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Hankel Type Operators And Invariant Subspaces

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Let $\mathcal{H}$ be a Hilbert space and M a closed subspace of $\mathcal{H}$. P^M denotes the orthogonal projection onto M. When $M_1 \subseteq M_2$ are closed subspaces of $\mathcal{H}$, the following operator is called a Hankel type operator :

$$G_{M_1,M_2} = (I - P^{M_2})GP^{M_1}$$

where G is a bounded linear operator on $\mathcal{H}$. When $M_1 = M_2$, we write $G_{M_1,M_2} = G_{M_1}$ simply. In this lecture, we are interested in when G_{M_1,M_2} is of finite rank. Since $G_M = 0$ means $GM \subset M$, finite rank G_M means that M is closed to an invariant subspace of G.

Let $\Gamma = \Gamma_z = \{z \in \mathbb{C} : |z| = 1\}$ and $\Gamma^2 = \Gamma \times \Gamma = \Gamma_z \times \Gamma_w$. $L^p = L^p(\Gamma^2)$ and $L^p(\Gamma)$ are Lebesgue spaces, $H^p = H^p(\Gamma^2)$ and $H^p(\Gamma)$ denotes usual Hardy spaces. For a function ϕ in L^∞, L_ϕ is a multiplication operator on L^2 and T_ϕ is a Toeplitz operator on H^2. A Hankel operator H_ϕ is just $G_{H^2,(H^2)^\perp}$ for $G = L_\phi$.

In one variable case, Kronecker [5] determined the symbol ϕ when H_ϕ is of finite rank and Beurling [2] described M when $G_M = 0$ for $G = T_z$. In two variable case, we [3] also can describe the symbol ϕ when H_ϕ is of finite rank in very different form. However we still can not describe M completely when $G_M = 0$ for $G = T_z$ and $G = T_w$. Many mathematicians believe that it is not possible to describe M as in one variable case (cf. [1], [4], [6], [7], [8]).

This lecture is an expository one. However a few new results are given.

Let M be a closed subspace in H^2 and $N = H^2 \ominus M$. Suppose $G_{M,N} = 0$ for $G = T_Q$ and Q is inner in H^2. In a half of this lecture, we give and use the following decomposition : If $G = T_Q^*$ then

$$M = (\sum_{j=0}^{\infty} \oplus T_{Q^j} K) \oplus M_\infty \text{ and } N = [\bigcup_{j=1}^{\infty} T_{Q^j}^* K] \oplus N_\infty$$

where $K = [G_M^* H^2] \subset M \ominus QM$, $T_Q M_\infty \subset M_\infty$, $T_Q^* M_\infty \subset M_\infty$ and $T_Q N_\infty \subset M_\infty$. If $G_M = 0$ for $G = T_Q^*$ then $M = M_\infty$ or equivalently $N = N_\infty$.

In one variable case, when $zM \subset M$, $G_M = 0$ for $G = T_Q^*$ if and only if $M = H^2(\Gamma)$. In two variable case, when $zM \subset M$ and $wM \subset M$, $G_M = 0$ for $G = T_Q^*$ if and only if

$$M = \sum_{j=0}^{\infty} \oplus Q^j S \text{ and } S \subset H^2(\Gamma^2) \ominus QH^2(\Gamma^2).$$

If $G_M = 0$, we can describe M completely in case $Q(z,w) = w$ or $Q(z,w) = zw$. Moreover we consider M when G_M is of finite rank.

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