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From Entropy to Relative Entropy

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It was conjectured by Diósi, Feldmann and Kosloff, based on thermodynamical considerations, that the Umegaki relative entropy $S(\sigma\|\rho)$ of density matrices ρ and σ shows up from the von Neumann entropy of the mixed state

$$R_n := \frac{1}{n}\left(\sigma \otimes \rho^{\otimes(n-1)} + \rho \otimes \sigma \otimes \rho^{\otimes(n-2)} + \cdots + \rho^{\otimes(n-1)} \otimes \sigma\right)$$

as follows:

$$S(R_n) - (n-1)S(\rho) - S(\sigma) \longrightarrow S(\sigma\|\rho) \quad \text{as } n \to \infty. \tag{1}$$

Here, recall that $S(\sigma) = -\mathrm{Tr}\,\sigma \log \sigma$ and

$$S(\sigma\|\rho) = \begin{cases} \mathrm{Tr}\,\sigma(\log \sigma - \log \rho) & \text{if } \mathrm{supp}\,\sigma \le \mathrm{supp}\,\rho, \\ +\infty & \text{otherwise.} \end{cases}$$

First, we give an analytic proof of (1) for the case $\mathrm{supp}\,\sigma \le \mathrm{supp}\,\rho$, using an inequality between the Umegaki and the Belavkin-Staszewski relative entropies, and the weak law of large numbers in the quantum case. Secondly, we clarify that the problem is related to the theory of classical-quantum channels. The essential observation is the fact that $S(R_n) - (n-1)S(\rho) - S(\sigma)$ in the conjecture is a Holevo quantity (classical-quantum mutual information) for a certain channel for which the relative entropy emerges as the capacity per unit cost.

The two different proofs lead to two different generalizations of the conjecture.

This is a joint work with I. Csiszár and D. Petz, Alfréd Rényi Institute of Mathematics, Hungary.