



세종대학교
SEJONG UNIVERSITY

Spectral properties of m -complex symmetric operators

Muneo Chō, Eungil Ko, and Ji Eun Lee*

Mathematics-Applied Statistics, Sejong University, Seoul

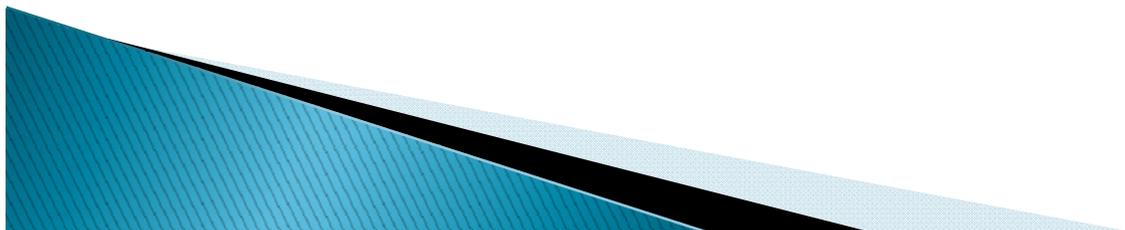
2015 KOTAC,
Chungnam National University,
June 18-19(Thu-Fri), 2015





Table of contents

- ① Study
- ② Introduction
 - m -complex symmetric operator
 - Local spectral theory
- ③ m -complex symmetric operators
- ④ Perturbation of m -complex symmetric operators by nilpotent operators
- ⑤ Reference



Study



Study

- J. W. Helton, *Operators with a representation as multiplication by x on a Sobolev space*, Colloquia Math. Soc. Janos Bolyai **5**, Hilbert Space Operators, Tihany, Hungary (1970), 279-287.
- M. Chō, S. Ota, K. Tanahashi, and M. Uchiyama, *Spectral properties of m -isometric operators*, Functional Analysis, Application and Computation 4:2 (2012), 33-39.
- M. Chō, E. Ko and J. Lee, *On m -complex symmetric operators*, Mediterranean Journal of Mathematics, accepted.
- M. Chō, E. Ko and J. Lee, *Perturbation of m -complex symmetric operators by nilpotent operators*, preprint.



Motivation

- \mathcal{H} : a complex (separable) Hilbert space
- $\mathcal{L}(\mathcal{H})$: the algebra of all bounded linear operators on \mathcal{H} .

Motivation

In 1970, J. W. Helton initiated the study of operators $T \in \mathcal{L}(\mathcal{H})$ which satisfy an identity of the form

$$\sum_{j=0}^m (-1)^{m-j} \binom{m}{j} T^{*j} T^{m-j} = 0. \quad (1)$$

m-complex symmetric operator

- A *conjugation* on \mathcal{H} is an antilinear operator $C : \mathcal{H} \rightarrow \mathcal{H}$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$.
- An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be **complex symmetric** if there exists a conjugation C on \mathcal{H} such that $T = CT^*C$.

Definition

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-complex symmetric operator* if there exists some conjugation C such that

$$\sum_{j=0}^m (-1)^{m-j} \binom{m}{j} T^{*j} C T^{m-j} C = 0$$

for some positive integer m . In this case, we say that T is *m-complex symmetric* with conjugation C .



Examples

- Normal operators, Algebraic operator of order 2, Aluthge transform of CSO, Truncated Toeplitz operator, Finite Toeplitz matrices, and Hankel matrices.
- (S. R. Garcia and M. Putinar, 2006);
If T is nilpotent of order 2, then T is complex symmetric.

Example

Let $T \in \mathcal{L}(\mathcal{H})$ and let C be a conjugation on \mathcal{H} . If T is nilpotent of order $k > 2$ and $T^* \neq CTC$, then T is a $(2k - 1)$ -complex symmetric operator with conjugation C .



Example

Example

Let C be a conjugation given by $C(z_1, z_2, z_3) = (\bar{z}_1, \bar{z}_2, \bar{z}_3)$ on \mathbb{C}^3 .

If $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ on \mathbb{C}^3 , then $T^3 = 0$ and T is a not complex

symmetric operator by [GP]. Hence T is a 5-complex symmetric operator with conjugation C . However, since $T^3 = 0$, we have

$$\sum_{j=0}^4 (-1)^{4-j} \binom{4}{j} T^{*j} C T^{4-j} C = 6 T^{*2} C T^2 C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 24 \end{pmatrix} \neq 0.$$

So it is not a 4-complex symmetric operator.



Example

Example

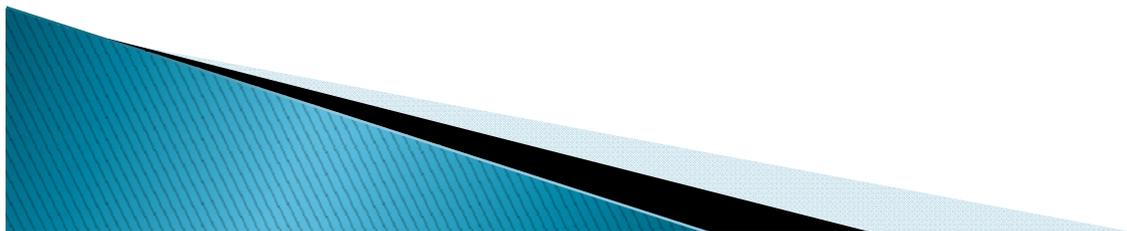
Let C be a conjugation on \mathcal{H} and a self-adjoint operator $R \in \mathcal{L}(\mathcal{H})$ be complex symmetric with C , i.e., $R = CRC$. If $RQ = QR$, $Q^* \neq CQC$ and $Q^k = 0$ for some $k > 2$, then an operator $T = R + Q$ is $(2k - 1)$ -complex symmetric with conjugation C .

m -complex symmetric operators

- Set $\Delta_m(T) := \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} T^{*j} C T^{m-j} C$.
- T is an m -complex symmetric operator with conjugation C if and only if $\Delta_m(T) = 0$.
- Note that

$$T^* \Delta_m(T) - \Delta_m(T)(CTC) = \Delta_{m+1}(T). \quad (2)$$

- If T is m -complex symmetric with conjugation C , then T is n -complex symmetric with conjugation C for all $n \geq m$.
- A 1-complex symmetric operator is complex symmetric.





SVEP

Single-valued extension property

- We say that an operator T has the *single-valued extension property at λ* (abbreviated SVEP at λ) if for every open set U containing λ the only analytic function $f : U \rightarrow \mathcal{H}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U .

- T has **SVEP** if T has SVEP at every point $\lambda \in \mathbb{C}$.

Property (β) and Decomposable

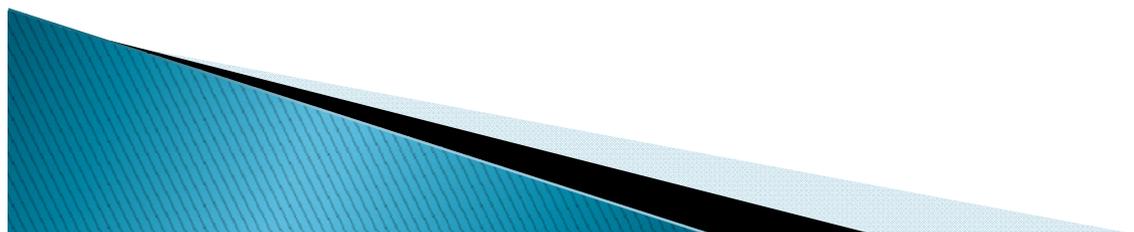
Property (β) [1959, E. Bishop]

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the property (β)* if for every open subset G of \mathbb{C} and every sequence $f_n : G \rightarrow \mathcal{H}$ of \mathcal{H} -valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G , then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G .

Decomposable [1963, C. Foias]

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of \mathbb{C} there are T -invariant subspaces \mathcal{M} and \mathcal{N} such that $\mathcal{H} = \mathcal{M} + \mathcal{N}$, $\sigma(T|_{\mathcal{M}}) \subset \overline{U}$, and $\sigma(T|_{\mathcal{N}}) \subset \overline{V}$.

Decomposable \Rightarrow Property (β) \Rightarrow SVEP. $\langle \equiv \rangle$



m-complex symmetric operators

- Two vectors x and y are C -orthogonal if $\langle Cx, y \rangle = 0$.

Theorem

Let $T \in \mathcal{L}(\mathcal{H})$ be an m -complex symmetric operator with conjugation C .

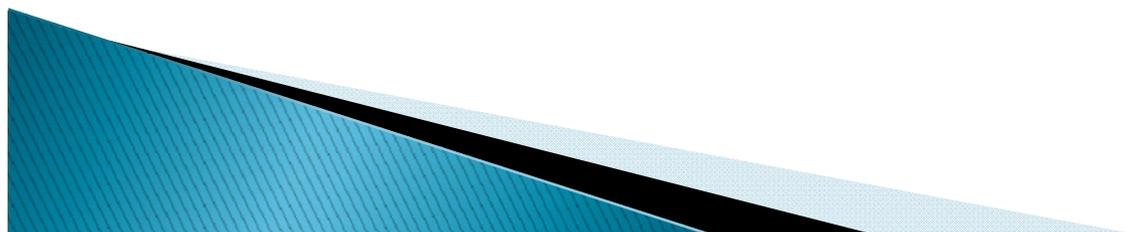
- (i) If λ is an eigenvalue of T , then $\bar{\lambda}$ is an eigenvalue of T^* .
- (ii) Eigenvectors of T corresponding to distinct eigenvalues are C -orthogonal.
- (iii) If $\lambda \in \sigma_{ap}(T)$, then $\bar{\lambda} \in \sigma_{ap}(T^*)$.
- (iv) Let $\lambda \neq \mu$. If $\{x_n\}, \{y_n\}$ are sequences of unit vectors such that $\lim_{n \rightarrow \infty} (T - \lambda)x_n = 0$ and $\lim_{n \rightarrow \infty} (T - \mu)y_n = 0$, then $\lim_{n \rightarrow \infty} \langle Cx_n, y_n \rangle = 0$.

m-complex symmetric operators

- $T \in \mathcal{L}(\mathcal{H})$ is said to be *isoloid* if for any $\lambda \in \text{iso } \sigma(T)$, $\lambda \in \mathbb{C}$ is an eigenvalue of T , where $\text{iso } \Delta$ denotes the set of all isolated points of Δ .

Corollary

Let $T \in \mathcal{L}(\mathcal{H})$ be m-complex symmetric with conjugation C . If T is isoloid, then T^ is also isoloid.*



m -complex symmetric operators

Theorem

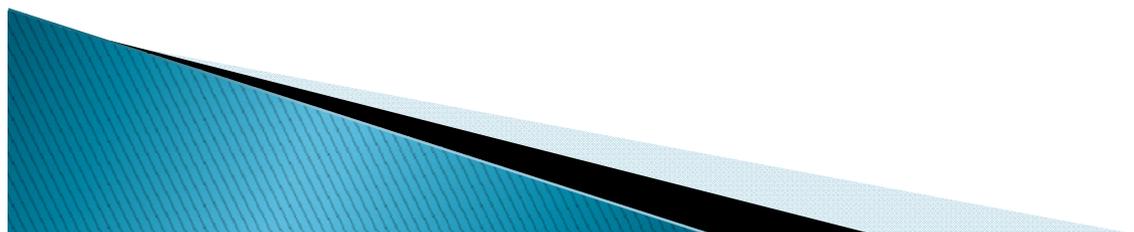
If $\{T_k\}$ is a sequence of m -complex symmetric operators with conjugation C such that $\lim_{k \rightarrow \infty} \|T_k - T\| = 0$, then T is also m -complex symmetric with conjugation C .

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$ be invertible and let C be a conjugation on \mathcal{H} .

(i) If $T^{*j}CT^{m-j}C = CT^{m-j}CT^{*j}$ for $j = 0, 1, \dots, m$, then T is m -complex symmetric with conjugation C if and only if $CT^{*-1}C$ is m -complex symmetric with conjugation C .

(ii) T is m -complex symmetric with conjugation C if and only if T^{-1} is m -complex symmetric with conjugation C .



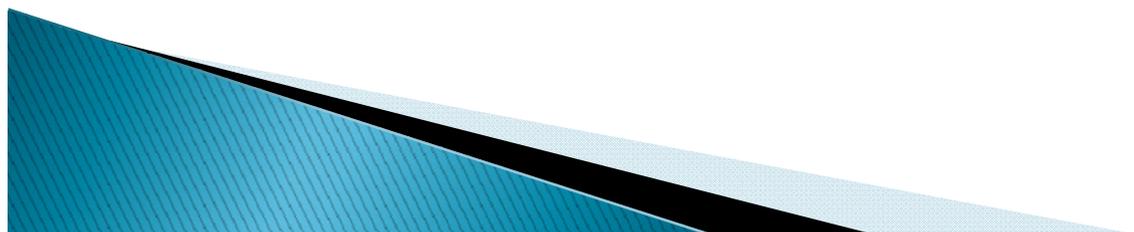
m-complex symmetric operators

Theorem

If $T \in \mathcal{L}(\mathcal{H})$ is an m -complex symmetric operator with conjugation C , then T^n is also m -complex symmetric with conjugation C for some $n \in \mathbb{N}$.

Corollary

*Let $T \in \mathcal{L}(\mathcal{H})$ be m -complex symmetric with conjugation C . If $\lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0$, then $\lim_{n \rightarrow \infty} \|T^{*mn} Cx\|^{\frac{1}{n}} = 0$.*



m-complex symmetric operators

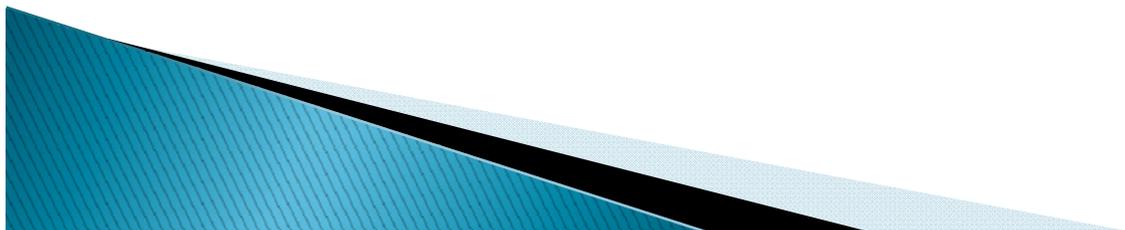
- Set $\Delta_m(T) := \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} T^{*j} C T^{m-j} C$.

Theorem

If $\Delta_m(T)$ is hyponormal and $\Delta_{m+1}(T) = 0$, then $\ker(\Delta_m(T) - \lambda) \cap \ker(\Delta_1(T) - \lambda) = \{0\}$ for any nonzero $\lambda \in \mathbb{C}$.

Corollary

Let C be a conjugation on \mathcal{H} . Suppose that H and K are self-adjoint operators which satisfy $HCK = KCH$ and $CRC \geq R$, where $R = i(HK - KH)$. For an operator $T = H + iK$, if $\Delta_2(T) = 0$, then $\ker(\Delta_1(T) - \lambda) = \{0\}$ for any nonzero $\lambda \in \mathbb{C}$.



m-complex symmetric operators

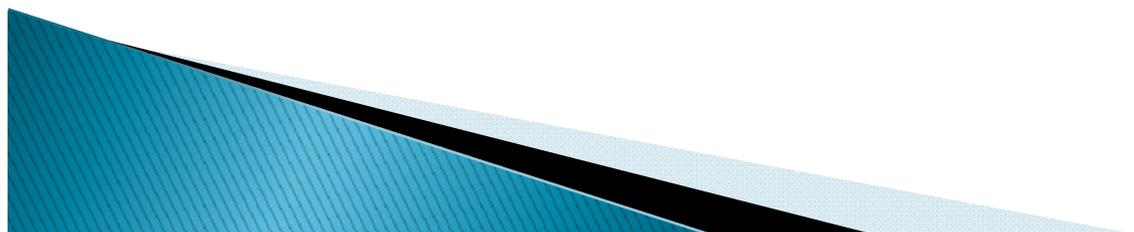
Theorem

Let $T \in \mathcal{L}(\mathcal{H})$ be an m -complex symmetric operator with conjugation C . Then T^ has the property (β) if and only if T is decomposable.*

Corollary

Let $T \in \mathcal{L}(\mathcal{H})$ be m -complex symmetric operators.

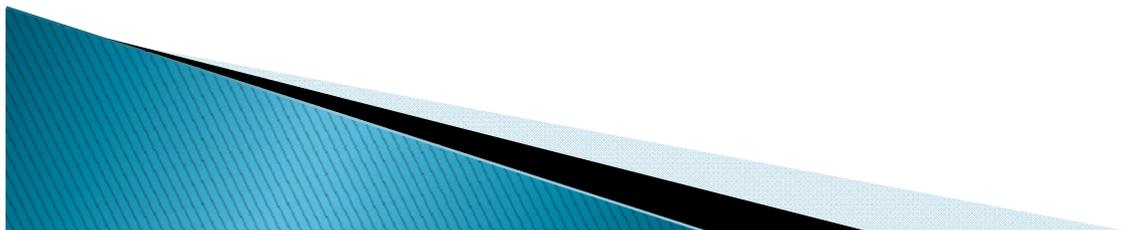
- (i) If T^* is hyponormal, i.e. $TT^* \geq T^*T$, then T is decomposable.*
- (ii) If T^* has the property (β) and $\sigma(T)$ has nonempty interior, then T has a nontrivial invariant subspace.*





Local spectrum

- $\rho_T(x) = \{\lambda_0 \in \mathbb{C} : \exists \text{ an } \mathcal{H}\text{-valued analytic function } f \text{ defined in a neighborhood of } \lambda_0 \text{ s.t. } (\lambda I - T)f(\lambda) \equiv x\}$
: the *local resolvent set* of x .
- $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$: the *local spectrum* of T at x .
- $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ where $F \subset \mathbb{C}$
: the *local spectral subspace* of T .

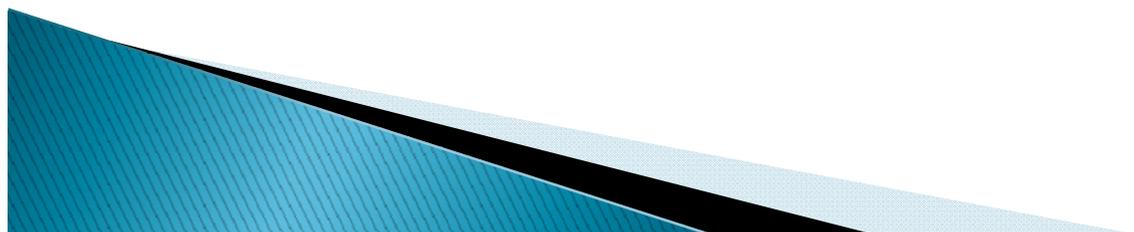


m-complex symmetric operators

- For $D \subset \mathbb{C}$, we denote $D^* = \{\bar{z} : z \in D\}$.

Theorem

Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C . If T^ has the single-valued extension property, then T has the single-valued extension property. Moreover, in this case, $\sigma_{T^*}(x) \subset \sigma_T(Cx)^*$ for all $x \in \mathcal{H}$. Furthermore, $CH_T(F) \subset H_{T^*}(F^*)$ where $F^* := \{\bar{z} : z \in F\}$ for any set F in \mathbb{C} .*



m-complex symmetric operators

- Assume that T has the single-valued extension property. If there exists a constant k such that for every $x, y \in \mathcal{H}$ with $\sigma_T(x) \cap \sigma_T(y) = \emptyset$ we have

$$\|x\| \leq k \|x + y\|$$

where k is independent of x and y , we say that an operator T satisfies *Dunford's boundedness condition (B)*.

Corollary

Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C . If T^ has the single-valued extension property and the Dunford's boundedness condition (B), then T also has the Dunford's boundedness condition (B).*

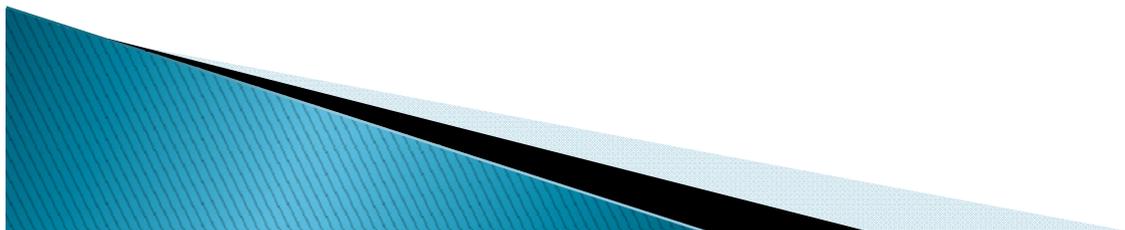
Perturbation of m -CSOs by nilpotent operators

Theorem

Let C be a conjugation on \mathcal{H} . If $T \in \mathcal{L}(\mathcal{H})$ is m -complex symmetric with the conjugation C and N is nilpotent of order $n > 2$ with $TN = NT$ and $N^ \neq CNC$, then $T + N$ is $(2n + m - 2)$ -complex symmetric with conjugation C .*

Corollary

*Let C be a conjugation on \mathcal{H} and let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric with C , i.e., $T = CT^*C$. Then an operator $R = T + N$ is $(2n - 1)$ -complex symmetric with conjugation C where $TN = NT$, $N^* \neq CNC$, and $N^n = 0$ for some $n > 2$.*



Perturbation of m -CSOs by nilpotent operators

Example

Let C be a conjugation operator given by

$C(z_1, z_2, z_3) = (\bar{z}_1, \bar{z}_2, \bar{z}_3)$ on \mathbb{C}^3 . If $R = I + N$ where

$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ on \mathbb{C}^3 , then $N^3 = 0$ and $N^* \neq CNC$. Hence

$$\Delta_4(R) = \Delta_4(N) = 6N^{*2}CN^2C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 54 \end{pmatrix} \neq 0.$$

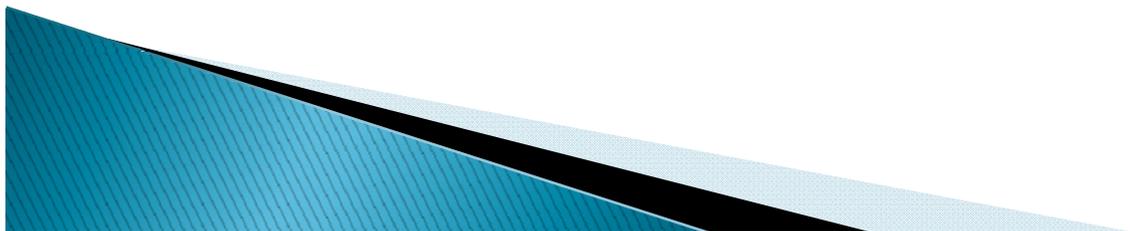
So R is not a 4-complex symmetric operator. Hence $R = I + N$ is 5-complex symmetric.

Perturbation of m -CSOs by nilpotent operators

Theorem

If $T \in \mathcal{L}(\mathcal{H})$ is m -complex symmetric with the conjugation C and N is a nilpotent operator of order n with $TN = NT$, then the following statements are equivalent:

- (i) T is decomposable.
- (ii) T^* has the property (β) .
- (iii) $T + N$ is decomposable.
- (iv) $T^* + N^*$ has the property (β) .

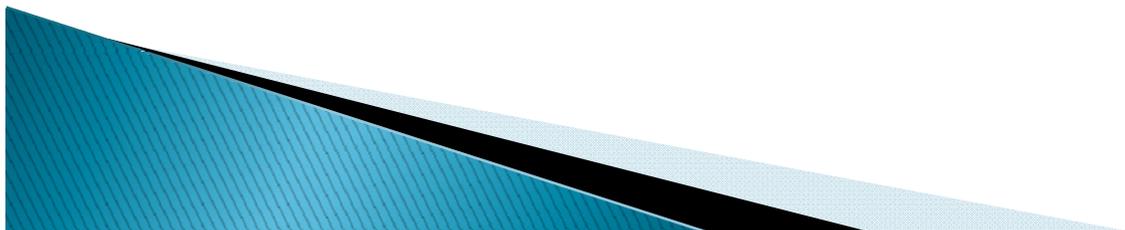


Perturbation of m -CSOs by nilpotent operators

Corollary

If $T \in \mathcal{L}(\mathcal{H})$ is complex symmetric with the conjugation C and N is a nilpotent operator of order n with $TN = NT$, then the following statements are equivalent:

- (i) T is decomposable.*
- (ii) T^* has the property (β) .*
- (iii) T has the property (β) .*
- (iv) $T + N$ is decomposable.*
- (v) $T^* + N^*$ has the property (β) .*
- (vi) $T + N$ has the property (β) .*

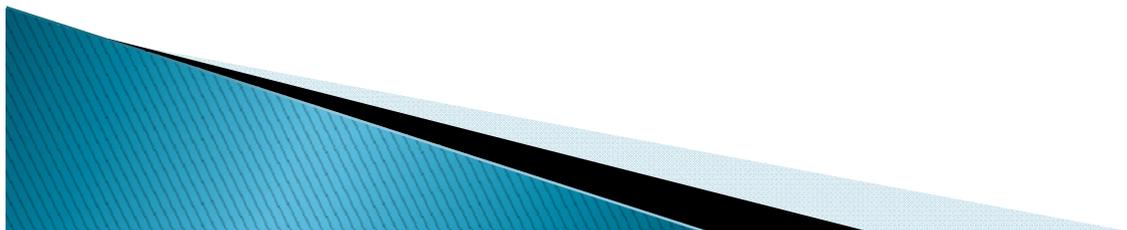


Perturbation of m -CSOs by nilpotent operators

- $iso\Delta$ denotes the set of all isolated points of Δ
- $\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < \dim \ker(T - \lambda) < \infty\}$
- *Weyl's theorem holds for $T \in \mathcal{L}(\mathcal{H})$ if*

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$$

- *Browder's theorem holds for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma_b(T) = \sigma_w(T)$.*



Perturbation of m -CSOs by nilpotent operators

Corollary

Let $R = T + N \in \mathcal{L}(\mathcal{H})$ where T is m -complex symmetric and $N^n = 0$ with $TN = NT$. If T^ has the property (β) , then the following assertions hold:*

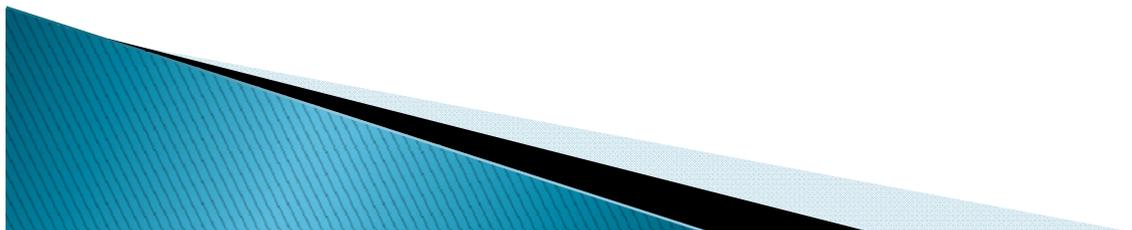
- (i) R and R^* have the property (β) and the single-valued extension property.*
- (ii) If $\sigma(R)$ has nonempty interior, then R has a nontrivial invariant subspace.*
- (iii) $H_R(F)$ is a hyperinvariant subspace for R .*
- (iv) If f is any function analytic on a neighborhood of $\sigma(R)$, then both Weyl's and Browder's theorems hold for $f(R)$ and $\sigma_w(f(R)) = \sigma_b(f(R)) = f(\sigma_w(R)) = f(\sigma_b(R))$.*

Perturbation of m -CSOs by nilpotent operators

Proposition

Let $R = T + N$ be an operator in $\mathcal{L}(\mathcal{H})$ with the same hypotheses as in Theorem 4. Then the following properties hold:

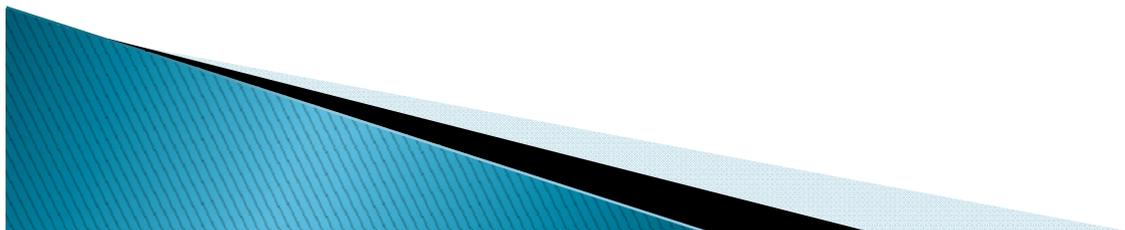
- (i) $\sigma_p(R) \subset \sigma_p(T^*)^* \cup \{0\}$, $\Gamma(R^*)^* \subset \Gamma(T) \cup \{0\}$, and $\sigma_{ap}(R) \subseteq \sigma_{ap}(T^*)^* \cup \{0\}$.
- (ii) $\sigma_{le}(R) \subset \sigma_{le}(T)$ and $\sigma_{re}(R^*)^* \subset \sigma_{re}(T^*)^*$.





Reference

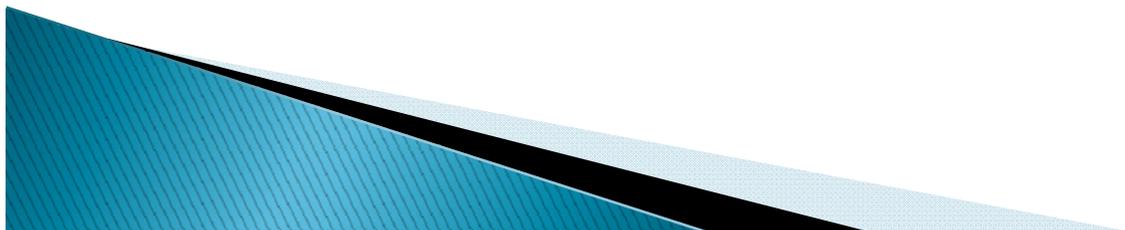
- P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Academic Pub. 2004
- M. Chō, S. Ota, K. Tanahashi, and M. Uchiyama, *Spectral properties of m -isometric operators*, *Functional Analysis, Application and Computation* 4:2 (2012), 33-39.
- I. Colojoara and C. Foias, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.
- J. Eschmeier, *Invariant subspaces for operators with Bishop's property (β) and thick spectrum*, *J. Funct. Anal.* **94**(1990), 196-222.
- S. R. Garcia, *Aluthge transforms of complex symmetric operators*, *Integr. equ. oper. theory.* **60**(2008), 357-367.





Reference

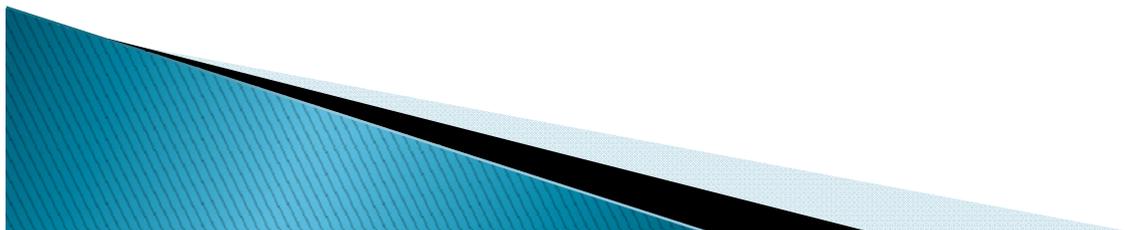
- S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358**(2006), 1285-1315.
- _____, *Complex symmetric operators and applications II*, Trans. Amer. Math. Soc. **359**(2007), 3913-3931.
- S. R. Garcia and W. R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362**(2010), 6065-6077.
- J. W. Helton, *Operators with a representation as multiplication by x on a Sobolev space*, Colloquia Math. Soc. Janos Bolyai **5**, Hilbert Space Operators, Tihany, Hungary (1970), 279-287.





Reference

- S. Jung, E. Ko, M. Lee, and J. Lee, *On local spectral properties of complex symmetric operators*, J. Math. Anal. Appl. **379**(2011), 325-333.
- S. Jung, E. Ko, and J. Lee, *On scalar extensions and spectral decompositions of complex symmetric operators*, J. Math. Anal. Appl. **382**(2011), 252-260.
- _____, *On complex symmetric operator matrices*, J. Math. Anal. Appl. **406**(2013), 373-385.
- K. Laursen and M. Neumann, *An introduction to local spectral theory*, Clarendon Press, Oxford, 2000.
- X. Wang and Z. Gao, *A note on Aluthge transforms of complex symmetric operators and applications*, Int. Eq. Op. Th. **65**(2009), 573-580.



Thank you!



세종대학교
SEJONG UNIVERSITY

Thank you for your attention !

jieunlee7@sejong.ac.kr; jieun7@ewhain.net

