Abstracts of KOTAC
Volume 12(2010), 23–24

Double piling structure of matrix monotone functions and of matrix convex functions

Hiroyuki Osaka

Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga 525-8577, Japan
osaka@se.ritsumei.ac.jp

A real valued continuous function f on I is said to be operator monotone if for every selfadjoint operators a, b on a Hilbert space H ($\dim H = +\infty$) such that $a \leq b$ and $\sigma(a), \sigma(b) \subseteq I$ we have $f(a) \leq f(b)$.

Let $n \in \mathbb{N}$ and M_n be the algebra of $n \times n$ matrices. We call a function f matrix monotone of order n or n-monotone in short whenever the inequality $f(a) \leq f(b)$ holds for every pair of selfadjoint matrices $a, b \in M_n$ such that $a \leq b$ and all eigenvalues of a and b are contained in I. Matrix convex (concave) functions on I are similarly defined as above as well as operator convex (concave) functions. We denote the spaces of operator monotone functions and of operator convex functions by $P_\infty(I)$ and $K_\infty(I)$ respectively. The spaces for n-monotone functions and n-convex functions are written as $P_n(I)$ and $K_n(I)$. We have then

$$P_1(I) \supseteq \cdots \supseteq P_{n-1}(I) \supseteq P_n(I) \supseteq P_{n+1}(I) \supseteq \cdots \supseteq P_\infty(I)$$
$$K_1(I) \supseteq \cdots \supseteq K_{n-1}(I) \supseteq K_n(I) \supseteq K_{n+1}(I) \supseteq \cdots \supseteq K_\infty(I)$$

Here we meet the facts that $\cap_{n=1}^\infty P_n(I) = P_\infty(I)$ and $\cap_{n=1}^\infty K_n(I) = K_\infty(I)$. We regard these two decreasing sequences as noncommutative counterpart of the following classical piling sequence $\{C^n(I), C^\infty(I), Anal(I)\}$,

$$C^1(I) \supsetneqq \cdots \supsetneqq C^{n-1}(I) \supsetneqq C^n(I) \supsetneqq C^{n+1}(I) \cdots,$$

where $C^n(I)$ means the set of all n-times continuously differentiable functions and $Anal(I)$ means the set of analytic functions. This sequence ends down to $C^\infty(I)$, i.e., the set of infinitely differentiable functions, and then we usually find further smaller set $Anal(I)$.

We could understand that the class of operator monotone functions $P_\infty(I)$ corresponds to the class $\{C^\infty(I), Anal(I)\}$ by the famous characterization of those functions by Loewner as the restriction of Pick functions.

There are basic equivalent assertions known for operator monotone functions and operator convex functions in two papers by Hansen and Pedersen. In this talk we consider their results as correlation problem between two sequences of matrix n-monotone functions and matrix n-convex functions, and we focus the following 3 assertions at each label n among them:

(i) $f(0) \leq 0$ and f is n-convex in $[0, \alpha)$,

(ii) For each matrix a with its spectrum in $[0, \alpha)$ and a contraction c in the matrix algebra M_n,
$$f(c^\star ac) \leq c^\star f(a)c,$$

(iii) The function $f(t)/t$ $(= g(t))$ is n-monotone in $(0, \alpha)$.

We show that for any $n \in \mathbb{N}$ two conditions (ii) and (iii) are equivalent. The assertion that f is n-convex with $f(0) \leq 0$ implies that $g(t)$ is $(n-1)$-monotone holds. The implication from (iii) to (i) does hold for $n \geq 2$ when g is n-monotone and n-convex in $(0, \alpha)$.

The partial part is a joint work with Jun Tomiyama.