaces and if $F$ is a function with values in $\mathcal{B}(E, D)$ such that $F(\cdot)e \in H^2_D(\mathbb{T})$ for each $e \in E$, then $F$ is called a strong $H^2$-function. The strong $H^2$-function $F$ is called a (strong) inner function if $F(\cdot)$ is a unitary operator from $D$ into $E$. Write $\mathcal{P}_E$ for the set of all polynomials with values in $E$, i.e., $p(\zeta) = \sum_{k=0}^{n} \hat{p}(k)\zeta^k$, $\hat{p}(k) \in E$. Then the function $Fp = \sum_{k=0}^{n} F\hat{p}(k)z^k$ belongs to $H^2_D(\mathbb{T})$. The strong $H^2$-function $F$ is called outer if 
\[
\text{cl } F : \mathcal{P}_E = H^2_D(\mathbb{T}).
\]
Note that if $\dim D = \dim E = n < \infty$, then evidently, every $F \in H^2_D$ is a strong $H^2$-function. We then have an analogue of the scalar Inner-Outer Factorization Theorem.

Inner-Outer Factorization. (cf. [Ni]) Every strong $H^2$-function $F$ with values in $\mathcal{B}(E, D)$ can be expressed in the form 
\[
F = F^i F^c,
\]
where $F^c$ is an outer function with values in $\mathcal{B}(E, D')$ and $F^i$ is an inner function with values in $\mathcal{B}(D', D)$ for some Hilbert space $D'$.

We are now ready to prove the main result of this paper.

Theorem 3.5. (Abrahamse’s Theorem for Matrix-Valued Symbols) Suppose $\Phi := \Phi^* + \Phi_+ \in L^\infty_{\mathcal{M}_n}$ is such that $\Phi$ and $\Phi^*$ are of bounded type. In view of Lemma 3.1, we may write 
\[
\Phi_+ = A^* \Theta_0 \Theta_2 \quad \text{and} \quad \Phi_- = B^* \Theta_2,
\]
where $\Theta_i = \theta_i I_n$ with an inner function $\theta_i$ ($i = 0, 2$) and $A, B \in H^2_{\mathcal{M}_n}$. Assume that $A, B$ and $\Theta_2$ are left coprime. If 
(i) $T\Phi$ is hyponormal; and 
(ii) $\text{ker } [T^*_\Phi, T\Phi]$ is invariant under $T\Phi$
then $T\Phi$ is normal or analytic. Hence, in particular, if $T\Phi$ is subnormal then it is normal or analytic.
Remark 3.6. We note that if $n = 1$ (i.e., $T_\Phi$ is a scalar Toeplitz operator) then $\Phi_+ = \overline{\theta_0}\theta_2$ and $\Phi_- = b\theta_2$ with $a, b \in H^2$. Thus, we can always arrange that $a, b$ and $\theta$ are coprime. Consequently, if $n = 1$ then our matrix version reduces to the original Abrahamse’s Theorem.

Proof. If $\Theta_2$ is constant then $\Phi_- = 0$, so that $T_\Phi$ is analytic. Suppose that $\Theta_2$ is nonconstant.

We split the proof into three steps.

STEP 1: We first claim that

$$\Theta_0 H_{\mathbb{C}^n}^2 \subseteq \ker [T_\Phi^*, T_\Phi].$$

(8)

To see this, we observe that

$$[T_\Phi^*, T_\Phi] = H_{\Phi_+^*}^* H_{\Phi_+} - H_{\Phi_-^*}^* H_{\Phi_-} = H_{A_0 \theta_2^* \theta_2^*}^* H_{A_0 \theta_2^* \theta_2^*} + H_{\Phi_2}^* B H_{\theta_2^*}^* B,$$

(9)

which implies that

$$\Theta_0 \Theta_2 H_{\mathbb{C}^n}^2 \subseteq \ker [T_\Phi^*, T_\Phi].$$

(10)

On the other hand, since $\Theta_0 \Theta_2$ is diagonal, we have that for all $g \in \mathcal{P}_{\mathbb{C}^n},$

$$T_\Phi(\Theta_0 \Theta_2 g) = P_{\mathcal{H}_{\theta_2}(\Theta_0 \Theta_2)}(\Theta_0 B g + \hat{a}_0, \theta_0 \Theta_2 g)$$

$$= \Theta_0 B g + \Theta_0 \Theta_2 \Phi_+ g$$

$$= P_{\mathcal{H}_{\theta_2}(\Theta_0 \Theta_2)}(\Theta_0 B g) + P_{\mathcal{H}_{\theta_2}(\Theta_0 \Theta_2)}(\Theta_0 B g) + \Theta_0 \Theta_2 \Phi_+ g.$$ Since $\mathcal{H}_{\theta_22} = \mathcal{H}_{\theta_2} \oplus \Theta_0 \mathcal{H}_{\theta_2},$ it follows that

$$P_{\mathcal{H}_{\theta_2}(\Theta_0 B g)} = P_{\mathcal{H}_{\theta_2}(\Theta_0 B g)}.$$ We thus have

$$T_\Phi(\Theta_0 \Theta_2 g) = P_{\mathcal{H}_{\theta_2}(\Theta_0 B g)} + P_{\mathcal{H}_{\theta_2}(\Theta_0 \Theta_2)}(\Theta_0 B g) + \Theta_0 \Theta_2 \Phi_+ g.$$ (11)

We claim that

$$\mathcal{H}_{\theta_2} = \text{cl} \left\{ P_{\mathcal{H}_{\theta_2}}(B g) : g \in \mathcal{P}_{\mathbb{C}^n} \right\}.$$ (12)

In view of the above mentioned Inner-Outer Factorization, let $B = B^* B^*$ be the inner-outer factorization of $B$ (as a strong $H^2$-function), where $B^* \in H_{M_{ax}, r}^\infty$ and $B^* \in H_{M_{ax}, r}^2$. Since $B$ and $\Theta_2$ are left coprime, $B^*$ and $\Theta_2$ are left coprime. Thus it follows from the Beurling-Lax-Halmos Theorem that

$$\Theta_2 H_{\mathbb{C}^n}^2 \bigcap \text{cl} \mathcal{P}_{\mathbb{C}^n} = \Theta_2 H_{\mathbb{C}^n}^2 \bigcap B^*(\text{cl} \mathcal{P}_{\mathbb{C}^n}) = \Theta_2 H_{\mathbb{C}^n}^2 \bigcap B^* H_{\mathbb{C}^n}^2 = H_{\mathbb{C}^n}^2,$$

given (12). Thus we have

$$\Theta_0 \mathcal{H}_{\theta_2} = \text{cl} \left\{ P_{\mathcal{H}_{\theta_2}}(B g) : g \in \mathcal{P}_{\mathbb{C}^n} \right\} = \text{cl} \left\{ P_{\mathcal{H}_{\theta_2}(\Theta_0 B g)} : g \in \mathcal{P}_{\mathbb{C}^n} \right\}.$$ (13)

If $\ker [T_\Phi^*, T_\Phi]$ is invariant under $T_\Phi$ then since $\ker [T_\Phi^*, T_\Phi]$ is a closed subspace it follows from (10) - (13) that

$$\Theta_0 \mathcal{H}_{\theta_2} \subseteq \ker [T_\Phi^*, T_\Phi].$$

We thus have

$$\Theta_0 H_{\mathbb{C}^n}^2 = \Theta_0 \mathcal{H}_{\theta_2} \oplus \Theta_0 \Theta_2 H_{\mathbb{C}^n}^2 \subseteq \ker [T_\Phi^*, T_\Phi],$$

which proves (8).

STEP 2: We next claim that

$$\mathcal{E}(\Phi) \text{ contains an inner function } K.$$ (14)
To see this, we first observe that if $K \in \mathcal{E}(\Phi)$ then by (4),

$$
[T^*_\Phi, T_\Phi] = H^*_{\Phi^+} H_{\Phi^+} - H_{K^{\Phi^+}} H_{K^{\Phi^+}} = H^*_{\Phi^+} (I - T^-_K T^*_K) H_{\Phi^+},
$$

so that

$$
\ker [T^*_\Phi, T_\Phi] = \ker (I - T^-_K T^*_K) H_{\Phi^+}.
$$

Thus by (8),

$$
\{0\} = (I - T^-_K T^*_K) H_{A\Theta_2^* \Theta_2^*} = (I - T^-_K T^*_K) H_{A\Theta_2^*} = (I - T^-_K T^*_K) H_{\Theta_2^*},
$$

which implies

$$
\text{cl ran } H_{A\Theta_2^*} \subseteq \ker (I - T^-_K T^*_K).
$$

Since by assumption, $A$ and $\Theta_2$ are left coprime, and hence $\tilde{A}$ and $\tilde{\Theta}_2$ are are right coprime, it follows from Lemma 1.1 that

$$
\text{cl ran } H_{\Theta_2} = \left(\ker H_{\tilde{A}\tilde{\Theta}_2^*}\right)^\perp = \left(\tilde{\Theta}_2 H_{\tilde{C}_n}\right)^\perp = H_{\tilde{\Theta}_2},
$$

which together with (16) implies

$$
H_{\tilde{\Theta}_2} \subseteq \ker (I - T^-_K T^*_K).
$$

We thus have

$$
F = T^-_K T^*_K F \quad \text{for each } F \in H_{\tilde{\Theta}_2}. \quad (19)
$$

But since $\|\tilde{K}\|_\infty = \|K\|_\infty \leq 1$, we have

$$
\|P_n(\tilde{K}^* F)\|_2 \leq \|\tilde{K}^* F\|_2 \leq \|F\|_2 \leq \|T^-_K T^*_K F\|_2 = \|\tilde{K} P_n(\tilde{K}^* F)\|_2 \leq \|P_n(\tilde{K}^* F)\|_2,
$$

which gives

$$
\|P_n(\tilde{K}^* F)\|_2 = \|\tilde{K}^* F\|_2,
$$

which implies $\tilde{K}^* F \in H^2_{\tilde{C}_n}$. Therefore by (19), we have

$$
F = \tilde{K} \tilde{K}^* F \quad \text{for each } F \in H_{\tilde{\Theta}_2}.
$$

In view of Lemma 3.4, we can choose an outer function $f \in H_{\tilde{\Theta}_2}$ that is invertible in $H^\infty$. For each $j = 1, 2, \cdots, n$, we define

$$
F_j := (0, \cdots, 0, f, 0, \cdots, 0)^t \quad \text{(where $f$ is the $j$-th component)}.
$$

Then $F_j \in H_{\tilde{\Theta}_2}$ for each $j = 1, 2, \cdots, n$, so that $(I - \tilde{K} \tilde{K}^*) F_j = 0$ for each $j = 1, 2, \cdots, n$. If we write $I - \tilde{K} \tilde{K}^* \equiv [q_{ij}]_{1 \leq i, j \leq n} \in L^\infty_{\Delta_n}$, then $q_{ij} f = 0$ for each $i, j = 1, 2, \cdots, n$, so that $q_{ij} = 0$ for each $i, j = 1, 2, \cdots, n$, because $f$ is invertible. Therefore we have $K^* K = \tilde{I} = I$, which implies that $K$ is an inner function. This proves (14).

STEP 3: Now since $K$ is inner it follows from (3) that

$$
I - T^-_K T^*_K = T^-_K \tilde{K} - T^-_K T^*_K = H^*_{\tilde{K}} H_{\tilde{K}}.
$$

Thus by (18), we have

$$
H_{\tilde{\Theta}_2} \subseteq \ker H^*_{\tilde{K}^*} H_{\tilde{K}} = \ker H_{\tilde{K}}.
$$

Write $K := [k_{ij}]_{1 \leq i, j \leq n} \in H^\infty_{\Delta_n}$. Since by Lemma 3.4, $H_{\tilde{\Theta}_2}$ contains an outer function $h$ which is invertible in $H^\infty$, it follows from (20) that

$$
k_{ij}(\overline{\tau}) h \in H^2 \quad \text{for each } i, j = 1, 2, \cdots, n,
$$

so that $k_{ij}(\overline{\tau}) \in \frac{1}{2} H^2 \subseteq H^2$ for each $i, j = 1, 2, \cdots, n$. Therefore each $k_{ij}$ is constant and hence, $K$ is constant. Therefore by (15), $[T^*_\Phi, T_\Phi] = 0$, i.e., $T_\Phi$ is normal. This completes the proof.
Remark 3.7. Theorem 3.5 may fail if the condition “$B$ and $\Theta_2$ are left coprime” is dropped even though $A$ and $\Theta_2$ are left coprime. To see this, let $\theta$ be a nonconstant finite Blaschke product. Consider the matrix-valued function

$$
\Phi = \begin{bmatrix} 2\theta + \overline{\theta} & \overline{\theta} \\ \theta & 2\theta + \overline{\theta} \end{bmatrix}.
$$

Write

$$
\Theta := \begin{bmatrix} \theta & 0 \\
0 & \theta \end{bmatrix}.
$$

Then

$$
\Phi_+ = 2\Theta \quad \text{and} \quad \Phi_- = \begin{bmatrix} \theta & \theta \\
\theta & \theta \end{bmatrix}^* \Theta.
$$

A direct calculation shows that $\Phi$ is normal. Put $K := \begin{bmatrix} 1 & 1 \\
1 & 1 \end{bmatrix}$. Remember that for a matrix-valued function $A$, we define $||A||_\infty := \sup_{t \in \mathbb{T}} ||A(t)||$ (where $||\cdot||$ means the operator norm). Then $||K||_\infty = 1$ and $\Phi - K\Phi^* \in H_\infty^{2,2}$, so that $T_\Phi$ is hyponormal. Observe that

$$
\begin{bmatrix} T_\Phi^*, T_\Phi \end{bmatrix} = H_{\Phi_+}^* H_{\Phi_-} - H_{\Phi_+}^* H_{\Phi_-}^*
$$

$$
= 4 \begin{bmatrix} H_\Phi^* H_\Phi & 0 \\
0 & H_\Phi^* H_\Phi \end{bmatrix} - 2 \begin{bmatrix} H_\Phi^* H_\Phi & H_\Phi^* H_\Phi \\
H_\Phi^* H_\Phi & H_\Phi^* H_\Phi \end{bmatrix}
$$

$$
= 2 \begin{bmatrix} H_\Phi^* H_\Phi & -H_\Phi^* H_\Phi \\
-H_\Phi^* H_\Phi & H_\Phi^* H_\Phi \end{bmatrix}
$$

$$
= 2 \begin{bmatrix} P_{H_\Phi} & -P_{H_\Phi} \\
-P_{H_\Phi} & P_{H_\Phi} \end{bmatrix},
$$

which gives

$$
\ker \begin{bmatrix} T_\Phi^* & T_\Phi \end{bmatrix} = \ker \begin{bmatrix} P_{H_\Phi} & -P_{H_\Phi} \\
-P_{H_\Phi} & P_{H_\Phi} \end{bmatrix} = \left\{ \begin{bmatrix} f \\
g \end{bmatrix} : P_{H_\Phi} f = P_{H_\Phi} g \right\}
$$

$$
= \Theta H_{\Theta}^{2,2} \oplus \left\{ f \oplus f : f \in H_\Theta \right\}.
$$

We now claim that $\ker \begin{bmatrix} T_\Phi^* & T_\Phi \end{bmatrix}$ is invariant under $T_\Phi$. To show this we suppose

$$
F = \begin{bmatrix} f \\
g \end{bmatrix} \in \ker \begin{bmatrix} T_\Phi^* & T_\Phi \end{bmatrix}.
$$

Then

$$
T_\Phi F = \begin{bmatrix} 2T_\theta + T_\overline{\theta} & T_\overline{\theta} \\
T_\theta + 2T_\overline{\theta} & T_\theta 
\end{bmatrix} \begin{bmatrix} f \\
g \end{bmatrix} = \begin{bmatrix} 2T_\theta f + T_\overline{\theta}(f + g) \\
2T_\theta g + T_\overline{\theta}(f + g) \end{bmatrix}.
$$

Observe that

$$
P_{H_\Theta}(2T_\theta f + T_\overline{\theta}(f + g)) = P_{H_\Theta}^*(f + g) = P_{H_\Theta}(2T_\theta g + T_\overline{\theta}(f + g)),
$$

which implies that $\ker \begin{bmatrix} T_\Phi^* & T_\Phi \end{bmatrix}$ is invariant under $T_\Phi$. But since

$$
\left\{ f \oplus f : f \in H_\Theta \right\} \subseteq H_{\Theta^2} , \quad \text{and hence, } \ker \begin{bmatrix} T_\Phi^* & T_\Phi \end{bmatrix} \neq H_{\Theta^2}^{2,2},
$$

we can see that $T_\Phi$ is not normal. Note that by Lemma 3.3, $[1 \ 1]$ and $\Theta$ are not left coprime.

If, in the left coprime factorization $\Phi_- = B^*\Theta_2 (\Theta_2 = \theta_2 I_n)$ of Theorem 3.5, $\theta_2$ has a Blaschke factor, then the assumption of the “left coprime factorization” for the analytic part $\Phi_+$ of $\Phi$ can be dropped in Theorem 3.5.
Corollary 3.8. Suppose \( \Phi = \Phi^- + \Phi^+ \in L^\infty_{M_n} \) is such that \( \Phi \) and \( \Phi^- \) are of bounded type. In view of (5), we may write
\[
\Phi^- = B^* \Theta,
\]
where \( \Theta := \theta I_n \) with an inner function \( \theta \). Assume that \( B \) and \( \Theta \) are left coprime. Assume also that \( \theta \) contains a Blaschke factor. If
(i) \( T_\Phi \) is hyponormal; and
(ii) \( \ker [T_\Phi^*, T_\Phi] \) is invariant under \( T_\Phi \)
then \( T_\Phi \) is normal or analytic. Hence, in particular, if \( T_\Phi \) is subnormal then it is normal or analytic.

Proof. For notational convenience, we let \( \Theta_2 := \Theta \) and \( \theta_2 := \theta \). Now suppose \( \theta_2 \) has a Blaschke factor \( b_\alpha \) and write \( B_\alpha := b_\alpha I_n \). By assumption, \( B \) and \( B_\alpha \) are left coprime, so that by Lemma 3.3, \( B \) and \( B_\alpha \) are right coprime. Thus, in view of Lemmas 3.1 and 3.2, we can write
\[
\Phi^+ = A^* \Theta_1 \Theta_2 = B_\alpha \Delta_1 A_r^* \quad \text{and} \quad \Phi^- = B^* \Theta_2,
\]
where \( \Theta_i := \theta_i I_n \) with an inner function \( \theta_i \) \((i = 0, 2)\), \( A_r \) and \( B_\alpha \Delta_1 \) are right coprime, and \( B \) and \( \Theta_2 \) are left coprime. In particular, we note that \( A \) and \( B_\alpha \) are right coprime, so that again by Lemma 3.3, \( A \) and \( B_\alpha \) are left coprime. On the other hand, an analysis for the proof of STEP 1 of Theorem 3.5 shows that \( \Theta_0 H^2_{C^n} \subseteq \ker [T_\Phi^*, T_\Phi] \). (21)
(Note that we didn’t employ the assumption “\( A \) and \( \Theta_2 \) are left coprime” to get (21) in the proof of STEP 1 of Theorem 3.5.) Thus if \( K \in E(\Phi) \) then by the same argument as (16), we have
\[
\text{cl ran } H A \Theta_2 \subseteq \ker (I - T_K T_K^*).
\]
(22)
Since \( A \) and \( B_\alpha \) is left coprime it follows that \( \tilde{A} \) and \( \tilde{B}_\alpha \) are right coprime. Thus we can write
\[
\tilde{A} \tilde{\Theta}_2 = \tilde{A}_1 \tilde{\Omega} \tilde{B}_\alpha \quad \left( \text{right coprime factorization} \right)
\]
for some inner function \( \Omega \) and \( A_1 \in H^2_{M_n} \). It thus follows from Lemma 1.1 that
\[
\text{cl ran } H A \Theta_2 \left( \ker H A \Theta_2 \right)^\perp = \left( \tilde{B}_\alpha \tilde{\Omega} H^2_{C^n} \right)^\perp \supseteq \left( \tilde{B}_\alpha H^2_{C^n} \right)^\perp \Rightarrow \text{H} \tilde{B}_\alpha.
\]
Thus by (22),
\[
\text{H} \tilde{B}_\alpha \subseteq \ker (I - T_K T_K^*).
\]
Then the exactly same argument as the argument from (18) to the end of the proof of Theorem 3.5 with \( B_\alpha \) in place of \( \Theta_2 \) shows that \( T_\Phi \) is normal. (We again note that we didn’t employ the assumption “\( A \) and \( \Theta_2 \) are left coprime” there.) This completes the proof.

We thus have:

Corollary 3.9. Suppose \( \Phi = \Phi^- + \Phi^+ \in L^\infty_{M_n} \) is a matrix-valued rational function. In view of (5) and (2), we may write
\[
\Phi^- = B^* \Theta,
\]
where \( \Theta := \theta I_n \) with a finite Blaschke product \( \theta \). Assume that \( B(\alpha) \) is invertible for each \( \alpha \in Z(\theta) \). If \( T_\Phi \) is subnormal then \( T_\Phi \) is normal or analytic.

Proof. This follows at once from Corollary 3.8 together with Lemma 3.3.
4 Scalar Toeplitz operators with finite rank self-commutators

If $\Phi$ is normal and analytic then $[T_\Phi^*, T_\Phi] = H_\Phi^2$, so that by the Kronecker’s Lemma, $T_\Phi$ has a finite rank self-commutator if and only if $\Phi$ is rational. Therefore Corollary 3.6 illustrates the case of subnormal Toeplitz operators with finite rank self-commutators. But it is still open whether subnormal (even scalar-valued) Toeplitz operators with finite rank self-commutators are either normal or analytic. We would like to state:

**Conjecture 4.1.** If $T_\Phi$ is a subnormal Toeplitz operator with finite rank self-commutator, then $T_\Phi$ is normal or analytic.

We need not expect that if $T_\Phi$ is a hyponormal Toeplitz operator with finite rank self-commutator then $\phi$ is of bounded type. Indeed, if $\psi$ is not of bounded type and $\phi = \overline{\psi} + z\psi$ (and hence $\phi$ is not of bounded type) then a straightforward calculation shows that $T_\phi$ is hyponormal and rank $[T_\phi^*, T_\phi] = 1$.

We would like to take this opportunity to give a positive evidence for Conjecture 4.1. First of all, we recall a theorem of Nakazi and Takahashi [NT, Theorem 10] which states that if $T_\phi$ is hyponormal then $[T_\phi^*, T_\phi]$ is of finite rank if and only if there exists a finite Blaschke product $b$ in $\mathcal{E}(\phi)$ such that the degree of $b$ equals the rank of $[T_\phi^*, T_\phi]$. In what follows we let $b\mathcal{M} := \{bf : f \in \mathcal{M}\}$.

**Theorem 4.2.** Suppose $T_\phi$ is a hyponormal Toeplitz operator with finite rank self-commutator. If $\ker [T_\phi^*, T_\phi]$ and $b\ker [T_\phi^*, T_\phi]$ (some $b \in \mathcal{E}(\phi)$) are invariant under $T_\phi$, then $T_\phi$ is normal or analytic.

**Proof.** Write $K := \ker [T_\phi^*, T_\phi]$ and $R := \text{ran}[T_\phi^*, T_\phi]$. If $\phi$ or $\overline{\phi}$ is of bounded type then by Abrahamse’s Theorem, $T_\phi$ is either normal or analytic. Suppose both $\phi$ and $\overline{\phi}$ are not of bounded type. We first claim that

$$\text{cl} \ H_\phi(\ker [T_\phi^*, T_\phi]) = H^2. \quad (23)$$

To see this we observe that by the Nakazi-Takahashi Theorem, there exists a finite Blaschke product $b \in \mathcal{E}(\phi)$ such that deg $(b) = \dim R$. Since

$$T_\phi^*T_\phi - T_\phi T_\phi^* = H_\phi^2H_\phi - H_\phi H_\phi^* = H_\phi^2H_\phi - H_\phi^2H_\phi = H_\phi^2H_\phi - H_\phi^2H_\phi^* = \tilde{b}H^2,$$

we have

$$\ker [T_\phi^*, T_\phi] = \ker H_\phi^2H_\phi = \ker (T_\phi^*T_\phi - \tilde{b}H^2),$$

which shows that $H_\phi^2H_\phi(K) = 0$, and hence $H_\phi^2(K) \subseteq \tilde{b}H^2$, so that $\text{cl} \ H_\phi^2K \subseteq \tilde{b}H^2$. But since $\dim R < \infty$ and by (1), $H_\phi$ is one-one and has dense range, we have

$$H^2 = \text{cl} H_\phi(K + R) = \text{cl} (H_\phiK + H_\phiR) = \text{cl} H_\phiK + H_\phiR.$$

We therefore have $\text{cl} H_\phiK = \tilde{b}H^2$ since $\dim H_\phiR = \text{deg}(b)$. Hence

$$\text{cl} \ H_\phiK = \text{cl} H_\phiK = \text{cl} T_\phi^*H_\phiK = T_\phi^*\tilde{b}H^2 = H^2,$$

which proves (23). On the other hand, we note that $\mathcal{E}(\phi)$ is a singleton set; otherwise, $\phi$ is of bounded type. Thus $\mathcal{E}(\phi)$ consists of only a finite Blaschke product $b$. We next argue that if $T_\phi(bK) \subseteq bH^2$ then

$$T_\phi(bk) = bT_\phi k \quad \text{for each} \ k \in K. \quad (24)$$

To see this, let $k \in K$ and write $k_1 := T_\phi k$. Thus $\phi k = k_1 + \overline{k_2}$ for some $k_2 \in H_\phi^2 = zH^2$. Then

$$T_\phi(bk) = P(b\phi k) = P(bk_2 + bk_1) = P(bk_2) + bk_1 = P(bk_2) + bT_\phi k.$$
Since, by assumption, \( T_{\phi}(bK) \subseteq bH^2 \), it follows that \( P(bK^2) \in bH^2 \). But since \( P(bK_2) \in (bH^2)^{\perp} \), we have \( P(bK^2) = 0 \), which proves (24). Since \( T_{\phi}T_b - T_bT_{\phi} = H_bH_{\phi} \), it follows from (23) and (24) that

\[
H_bH_{\phi} = H_b^2(clH_{\phi}K) = clH_b^2H_{\phi}K = cl(T_{\phi}T_b - T_bT_{\phi})K = 0,
\]

which implies that \( \bar{b}H^2 \subseteq H^2 \), so that \( b = e^{i\theta} \) for some \( \theta \in [0, 2\pi) \). Therefore \( \phi \) is of the form \( \phi = f + e^{i\theta}f \) for some \( f \in H^{\infty} \) and \( \theta \in [0, 2\pi) \) which implies that \( T_{\phi} \) is normal. 

We thus have:

**Corollary 4.3.** Suppose \( T_{\phi} \) is a subnormal Toeplitz operator with finite rank self-commutator. If \( b \ker [T_{\phi}^*, T_{\phi}] \) is invariant under \( T_{\phi} \) (some \( b \in \mathcal{E}(\phi) \)), then \( T_{\phi} \) is normal or analytic.

**Proof.** Since \( \ker [T^*, T] \) is invariant under \( T \) for every subnormal operator \( T \), the result follows at once from Theorem 4.2.

We were not unable to decide whether the condition “\( b \ker [T_{\phi}^*, T_{\phi}] \) (some \( b \in \mathcal{E}(\phi) \)) is invariant under \( T_{\phi} \)” can be dropped from Corollary 4.3: in other words, if \( T_{\phi} \) is a subnormal operator with finite rank self-commutator and \( b \in \mathcal{E}(\phi) \), is \( b \ker [T_{\phi}^*, T_{\phi}] \) invariant under \( T_{\phi} \)? If the answer to this question is affirmative we can conclude that Conjecture 4.1 is true.

**References**


Department of Mathematics, University of Iowa, Iowa City, IA 52242, U.S.A.
E-mail: raul-curto@uiowa.edu

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
E-mail: ihwang@skku.edu

Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail: wylee@snu.ac.kr