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Operator Radii

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On the space $B(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ there are two natural norms; the one is the operator norm $\|\cdot\|_\infty$ and the other is the numerical radius $w(\cdot)$. The operator norm is closely related to the notion of unitary dilation. Inspired by the discovery by Berger that the numerical radius is related to the notion of so-called unitary 2-dilation, for any $\rho > 1$ Sz.-Nagy and Foiaş introduced the notion of unitary ρ-dilation by the requirement that there is a unitary operator $\mathbf{U}$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that

$$T^k \;=\; \rho \mathbf{P}\mathbf{U}^k|_{\mathcal{H}}\ (k = 1, 2, \ldots) \qquad \mathbf{P} \text{ projection from } \mathcal{K} \text{ to } \mathcal{H}.$$

Holbrook defined ρ-radius $w_\rho(T)$ in such a way that T admits a unitary ρ-dilation if and only if $w_\rho(T) \le 1$, hence $w_1(T) = \|T\|_\infty$ and $w_2(T) = w(T)$. It is easy to see that $\rho \longmapsto w_\rho(T)$ is a decreasing function of $\rho \ge 1$ and $\lim_{\rho\to\infty} w_\rho(T)$ coincides with the spectral radius of T while $\rho \longmapsto \rho w_\rho(T)$ is an increasing function. When $\dim(\mathcal{H}) \ge 2$, the ρ-radius $w_\rho(T)$ becomes a norm if and only if $1 \le \rho \le 2$.

The artificialness for T to admit a unitary ρ-dilation was converted to a requirement inside $B(\mathcal{H})$ by Ando, Durszt, and Okubo-Ando as

$$w_\rho(T) \le 1 \qquad \Longleftrightarrow \qquad T = \rho f_\rho(T)^{1/2} W A^{1/2} \quad \exists\, 0 \le A \le I, \quad \exists\ \|W\|_\infty \le 1,$$
$$\text{where} \quad f_\rho(t) := \frac{1-t}{1+\rho(\rho-2)t} \quad (0 \le t \le 1).$$

In this talk the usefulness of this representation will be shown in proving the following:

(1) For any unital completely positive linear map $\Phi(\cdot)$

$$w_\rho\Big(\Phi(T)\Big) \;\le\; \frac{\max\{\rho, 2\}}{2} w_\rho(T).$$

(2) $w_\rho(T) \le 1,\ w_\rho(T^{-1}) \le 1\ \exists\, \rho \quad \Longrightarrow T$ is unitary.

(3) When $\dim(\mathcal{H}) = n < \infty$ and $\rho \ge 2$ the maximum in the class of unitarily invariant norms dominated by the ρ-radius $w_\rho(\cdot)$ is given by

$$\max\left\{\frac{\|T\|_\infty}{\rho}, \frac{\|T\|_1}{\frac{n}{2}\rho}\right\} \quad \text{if } n \text{ is even} \quad \text{and} \quad \max\left\{\frac{\|T\|_\infty}{\rho}, \frac{\|T\|_1}{\frac{n-1}{2}\rho + 1}\right\} \quad \text{if } n \text{ is odd.}$$

(4) When $2 \le \dim(\mathcal{H}) = n < \infty$ and $1 < \rho < 2$,

$$\max\left\{\frac{\|T\|_\infty}{\rho}, \frac{\|T\|_1}{n}\right\} \le w_\rho(T) \quad \forall\, T.$$

But the norm on the left-hand side is not maximum in the class of unitarily invariant norms, dominated by the ρ-radius $w_\rho(\cdot)$.