Abstracts of KOTAC
Volume 7 (2004), 7–7

ESTIMATIONS OF GENERALIZED QUASI-ARITHMETIC MEANS

Masatoshi Fujii
Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan
E-mail: mfujii@cc.osaka-kyoiku.ac.jp

and

Yuki Seo
Senior Highschool, Osaka Kyoiku University, Osaka 543-0054, Japan
E-mail: yukis@cc.osaka-kyoiku.ac.jp

Let $f(x)$ be a monotone increasing function and Φ a normalized positive linear map of $B(H)$ into $B(K)$. Then the quasi-arithmetic mean for a selfadjoint operator A on H is defined as

$$f^{-1}\left(\Phi(f(A))\right).$$

For example, put $f(x) = \log x$ and Φ is the normalized trace on $M_n(\mathbb{C})$. For $A = \mathrm{d}iag(a_1, \cdots, a_n) > 0$, $f^{-1}\left(\Phi(f(A))\right) = \sqrt{[n]a_1 \cdots a_n}$, i.e., the geometric mean of eigenvalues. On the other hand, it is the arithmetic mean for $f(x) = x$, and the harmonic one for $f(x) = -1/x$. In this case, the arithmetic-geometric mean inequality

$$f^{-1}\left(\Phi(f(A))\right) \leq \Phi(A) \tag{1}$$

holds. However, the inequality (1) does not always hold even if f is a monotone increasing operator concave function. Under such situation, we estimate the lower bound of the difference of the inequality (1) under a general setting. A general result is as follows:

Let A be a positive operator on H such that $mI \leq A \leq MI$. If f is a monotone increasing convex function on $[m, M]$ and g is a continuous function on $f([m, M])$, then for each $\lambda > 0$

$$g\left(\Phi(f(A))\right) - \lambda\Phi(A) \geq \min_{t\in[m,M]} \{\, g(\alpha_f t + \beta_f) - \lambda t\}$$

where $\alpha_f = (f(M) - f(m))/(M - m)$ and $\beta_f = (Mf(m) - mf(M))/(M - m)$.