Partially normal composition operators via directed trees

Eun Young Lee

Department of Mathematics, Kyungpook National University, Korea

ee-2220@hanmail.net

A pair \((V, E)\) is a directed graph if \(V\) is a nonempty and \(E\) is a subset of \(V \times V \setminus \{(v, v) : v \in V\}\). An element of \(V\) is called a node of \(G\). Set \(\text{Chi}(u) = \{v \in V : (u, v) \in E\}, u \in V\). For \(u \in V\), the cardinality of \(\text{Chi}(u)\) called to be the degree of \(u\). Put \(\text{Gen}^k(u) = \bigcup_{v \in \text{Gen}^{k-1}(u)} \text{Chi}(v), u \in V\), where \(\text{Gen}^0(u) = \{u\}\). The set \(\text{Gen}^k(u)\) is called generation \(k\) of \(u\). A graph \(G = (V, E, \mu)\) is a weighted directed graph if \((V, E)\) is a directed graph and \((V, \mathcal{P}(V), \mu)\) be a \(\sigma\)-finite measure space on \(V\), where \(\mathcal{P}(V)\) is the power set of \(V\). For a node \(u \in V\), we write \(m_u\) for the point mass \(\mu(u)\). If, for a given node \(u \in V\), there exists a unique node \(v \in V\) such that \((v, u) \in E\), then we say that \(u\) has a parent \(v\) and write \(\text{par}(u)\) for \(v\). A node \(v\) of \(G\) is called a root of \(G\) if there is no node \(u\) of \(G\) such that \((u, v)\) is an edge of \(G\). A graph \(G = (V, E, \mu)\) is a weighted directed tree if \(G\) is a weighted directed graph such that \(G\) is connected, \(G\) has no circuits, and each node \(v \in V \setminus \{\text{root}\}\) has a parent. Given a graph \(G\) and a sequence of graphs \(G_n\), we say that \(G\) is an equivalent-limit of \(\{G_n\}\), and write \(e\)-\(\lim G_n = G\) if there exist graphs \(H_1, \ldots, H_n, \ldots\) such that \(G_i \preceq H_i, i = 1, 2, \ldots, \) and \(\lim_{n \to \infty} H_n = G\). Let \(G\) be weighted directed trees and let \(T\) be the measurable transformation \(V\) corresponded by \(G\). The directed tree \(G\) is \(p\)-hyponormal if the composition operator \(CT\) relevant to \(G\) is \(p\)-hyponormal.

We introduce one of our results as following.

Theorem. Let \(M \in \mathbb{R}_+, D \in \mathbb{N}, D \geq 2, p \in \mathbb{R}_+\) and \(\epsilon > 0\) be arbitrary. Then there exists a \(p\)-hyponormal directed tree \(E = E(M, D, \epsilon, p)\) with \(D_E = D\) and \(M_E < M + \epsilon\), and such that for all trees \(G\) with \(D_G \leq D\) and \(M_G \leq M\), the following assertions hold.

(i) If \(G\) has no root, \(G\) is \(p\)-hyponormal if and only if for every node \(n\) of \(G\), \(S(n) = e\)-\(\lim_{j \to \infty} S_j\) with \(S_j\) a system in \(E\) for all \(j\).

(ii) If \(G\) has a root, then the test \((\dagger)\) for \(p\)-hyponormal of the root \(r\) holds and \(G\) has the property in (i).

In particular, if \(D = 1\), then we may obtain the analogous result with \(D_E = 2\). Also, \(E\) has the additional property that \(m_v \in \mathbb{Q}_+\) for all \(v \in E\).