Graduate Texts

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Lecture Notes on Operator Theory

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Preface

The present lectures are based on a graduate course delivered by the author at the Seoul National University, in the spring semester of 2008.

In these lectures I attempt to set forth some of the recent developments that had taken place in Operator Theory. In particular, I focus on the Fredholm and Weyl theory, hyponormal and subnormal theory, weighted shift theory, Toeplitz theory, and the invariant subspace problem.

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Contents

Chapter 1

Fredholm Theory

1.1 Introduction

If $k(x,y)$ is a continuous complex-valued function on $[a,b]\times[a,b]$ then $K : C[a,b] \to C[a,b]$ defined by

$$(Kf)(x) = \int_a^b k(x,y)f(y)dy$$

is a compact operator. The classical Fredholm integral equations is

$$\lambda f(x) - \int_a^b k(x,y)f(y)dy = g(x), \quad a \le x \le b,$$

where $g \in C[a,b]$, λ is a parameter and f is the unknown. Using I to be the identity operator on $C[a,b]$, we can recast this equation into the form $(\lambda I - K)f = g$. Thus we are naturally led to study of operators of the form $T = \lambda I - K$ on any Banach space X. Riesz-Schauder theory concentrates attention on these operators of the form $T = \lambda I - K$, $\lambda \neq 0$, K compact. The Fredholm theory concentrates attention on operators called Fredholm operators, whose special cases are the operators $\lambda I - K$. After we develop the "Fredholm Theory", we see the following result. Suppose $k(x,y) \in C[a,b]\times C[a,b]$ (or $L^2[a,b]\times L^2[a,b]$). The equation

$$\lambda f(x) - \int_a^b k(x,y)f(y)dy = g(x), \quad \lambda \neq 0 \tag{1.1}$$

has a unique solution in $C[a,b]$ for each $g \in C[a,b]$ if and only if the homogeneous equation

$$\lambda f(x) - \int_a^b k(x,y)f(y)dy = 0, \quad \lambda \neq 0 \tag{1.2}$$

has only the trivial solution in $C[a,b]$. Except for a countable set of λ, which has zero as the only possible limit point, equation (1.1) has a unique solution for every $g \in C[a,b]$. For $\lambda \neq 0$, the equation (1.2) has at most a finite number of linear independent solutions.

1.2 Preliminaries

Let X and Y be complex Banach spaces. Write $B(X,Y)$ for the set of bounded linear operators from X to Y and abbreviate $B(X,X)$ to $B(X)$. If $T \in B(X)$ write $\rho(T)$ for the resolvent set of T ; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T.

We begin with:

Definition 1.2.1. Let X be a normed space and let X^* be the dual space of X. If Y is a subset of X, then

$$Y^\perp = \{f \in X^* : \ f(x) = 0 \text{ for all } x \in Y\} = \{f \in X^* : Y \subset f^{-1}(0)\}$$

is called the *annihilator* of Y. If Z is a subset of X^* then

$$.^\perp Z = \{x \in X : \ f(x) = 0 \text{ for all } f \in Z\} = \bigcap_{f \in Z} f^{-1}(0)$$

is called the *back annihilator* of Z.

Even if Y and Z are not subspaces, and $Y^\perp$ and $.^\perp Z$ are closed subspaces.

Lemma 1.2.2. *Let $Y, Y' \subset X$ and $Z, Z' \subset X^*$. Then*
(a) $Y \subset .^\perp(Y^\perp), \quad Z \subset (^\perp Z)^\perp$;
(b) $Y \subset Y' \Longrightarrow (Y')^\perp \subset Y^\perp; \quad Z \subset Z' \Longrightarrow .^\perp(Z') \subset .^\perp Z$;
(c) $(^\perp(Y^\perp))^\perp = Y^\perp, \quad .^\perp((^\perp Z)^\perp) =^\perp Z$;
(d) $\{0\}^\perp = X^*, \quad X^\perp = \{0\}, \quad .^\perp\{0\} = X$.

Proof. This is straightforward. ☐

Theorem 1.2.3. *Let M be a subspace of X. Then*
(a) $X^*/M^\perp \cong M^*$;
(b) *If M is closed then* $(X/M)^* \cong M^\perp$;
(c) $.^\perp(M^\perp) = \operatorname{cl} M$.

Proof. See [Go, p.25]. ☐

Theorem 1.2.4. *If $T \in B(X,Y)$ then*
(a) $T(X)^\perp = (T^*)^{-1}(0)$;
(b) $\operatorname{cl} T(X) = .^\perp(T^{*-1}(0))$;
(c) $T^{-1}(0) \subset .^\perp T^*(Y^*)$;
(d) $\operatorname{cl} T^*(Y^*) \subset T^{-1}(0)^\perp$.

Proof. See [Go, p.59]. ☐

Theorem 1.2.5. *Let X and Y be Banach spaces and $T \in B(X,Y)$. Then the followings are equivalent:*

(a) *T has closed range;*

(b) *T^* has closed range;*

(c) *$T^*(Y^*) = T^{-1}(0)^{\perp}$;*

(d) *$T(X) = {}^{\perp}(T^{*-1}(0))$.*

Proof. (a) ⇔ (d): From Theorem 1.2.4 (b).

(a) ⇒ (c): Observe that the operator $T^{\wedge} : X/T^{-1}(0) \to TX$ defined by

$$x + T^{-1}(0) \mapsto Tx$$

is invertible by the Open Mapping Theorem. Thus we have

$$T^{-1}(0)^{\perp} \cong \left(X/T^{-1}(0)\right)^* \cong (TX)^* \cong T^*(Y^*).$$

(c) ⇒ (b): This is clear because $T^{-1}(0)^{\perp}$ is closed.

(b) ⇒ (a): Observe that if $T_1 : X \to \mathrm{cl}\,(TX)$ then $T_1^* : (\mathrm{cl}\,TX)^* \to X^*$ is one-one. Since $T^*(Y^*) = \mathrm{ran}\,T_1^*$, T_1^* has closed range. Therefore T_1^* is bounded below, so that T_1 is open; therefore TX is closed. □

Definition 1.2.6. If $T \in B(X,Y)$, write

$$\alpha(T) := \dim T^{-1}(0) \quad \text{and} \quad \beta(T) = \dim Y/\mathrm{cl}\,(TX).$$

Theorem 1.2.7. *If $T \in B(X,Y)$ has a closed range then*

$$\alpha(T^*) = \beta(T) \quad \textit{and} \quad \alpha(T) = \beta(T^*).$$

Proof. This follows form the following observation:

$$T^{*-1}(0) = (TX)^{\perp} \cong (Y/TX)^* \cong Y/TX$$

and

$$T^{-1}(0) \cong (T^{-1}(0))^* \cong X^*/T^{-1}(0)^{\perp} \cong X^*/T^*(Y^*).$$

□

1.3 Definitions and Examples

In the sequel X and Y denote complex Banach spaces.

Definition 1.3.1. An operator $T \in B(X,Y)$ is called a *Fredholm operator* if TX is closed, $\alpha(T) < \infty$ and $\beta(T) < \infty$. In this case we define the *index* of T by the equality

$$\operatorname{index}(T) := \alpha(T) - \beta(T).$$

In the below we shall see that the condition " TX is closed " is automatically fulfilled if $\beta(T) < \infty$.

Example 1.3.2. If X and Y are both finite dimensional then any operator $T \in B(X,Y)$ is Fredholm and

$$\operatorname{index}(T) = \dim X - \dim Y :$$

indeed recall the "rank theorem"

$$\dim X = \dim T^{-1}(0) + \dim TX,$$

which implies

$$\begin{aligned}\operatorname{index}(T) &= \dim T^{-1}(0) - \dim Y/TX \\ &= \dim X - \dim TX - (\dim Y - \dim TX) \\ &= \dim X - \dim Y.\end{aligned}$$

Thus in particular, if $T \in B(X)$ with $\dim X < \infty$ then T is Fredholm of index zero.

Example 1.3.3. If $K \in B(X)$ is a compact operator then $T = I - K$ is Fredholm of index 0. This follows from the Fredholm theory for compact operators.

Example 1.3.4. If U is the unilateral shift operator on ℓ^2, then

$$\operatorname{index} U = -1 \quad \text{and} \quad \operatorname{index} U^* = -1.$$

With U and U^*, we can build a Fredholm operator whose index is equal to an arbitrary prescribed integer. Indeed if

$$T = \begin{bmatrix} U^p & 0 \\ 0 & U^{*q} \end{bmatrix} : \ell^2 \oplus \ell^2 \ \to \ell^2 \oplus \ell^2,$$

then T is Fredholm, $\alpha(T) = q$, $\beta(T) = p$, and hence $\operatorname{index} T = q - p$.

1.4 Operators with Closed Ranges

If $T \in B(X,Y)$, write

$$\text{dist}\big(x,\ T^{-1}(0)\big) = \inf\big\{||x-y|| :\ Ty = 0\big\} \quad \text{for each } x \in X. \tag{1.3}$$

If $T \in B(X,Y)$, we define

$$\gamma(T) = \inf\Bigg\{c > 0 :\ ||Tx|| \geq c\, \text{dist}\,(x,\ T^{-1}(0)) \ \text{ for each } x \in X\Bigg\} :$$

we call $\gamma(T)$ the *reduced minimum modulus* of T.

Theorem 1.4.1. *If $T \in B(X,Y)$ then*

$$T(X) \textit{ is closed } \iff \gamma(T) > 0.$$

Proof. Consider $\widehat{X} = X/T^{-1}(0)$ and thus $\widehat{X}$ is a Banach space with norm $||\widehat{x}|| = \text{dist}\,(x,\ T^{-1}(0))$. Define $\widehat{T} : \widehat{X} \to Y$ by $\widehat{T}\widehat{x} = Tx$. Then $\widehat{T}$ is one-one and $\widehat{T}(\widehat{X}) = T(X)$.

($\Rightarrow$) Suppose TX is closed and thus $\widehat{T} : \widehat{X} \to TX$ is bijective. By the Open Mapping Theorem $\widehat{T}$ is invertible with inverse $\widehat{T}^{-1}$. Thus

$$||Tx|| = ||\widehat{T}\widehat{x}|| \geq \frac{1}{||\widehat{T}^{-1}||}||\widehat{x}|| = \frac{1}{||\widehat{T}^{-1}||}\text{dist}\,(x,\ T^{-1}(0)),$$

which implies that $\gamma(T) = \frac{1}{||\widehat{T}^{-1}||} > 0$.

($\Leftarrow$) Suppose $\gamma(T) > 0$. Let $Tx_n \to y$. Then by the assumption $||Tx_n|| \geq \gamma(T)\,||\widehat{x_n}||$, and hence, $||Tx_n - Tx_m|| \geq \gamma(T)\,||\widehat{x_n} - \widehat{x_m}||$, which implies that $(\widehat{x_n})$ is a Cauchy sequence in $\widehat{X}$. Thus $\widehat{x_n} \to \widehat{x} \in \widehat{X}$ because $\widehat{X}$ is complete. Hence $Tx_n = \widehat{T}\widehat{x_n} \to \widehat{T}\widehat{x} = Tx$; therefore $y = Tx$. □

Theorem 1.4.2. *If there is a closed subspace Y_0 of Y for which $T(X) \oplus Y_0$ is closed then T has closed range.*

Proof. Define $T_0 : X \times Y_0 \to Y$ by

$$T_0(x, y_0) = Tx + y_0.$$

The space $X \times Y_0$ is a Banach space with the norm defined by

$$||(x, y_0)|| = ||x|| + ||y_0||.$$

Clearly, T_0 is a bounded linear operator and $\text{ran}\,(T_0) = T(X) \oplus Y_0$, which is closed by hypothesis. Moreover, $\ker\,(T_0) = T^{-1}(0) \times \{0\}$. Theorem 1.4.1 asserts that there exists a $c > 0$ such that

$$||Tx|| = ||T_0(x,0)|| \geq c\, \text{dist}\Big((x,0),\ \ker T_0\Big) = c\, \text{dist}\,\big(x, T^{-1}(0)\big),$$

which implies that $T(X)$ is closed. □

Corollary 1.4.3. *If $T \in B(X,Y)$ then*

$$T(X) \text{ is complemented} \implies T(X) \text{ is closed.}$$

In particular, if $\beta(T) < \infty$ then $T(X)$ is closed.

Proof. If $T(X)$ is complemented then we can find a closed subspace Y_0 for which $T(X) \oplus Y_0 = Y$. Theorem 1.4.2 says that $T(X)$ is closed. □

To see the importance of Corollary 1.4.3, note that for a subspace M of a Banach space Y,

$$Y = M \oplus Y_0 \text{ does not imply that } M \text{ is closed.}$$

Take a non-continuous linear functional f on Y and put $M = \ker f$. Then there exists a one-dimensional subspace Y_0 such that $Y = M \oplus Y_0$ (recall that $Y/\ker(f)$ is one-dimensional). But $M = \ker f$ cannot be closed because f is continuous if and only if $f^{-1}(0)$ is closed.

Consequently, we don't guarantee that

$$\dim(Y/M) < \infty \implies M \text{ is closed.} \tag{1.4}$$

However Corollary 1.4.3 asserts that if M is a range of a bounded linear operator then (1.4) is true. Of course, it is true that

$$M \text{ is closed}, \ \ \dim(Y/M) < \infty \implies M \text{ is complemented.}$$

Theorem 1.4.4. *Let $T \in B(X,Y)$. If T maps bounded closed sets onto closed sets then T has closed range.*

Proof. Suppose $T(X)$ is not closed. Then by Theorem 1.4.1 there exists a sequence $\{x_n\}$ such that

$$Tx_n \to 0 \quad \text{and} \quad \text{dis}\big(x_n,\ T^{-1}(0)\big) = 1.$$

For each n choose $z_n \in T^{-1}(0)$ such that $||x_n - z_n|| < 2$. Let $V := \text{cl}\{x_n - z_n :\ n = 1, 2, \ldots\}$. Since V is closed and bounded in X, $T(V)$ is closed in Y by assumption. Note that $Tx_n = T(x_n - z_n) \in T(V)$. So $0 \in T(V)$ ($Tx_n \to 0 \in T(V)$) and thus there exists $u \in V \cap T^{-1}(0)$. From the definition of V it follows that

$$||u - (x_{n_0} - z_{n_0})|| < \frac{1}{2} \quad \text{for some } n_0,$$

which implies that

$$\text{dis}\big(x_{n_0},\ T^{-1}(0)\big) < \frac{1}{2}.$$

This contradicts the fact that $\text{dist}\big(x_n,\ T^{-1}(0)\big) = 1$ for all n. Therefore $T(X)$ is closed. □

Theorem 1.4.5. *Let $K \in B(X)$. If K is compact then $T = I - K$ has closed range.*

Proof. Let V be a closed bounded set in X and let

$$y = \lim_{n\to\infty} (I - K)x_n, \quad \text{where } x_n \in V. \tag{1.5}$$

We have to prove that $y = (I - K)x_0$ for some $x_0 \in V$. Since V is bounded and K is compact the sequence $\{Kx_n\}$ has a convergent subsequence $\{Kx_{n_i}\}$. By (1.5), we see that

$$x_0 := \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} \big((I - K)x_{n_i} + Kx_{n_i}\big) \text{ exists.}$$

But then $y = (I - K)x_0 \in (I - K)V$; thus $(I - K)V$ is closed. Therefore by Theorem 1.4.4, $I - K$ has closed range. □

Corollary 1.4.6. *If $K \in B(X)$ is compact then $I - K$ is Fredholm.*

Proof. From Theorem 1.4.5 we see that $(I - K)(X)$ is closed. Since $x \in (I - K)^{-1}(0)$ implies $x = Kx$, the identity operator acts as a compact operator on $(I - K)^{-1}(0)$; thus $\alpha(I - K) < \infty$. To prove that $\beta(I - K) < \infty$, recall that $K^* : X^* \to X^*$ is also compact. Since $(I - K)(X)$ is closed it follows from Theorem 1.2.7 that

$$\beta(I - K) = \alpha(I - K^*) < \infty.$$

□

1.5 The Product of Fredholm Operators

Let $T \in B(X,Y)$. Suppose $T^{-1}(0)$ and $T(X)$ are complemented by subspaces X_0 and Y_0; i.e.,

$$X = T^{-1}(0) \oplus X_0 \quad \text{and} \quad Y = T(X) \oplus Y_0.$$

Define $\widetilde{T} : X_0 \times Y_0 \to Y$ by

$$\widetilde{T}(x_0, y_0) = Tx_0 + y_0.$$

The space $X_0 \times Y_0$ is a Banach space with the norm defined by $||(x,y)|| = ||x|| + ||y||$ and $\widetilde{T}$ is a bijective bounded linear operator. We call $\widetilde{T}$ the *bijection associated with* T. If T is Fredholm then such a bijection always exists and Y_0 is finite dimensional. If we identify $X_0 \cong X_0 \times \{0\}$ then the operator $T_0 : X_0 \to Y$ defined by

$$T_0 x = Tx$$

is a common restriction of T and $\widetilde{T}$ to X_0 $(= X_0 \times \{0\})$.

Note that

(a) $\frac{1}{||\widetilde{T}^{-1}||} = \gamma(T)$;

(b) If $\widehat{T} : X/T^{-1}(0) \to TX$ then $\widehat{T} \cong \widetilde{T}$.

Lemma 1.5.1. *Let* $T \in B(X,Y)$ *and* $M \subset X$ *with* $\operatorname{codim} M = n < \infty$. *Then*

$$T \textit{ is Fredholm} \iff T_0 := T|_M \textit{ is Fredholm},$$

in which case, $\operatorname{index} T = \operatorname{index} T_0 + n$.

Proof. It suffices to prove the lemma for $n = 1$. Put $X := M \oplus \operatorname{span}\{x_1\}$. We consider two cases:

(Case 1) Assume $Tx_1 \notin T_0(M)$. Then $TX = T_0M \oplus \operatorname{span}\{Tx_1\}$ and $T^{-1}(0) = T_0^{-1}(0)$. Hence

$$\beta(T_0) = \beta(T) + 1 \quad \text{and} \quad \alpha(T_0) = \alpha(T). \tag{1.6}$$

(Case 2) Assume $Tx_1 \in T_0(M)$. Then $TX = T_0M$, and hence there exists $u \in M$ such that $Tx_1 = T_0 u$. Thus $T^{-1}(0) = T_0^{-1}(0) \oplus \operatorname{span}\{x_1 - u\}$. Thus

$$\beta(T_0) = \beta(T) \quad \text{and} \quad \alpha(T_0) = \alpha(T) - 1. \tag{1.7}$$

From (1.6) and (1.7) we have the result. □

Theorem 1.5.2. (Index Product Theorem) *If* $T \in B(X,Y)$ *and* $S \in B(Y,Z)$ *then*

$$S \textit{ and } T \textit{ are Fredholm} \implies ST \textit{ is Fredholm with}$$
$$\operatorname{index}(ST) = \operatorname{index} S + \operatorname{index} T.$$

Proof. Let $\widetilde{T}$ be a bijection associated with T, X_0, and Y_0: i.e., $X = T^{-1}(0) \oplus X_0$ and $Y = T(X) \oplus Y_0$. Suppose $T_0 := T|_{X_0}$. Since $\widetilde{T}$ is invertible, $S\widetilde{T}$ is invertible and $\text{index}\,(S\widetilde{T}) = \text{index}\,S$. By identifying X_0 and $X_0 \times \{0\}$, we see that ST_0 is a common restriction of $S\widetilde{T}$ and ST to X_0. By Lemma 1.5.1, ST is Fredholm and

$$\begin{aligned}\text{index}\,(ST) &= \text{index}\,(ST_0) + \dim X/X_0\\ &= \text{index}(S\widetilde{T}) - \dim\Bigg(X_0 \times Y_0/X_0 \times \{0\}\Bigg) + \alpha(T)\\ &= \text{index}S - \dim Y_0 + \alpha(T)\\ &= \text{index}\,S - \beta(T) + \alpha(T)\\ &= \text{index}\,S + \text{index}\,T.\end{aligned}$$

□

The converse of Theorem 1.5.2 is not true in general. To see this, consider the following operators on ℓ^2:

$$\begin{aligned}T(x_1, x_2, x_3, \ldots) &= (0, x_1, 0, x_2, 0, x_3, \ldots)\\ S(x_1, x_2, x_3, \ldots) &= (x_2, x_3, x_4, \ldots).\end{aligned}$$

Then T ad S are not Fredholm, but $ST = I$. However, if $ST = TS$ then we have

$$ST \text{ is Fredholm} \implies S \text{ and } T \text{ are both Fredholm}$$

because $T^{-1}(0) \subset (ST)^{-1}(0)$ and $(ST)(X) = TS(X) \subset T(X)$.

Remark 1.5.3. For a time being, a Fredholm operator of index 0 will be called a Weyl operator. Then we have the following question: *Is there implication that if $ST = TS$ then*

$$S, T \textit{ are Weyl} \Longleftrightarrow ST \textit{ is Weyl}?$$

Here is the answer. The forward implication comes from the "Index Product Theorem" without commutativity condition. However the backward implication may fail even with commutativity condition. To see this, let

$$T = \begin{bmatrix} U & 0\\ 0 & I\end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} I & 0\\ 0 & U^*\end{bmatrix},$$

where U is the unilateral shift on ℓ_2. Evidently,

$$\begin{aligned}\text{index}\,(ST) &= \text{index}\begin{bmatrix} U & 0\\ 0 & U^*\end{bmatrix}\\ &= \text{index}\,U + \text{index}\,U^*\\ &= 0,\end{aligned}$$

but S and T are not Weyl.

1.6 Perturbation Theorems

We begin with:

Theorem 1.6.1. *Suppose $T \in B(X,Y)$ is Fredholm. If $S \in B(X,Y)$ with $||S|| < \gamma(T)$ then $T+S$ is Fredholm and*
(i) $\alpha(T+S) \le \alpha(T)$;
(ii) $\beta(T+S) \le \beta(T)$;
(iii) $\operatorname{index}(T+S) = \operatorname{index} T$.

Proof. Let $X = T^{-1}(0) \oplus X_0$ and $Y = T(X) \oplus Y_0$. Suppose $\widetilde{T}$ is the bijection with T, X_0 and Y_0. Put $R = T+S$ and define

$$\widetilde{R} : X_0 \times Y_0 \to Y \quad \text{by } \widetilde{R}(x_0, y_0) = Rx_0 + y_0.$$

By definition, $\widetilde{T}(x_0, y_0) = Tx_0 + y_0$. Since $\widetilde{T}$ is invertible and

$$||\widetilde{T} - \widetilde{R}|| \le ||T - R|| = ||S|| < \gamma(T) = \frac{1}{||\widetilde{T}^{-1}||},$$

we have that $\widetilde{R}$ is also invertible. Note that $R_0 : X_0 \to Y$ defined by

$$R_0 x = Rx$$

is a common restriction of R and $\widetilde{R}$ to X_0. By Lemma 1.5.1, R is Fredholm and

$$\begin{aligned} \operatorname{index} R &= \operatorname{index} R_0 + \alpha(T) \\ &= \operatorname{index} \widetilde{R} - \beta(T) + \alpha(T) \\ &= \operatorname{index} T \end{aligned}$$

which proves (iii). The invertibility of $\widetilde{R}$ implies that $X_0 \cap R^{-1}(0) = \{0\}$. Thus we have

$$\alpha(R) \le \dim X/X_0 = \alpha(T),$$

which proves (i). Note that (ii) is an immediate consequence of (i) and (iii). $\square$

The first part of Theorem 1.6.1 asserts that

the set of Fredholm operators forms an open set.

Theorem 1.6.2. *Let $T, K \in B(X,Y)$. Then*

$$T \text{ is Fredholm, } K \text{ is compact} \implies T+K \text{ is Fredholm with } \operatorname{index}(T+K) = \operatorname{index} T.$$

Proof. Let $X = T^{-1}(0) \oplus X_0$ and $Y = T(X) \oplus Y_0$. Define $\widetilde{T},\ \widetilde{K} : X_0 \times Y_0 \to Y$ by

$$\widetilde{T}(x_0, y_0) = Tx_0 + y_0, \quad \widetilde{K}(x_0, y_0) = Kx_0 + y_0.$$

Therefore $\widetilde{K}$ is compact since K is compact and $\dim Y_0 < \infty$. From $(\widetilde{T} + \widetilde{K})(x_0, 0) = (T + K)x_0$ and Lemma 1.5.1 it follows that

$$T + K \text{ is Fredholm} \iff \widetilde{T} + \widetilde{K} \text{ is Fredholm.}$$

But $\widetilde{T}$ is invertible. So

$$\widetilde{T} + \widetilde{K} = \widetilde{T}\left(I + \widetilde{T}^{-1}\widetilde{K}\right).$$

Observe that $\widetilde{T}^{-1}\widetilde{K}$ is compact. Thus by Corollary 1.4.6, $I + \widetilde{T}^{-1}\widetilde{K}$ is Fredholm. Hence $T + K$ is Fredholm.

To prove the statement about the index consider the integer valued function $F(\lambda) := \operatorname{index}(T + \lambda K)$. Applying Theorem 1.6.1 to $T + \lambda K$ in place of T shows that f is continuous on $[0, 1]$. Consequently, f is constant. In particular,

$$\operatorname{index} T = f(0) = f(1) = \operatorname{index}(T + K).$$

□

Corollary 1.6.3. *If* $K \in B(X)$ *then*

$$K \textit{ is compact} \implies I - K \textit{ is Fredholm with } \operatorname{index}(I - K) = 0.$$

Proof. Apply the preceding theorem with $T = I$ and note that $\operatorname{index} I = 0$. □

1.7 The Calkin Algebra

We begin with:

Theorem 1.7.1. *If $T \in B(X,Y)$ then*

$$T \text{ is Fredholm} \iff \exists\ S \in B(Y,X) \text{ such that } I-ST \text{ and } I-TS \text{ are finite rank.}$$

Proof. ($\Rightarrow$) Suppose T Fredholm and let

$$X = T^{-1}(0) \oplus X_0 \quad \text{and} \quad Y = T(X) \oplus Y_0.$$

Define $T_0 := T|_{X_0}$. Since T_0 is one-one and $T_0(X_0) = T(X)$ is closed

$$T_0^{-1} : T(X) \to X_0 \text{ is invertible.}$$

Put $S := T_0^{-1}Q$, where $Q : Y \to T(X)$ is a projection. Evidently, $S(Y) = X_0$ and $S^{-1}(0) = Y_0$. Furthermore,

$$\begin{aligned} &I - ST \text{ is the projection of } X \text{ onto } T^{-1}(0)\\ &I - TS \text{ is the projection of } Y \text{ onto } Y_0. \end{aligned}$$

In particular, $I - ST$ and $I - TS$ are of finite rank.

($\Leftarrow$) Assume $ST = I - K_1$ and $TS = I - K_2$, where K_1, K_2 are finite rank. Since

$$T^{-1}(0) \subset (ST)^{-1}(0) \quad \text{and} \quad (TS)X \subset T(X),$$

we have

$$\begin{aligned} \alpha(T) &\le \alpha(ST) = \alpha(I - K_1) < \infty\\ \beta(T) &\le \beta(TS) = \beta(I - K_2) < \infty, \end{aligned}$$

which implies that T is Fredholm. □

Theorem 1.7.1 remains true if the statement "$I - ST$ and $I - TS$ are of finite rank" is replaced by "$I - ST$ and $I - TS$ are compact operators." In other words,

$$T \text{ is Fredholm} \iff T \text{ is invertible modulo compact operators.}$$

Let $K(X)$ be the space of all compact operators on X. Note that $K(X)$ is a closed ideal of $B(X)$. On the quotient space $B(X)/K(X)$, define the product

$$[S][T] = [ST], \quad \text{where } [S] \text{ is the coset } S + K(X).$$

The space $B(X)/K(X)$ with this additional operation is an algebra, which is called the *Calkin algebra*, with identity $[I]$.

Theorem 1.7.2. (Atkinson's Theorem) *Let* $T \in B(X)$. *Then*

$$T \text{ is Fredholm} \iff [T] \text{ is invertible in } B(X)/K(X).$$

Proof. ($\Rightarrow$) If T is Fredholm then

$$\exists\ S \in B(X) \text{ such that } ST - I \text{ and } TS - I \text{ are compact.}$$

Hence $[S][T] = [T][S] = [I]$, so that $[S]$ is the inverse of $[T]$ in the Calkin algebra.

($\Leftarrow$) If $[S][T] = [T][S] = [I]$ then

$$ST = I - K_1 \quad \text{and} \quad TS = I - K_2,$$

where K_1, K_2 are compact operators. Thus T is Fredholm. □

Let $T \in B(X)$. The *essential spectrum* $\sigma_e(T)$ of T is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Fredholm}\}$$

We thus have

$$\sigma_e(T) = \sigma_{B(X)/K(X)}(T + K(X)).$$

Evidently $\sigma_e(T)$ is compact. If $\dim X = \infty$ then

$$\sigma_e(T) \neq \emptyset \quad (\text{because } B(X)/K(X) \neq \emptyset).$$

In particular, Theorem 1.6.2 implies that

$$\sigma_e(T) = \sigma_e(T + K) \quad \text{for every } K \in K(X).$$

Theorem 1.7.3. *If* $T \in B(X, Y)$ *then*

$$T \text{ is Weyl} \iff \exists \text{ a finite rank operator } F \text{ such that } T + F \text{ is invertible.}$$

Proof. ($\Rightarrow$) Let T be Weyl and put

$$X = T^{-1}(0) \oplus X_0 \quad \text{and} \quad Y = T(X) \oplus Y_0.$$

Since $\operatorname{index} T = 0$, it follows that

$$\dim T^{-1}(0) = \dim Y_0.$$

Thus there exists an invertible operator $F_0 : T^{-1}(0) \to Y_0$. Define $F := F_0(I - P)$, where P is the projection of X onto X_0. Obviously, $T + F$ is invertible.

($\Leftarrow$) Assume $S = T + F$ is invertible, where F is of finite rank. By Theorem 1.6.2, T is Fredholm and $\operatorname{index} T = \operatorname{index} S = 0$. □

The *Weyl spectrum*, $\omega(T)$, of $T \in B(X)$ is defined by

$$\omega(T) = \Big\{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\Big\}$$

Evidently, $\omega(T)$ is compact and in particular,

$$\omega(T) = \bigcap_{K \text{ compact}} \sigma(T + K).$$

Definition 1.7.4. An operator $T \in B(X, Y)$ is said to be *regular* if there is $T' \in B(Y, X)$ for which

$$T = TT'T; \tag{1.8}$$

then T' is called a *generalized inverse* of T. We can always arrange

$$T' = T'TT' : \tag{1.9}$$

indeed if (1.8) holds then

$$T'' = T'TT' \implies TT''T \quad \text{and} \quad T'' = T''TT''.$$

If T' satisfies (1.8) and (1.9) then it will be called a *generalized inverse of* T *in the strong sense.* Also $T \in B(X, Y)$ is said to be *decomposably regular* if there exists $T' \in B(Y, X)$ such that

$$T = TT'T \quad \text{and} \quad T' \text{ is invertible.}$$

The operator $S := T_0^{-1}Q$, which was defined in the proof of Theorem 1.7.1, is a generalized inverse of in the strong sense. Thus we have

$$T \text{ is Fredholm} \iff I - T'T \text{ and } I - TT' \text{ are finite rank.}$$

Generalized inverses are useful in solving linear equations. Suppose T' is a generalized inverse of T. If $Tx = y$ is solvable for a given $y \in Y$, then $T'y$ is a solution (not necessary the only one). Indeed,

$$\begin{aligned} Tx = y \text{ is solvable} &\implies \exists\, x_0 \text{ such that } Tx_0 = y \\ &\implies TT'y = TT'Tx_0 = Tx_0 = y. \end{aligned}$$

Theorem 1.7.5. *If $T \in B(X, Y)$, then*

$$T \textit{ is regular} \iff T^{-1}(0) \textit{ and } T(X) \textit{ are complemented.}$$

Proof. ($\Leftarrow$) If $X = X_0 \oplus T^{-1}(0)$ and $Y = Y_0 \oplus T(X)$ then $T' : Y \to X$ defined by

$$T'(Tx_0 + y_0) = x_0, \quad \text{where } x_0 \in X_0 \text{ and } y_0 \in Y_0$$

is a generalized inverse of T because for $x_0 \in X_0$ and $z \in T^{-1}(0)$,

$$TT'T(x_0 + z) = TT'(Tx_0) = Tx_0 = T(x_0 + z).$$

($\Rightarrow$) Assume T' is a generalized inverse of T: $TT'T = T$. Obviously, TT' and $T'T$ are both projections. Also,

$$\begin{gathered} T(X) = TT'T(X) \subset TT'(X) \subset T(X); \\ T^{-1}(0) \subset (T'T)^{-1}(0) \subset (TT'T)^{-1}(0) = T^{-1}(0), \end{gathered}$$

which gives

$$TT'(X) = T(X) \quad \text{and} \quad (T'T)^{-1}(0) = T^{-1}(0),$$

which implies that $T^{-1}(0)$ and $T(X)$ are complemented. □

Corollary 1.7.6. *If $T \in B(X, Y)$ then*

$$T \textit{ is Fredholm} \implies T \textit{ is regular.}$$

Theorem 1.7.7. *If $T \in B(X, Y)$ is Fredholm with $T = TT'T$, then T' is also Fredholm with*

$$\operatorname{index}(T') = -\operatorname{index}(T).$$

Proof. We first claim that

$$ST \text{ is Fredholm} \implies (S \text{ Fredholm} \iff T \text{ Fredholm}) : \tag{1.10}$$

indeed,

$$ST \text{ is Fredholm} \implies I - (ST)'(ST) \in K_0 \quad \text{and} \quad I - (ST)(ST)' \in K_0,$$

which implies

$$T \text{ is Fredholm} \iff I - T(ST)'S \in K_0 \iff S \text{ is Fredholm.}$$

Thus by (1.10), T' is Fredholm and by the index product theorem,

$$\operatorname{index}(T) = \operatorname{index}(TT'T) = \operatorname{index}(T) + \operatorname{index}(T') + \operatorname{index}(T).$$

□

Theorem 1.7.8. *If $T \in B(X,Y)$ is Fredholm with generalized inverse $T' \in B(Y,X)$ in the strong sense then*

$$\text{index}\,(T) = \dim T^{-1}(0) - \dim\,(T')^{-1}(0).$$

Proof. Observe that

$$(T')^{-1}(0) = (TT')^{-1}(0) \cong X/TT'(X) \cong X/T(X),$$

which gives that $\beta(T) = \alpha(T')$. $\square$

Theorem 1.7.9. *If $T \in B(X,Y)$ is Fredholm with generalized inverse $T' \in B(Y,X)$, then*

$$\text{index}\,(T) = \text{trace}\,(TT' - T'T).$$

Proof. If $T = TT'T$ is Fredholm then

$$I - T'T \text{ and } I - TT' \text{ are both finite rank.}$$

Observe that

$$\begin{aligned} \dim\,(I - T'T)(X) &= \dim\,(T'T)^{-1}(0) = \dim T^{-1}(0) = \alpha(T); \\ \dim\,(I - TT')(Y) &= \dim\,(TT')^{-1}(0) = \dim X/TT'(Y) = \dim X/T(X) = \beta(T). \end{aligned}$$

Thus we have

$$\begin{aligned} \text{trace}\,(TT' - T'T) &= \text{trace}\,\big((I - T'T) - (I - TT')\big) \\ &= \text{trace}\,(I - T'T) - \text{trace}\,(I - TT') \\ &= \text{rank}\,(I - T'T)(X) - \dim\,(I - TT')(X) \\ &= \alpha(T) - \beta(T) \\ &= \text{index}\,(T). \end{aligned}$$

$\square$

1.8 The Punctured Neighborhood Theorem

If $T \in B(X,Y)$ then

(a) T is said to be *upper semi-Fredholm* if $T(X)$ is closed and $\alpha(T) < \infty$;
(b) T is said to be *lower semi-Fredholm* if $T(X)$ is closed and $\beta(T) < \infty$.
(c) T is said to be *semi-Fredholm* if it is upper or lower semi-Fredholm.

Theorem 1.6.1 remains true for semi-Fredholm operators. Thus we have:

Lemma 1.8.1. *Suppose $T \in B(X,Y)$ is semi-Fredholm. If $||S|| < \gamma(T)$ then*
(i) *$T+S$ has a closed range;*
(ii) *$\alpha(T+S) \le \alpha(T)$, $\beta(T+S) \le \beta(T)$;*
(iii) $\operatorname{index}(T+S) = \operatorname{index} T$.

Proof. This follows from a slight change of the argument for Theorem 1.6.1. □

We are ready for the punctured neighborhood theorem; this proof is due to Harte and Lee [HaL1].

Theorem 1.8.2. (Punctured Neighborhood Theorem) *If $T \in B(X)$ is semi-Fredholm then there exists $\rho > 0$ such that $\alpha(T-\lambda I)$ and $\beta(T-\lambda I)$ are constant in the annulus $0 < |\lambda| < \rho$.*

Proof. Assume that T is upper semi-Fredholm and $\alpha(T) < \infty$. First we argue

$$(T-\lambda I)^{-1}(0) \subset \bigcap_{n=1}^{\infty} T^n(X) =: T^\infty(X). \tag{1.11}$$

Indeed,

$$\begin{aligned} x \in (T-\lambda I)^{-1}(0) &\implies Tx = \lambda x, \text{ and hence } x \in T(X) \\ &\implies \text{Note that } \lambda x = Tx \in T(TX) = T^2(X) \\ &\implies \text{By induction}, x \in T^n(X) \text{ for all } n. \end{aligned}$$

Next we claim that

$$T^\infty(X) \text{ is closed:}$$

indeed, since T^n is upper semi-Fredholm for all n, $T^n(X)$ is closed and hence $T^\infty(X)$ is closed.

If S commutes with T, so that also $S(T^\infty(X)) \subset T^\infty(X)$, we shall write $\widetilde{S} : T^\infty(X) \to T^\infty(X)$. We claim that

$$\widetilde{T} : T^\infty(X) \to T^\infty(X) \text{ is onto.} \tag{1.12}$$

To see this, let $y \in T^\infty(X)$ and thus

$$\exists\ x_n \in T^n(X) \text{ such that } Tx_n = y \quad (n = 1, 2, \ldots).$$

Since $T^{-1}(0)$ is finite dimensional and $T^n(X) \supset T^{n+1}(X)$,

$$\exists n_0 \in \mathbb{N} \text{ such that } T^{-1}(0) \cap T^{n_0}(X) = T^{-1}(0) \cap T^n(X) \quad \text{for } n \geq n_0.$$

From the fact that $T^n(X) \subset T^{n_0}(X)$, we have

$$x_n - x_{n_0} \in T^{-1}(0) \cap T^{n_0}(X) = T^{-1}(0) \cap T^n(X) \subset T^n(X).$$

Hence

$$x_{n_0} \in \bigcap_{n \geq n_0} T^n(X) = T^\infty(X) \quad \text{and} \quad Tx_{n_0} = y,$$

which says that $\widetilde{T}$ is onto. This proves (1.12). Now observe

$$\dim (T-\lambda I)^{-1}(0) = \dim \widetilde{T-\lambda I}^{\,-1}(0) = \operatorname{index} \widetilde{T-\lambda I} = \operatorname{index} \widetilde{T} : \tag{1.13}$$

the first equality comes from (1.11), the second equality follows from the fact that $\beta(\widetilde{T-\lambda I}) \leq \beta(\widetilde{T}) = 0$ by Lemma 1.8.1, and the third equality follows the observation that $\widetilde{T}$ is semi-Fredholm. Since the right-hand side of (1.13) is independent of λ, $\alpha(T-\lambda I)$ is constant and hence also is $\beta(T-\lambda I)$.

If instead $\beta(T) < \infty$, apply the above argument with T^*. $\square$

Theorem 1.8.3. *Define*

$$U := \left\{ \lambda \in \mathbb{C} : \ T - \lambda I \text{ is semi-Fredholm} \right\}.$$

Then

- (i) *U is an open set;*
- (ii) *If C is a component of U then on C, with the possible exception of isolated points,*

  $$\alpha(T-\lambda I) \text{ and } \beta(T-\lambda I) \text{ have constant values } n_1 \text{ and } n_2, \text{ respectively.}$$

  At the isolated points,

  $$\alpha(T-\lambda I) > n_1 \quad \text{and} \quad \beta(T-\lambda I) > n_2.$$

Proof. (i) For $\lambda \in U$ apply Lemma 1.8.1 to $T - \lambda I$ in place of T.

(ii) The component C is open since any component of an open set in $\mathbb{C}$ is open. Let $\alpha(\lambda_0) = n_1$ be the smallest integer which is attained by

$$\alpha(\lambda) = \alpha(T-\lambda I) \quad \text{on } C.$$

Suppose $\alpha(\lambda') \neq n_1$. Since C is connected there exists an arc Γ lying in C with endpoints λ_0 and λ'. It follows from Theorem 1.8.2 and the fact that C is open that for each $\mu \in \Gamma$, there exists an open ball $S(\mu)$ in C such that

$$\alpha(\lambda) \text{ is constant on the set } S(\mu) \text{ with the point } \mu \text{ deleted.}$$

Since Γ is compact and connected there exist points $\lambda_1, \lambda_2, \cdots, \lambda_n = \lambda'$ on Γ such that

$$S(\lambda_0), S(\lambda_1), \ldots, S(\lambda_n) \text{ cover } \Gamma \quad \text{and} \quad S(\lambda_i) \cap S(\lambda_{i+1}) \neq \emptyset \ (0 \leq i \leq n-1) \quad (1.14)$$

We claim that $\alpha(\lambda) = \alpha(\lambda_0)$ on all of $S(\lambda_0)$. Indeed it follows from the Lemma 1.8.1 that

$$\alpha(\lambda) \leq \alpha(\lambda_0) \text{ for } \lambda \text{ sufficiently close to } \lambda_0.$$

Therefore, since $\alpha(\lambda_0)$ is the minimum of $\alpha(\lambda)$ on C,

$$\alpha(\lambda) = \alpha(\lambda_0) \text{ for } \lambda \text{ sufficiently close to } \lambda_0.$$

Since $\alpha(\lambda)$ is constant for all $\lambda \neq \lambda_0$ in $S(\lambda_0)$, which is $\alpha(\lambda_0)$. Now $\alpha(\lambda)$ is constant on the set $S(\lambda_i)$ with the point λ_i deleted $(1 \leq i \leq n)$. Hence it follows from (1.14) and the observation $\alpha(\lambda) = \alpha(\lambda_0)$ for all $\lambda \in S(\lambda_0)$ that $\alpha(\lambda) = \alpha(\lambda_0)$ for all $\lambda \neq \lambda'$ in $S(\lambda')$ and $\alpha(\lambda') > n_1$. The result just obtained can be applied to the adjoint. This completes the proof. □

1.9 The Riesz-Schauder (or Browder) Theory

An operator $T \in B(X)$ is said to be *quasinilpotent* if

$$\|T^n\|^{\frac{1}{n}} \longrightarrow 0$$

and is said to be *nilpotent* if

$$T^n = 0 \quad \text{for some } n.$$

An example for quasinilpotent but not nilpotent:

$$\begin{aligned} T : \ell^2 &\rightarrow \ell^2 \\ T(x_1, x_2, x_3, \ldots) &\longmapsto (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots). \end{aligned}$$

An example for quasinilpotent but neither nilpotent nor compact:

$$T = T_1 \oplus T_2 : \ell^2 \oplus \ell^2 \longrightarrow \ell^2 \oplus \ell^2,$$

where

$$\begin{aligned} T_1 &: (x_1, x_2, x_3, \ldots) \longmapsto (0, x_1, 0, x_3, 0, x_5, \ldots) \\ T_2 &: (x_1, x_2, x_3, \ldots) \longmapsto (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots). \end{aligned}$$

Remember that if $T \in B(X)$ we define $L_T, R_T \in B(B(X))$ by

$$L_T(S) := TS \quad \text{and} \quad R_T(S) := ST \quad \text{for } S \in B(X).$$

Lemma 1.9.1. *We have:*
(a) *L_T is 1-1 $\Longleftrightarrow$ T is 1-1;*
(b) *R_T is 1-1 $\Longleftrightarrow$ T is dense;*
(c) *L_T is bounded below $\Longleftrightarrow$ T is bounded below;*
(d) *R_T is bounded below $\Longleftrightarrow$ T is open.*

Proof. See [Be3]. □

Theorem 1.9.2. *If $T \in B(X)$, then*
(a) *T is nilpotent $\Longrightarrow$ T is neither 1-1 nor dense;*
(b) *T is quasinilpotent $\Longrightarrow$ T is neither bounded below nor open.*

Proof. By Lemma 1.9.1,

$$\begin{aligned}
\text{(a) } T \text{ is nilpotent} &\Longrightarrow T^{n+1} = 0 \neq T^n \\
&\Longrightarrow L_T(T^n) = R_T(T^n) = 0 \neq T^n \\
&\Longrightarrow L_T \text{ and } R_T \text{ are not 1-1} \\
&\Longrightarrow T \text{ is not 1-1 and not dense.}
\end{aligned}$$

$$\begin{aligned}
\text{(b) } T \text{ is quasinilpotent} &\Longrightarrow \forall\, \varepsilon > 0,\ \exists\, n \in \mathbb{N} \text{ such that } \|T^n\|^{\frac{1}{n}} \geq \varepsilon > \|T^{n+1}\|^{\frac{1}{n+1}} \\
&\Longrightarrow \|L_T(T^n)\| = \|R_T(T^n)\| < \varepsilon \|T^n\| \\
&\Longrightarrow L_T \text{ and } R_T \text{ are not bounded below} \\
&\Longrightarrow T \text{ is not bounded below and not open.}
\end{aligned}$$

□

We would remark that

$$\{\text{quasinilpotents}\} \subseteq \partial B^{-1}(X).$$

Observe that quasinilpotents of finite rank or cofinite rank are nilpotents.

Definition 1.9.3. An operator $T \in B(X)$ is said to be *quasipolar* [*polar*, resp.] if there is a projection P commuting with T for which T has a matrix representation

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} P(X) \\ P^{-1}(0) \end{bmatrix} \rightarrow \begin{bmatrix} P(X) \\ P^{-1}(0) \end{bmatrix},$$

where T_1 is invertible and T_2 is quasinilpotent [nilpotent, resp.]

Definition 1.9.4. An operator $T \in B(X)$ is said to be *simply polar* if there is $T' \in B(X)$ for which

$$T = TT'T \quad \text{with} \quad TT' = T'T$$

Proposition 1.9.5. *Simply polar operators are decomposably regular.*

Proof. Assume $T = TT'T$ with $TT' = T'T$. Then

$$T'' = T' + (1 - T'T) \implies \begin{cases} T = TT''T \\ (T'')^{-1} = T + (1 - T'T) \end{cases}.$$

□

Theorem 1.9.6. *If $T \in B(X)$ then*

$$T \text{ is quasipolar but not invertible} \iff 0 \in \operatorname{iso} \sigma(T)$$

Proof. ($\Rightarrow$) If T is quasipolar we may write

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} P(X) \\ P^{-1}(0) \end{bmatrix} \;\rightarrow\; \begin{bmatrix} P(X) \\ P^{-1}(0) \end{bmatrix},$$

where T_1 is invertible and T_2 is quasinilpotent. Thus for sufficiently small $\lambda \neq 0$, $T_1 - \lambda I$ and $T_2 - \lambda I$ are both invertible, which implies that $0 \in \operatorname{iso} \sigma(T)$

($\Leftarrow$) If $0 \in \operatorname{iso} \sigma(T)$, construct open discs D_1 and D_2 such that D_1 contains 0, D_2 contains the spectrum $\sigma(T)$ and $D_1 \cap D_2 = \emptyset$. If we define $f : D_1 \cup D_2 \longrightarrow \mathbb{C}$ by setting

$$f(\lambda) = \begin{cases} 0 & \text{on} \quad D_1 \\ 1 & \text{on} \quad D_2 \end{cases}$$

then f is analytic on $D_1 \cup D_2$ and $f(\lambda)^2 = f(\lambda)$. Observe that

$$P = P_{D_2} = f(T) = \frac{1}{2\pi i} \int_{\partial D_2} (\lambda - T)^{-1} d\lambda$$

and $PT = TP$. Thus we may write

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : P(X) \oplus P^{-1}(0) \;\longrightarrow\; P(X) \oplus P^{-1}(0),$$

where $\sigma(T_1) = \sigma(T) \setminus \{0\}$ and $\sigma(T_2) = \{0\}$. Therefore T_1 is invertible and T_2 is quasinilpotent; so that T is quasipolar.

□

Theorem 1.9.7. *If* $T \in B(X)$ *then*

$$T \text{ is simply polar} \iff T(X) = T^2(X), \quad T^{-1}(0) = T^{-2}(0)$$

Proof. ($\Rightarrow$) Observe

$$\begin{aligned} T(X) &= TT'T(X) = T^2T'(X) \subseteq T^2(X) \subseteq TX; \\ T^{-1}(0) &= (TT'T)^{-1}(0) = (T'T^2)^{-1}(0) \supseteq T^{-2}(0) \supseteq T^{-1}(0). \end{aligned}$$

($\Leftarrow$) (i) $x \in TX \cap T^{-1}(0) \;\Rightarrow\; x = Ty$ for some $y \in X$ and $Tx = 0$
$\Rightarrow\; T^2 y = 0 \;\Rightarrow\; y \in T^{-2}(0) = T^{-1}(0)$
$\Rightarrow\; Ty = 0 \;\Rightarrow\; x = 0,$
which gives $TX \cap T^{-1}(0) = \{0\}$.

(ii) By assumption, $T(T(X)) = T(X)$. Let $T_1 := T|_{T(X)}$, so that $T_1(X) = T^2(X) = T(X)$. Thus for all $x \in X$,

$$\exists\, y \in T(X) \text{ such that } Tx = T_1 y = Ty.$$

Define $z = x - y$, and hence $z \in T^{-1}(0)$. Thus $X = T(X) + T^{-1}(0)$. In particular, $T(X)$ is closed by Theorem 1.4.2, so that

$$X = T(X) \oplus T^{-1}(0).$$

Therefore we can find a projection $P \in B(X)$ for which

$$P(X) = T(X) \quad \text{and} \quad P^{-1}(0) = T^{-1}(0).$$

We thus write

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} P(X) \\ P^{-1}(0) \end{bmatrix} \to \begin{bmatrix} P(X) \\ P^{-1}(0) \end{bmatrix},$$

where T_1 is invertible because $T_1 := T|_{TX}$ is 1-1 and onto since $T(X) = T^2(X)$. If we put

$$T' = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

then $TT'T = T$ and

$$TT' = T'T = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = P,$$

which says that T is simply polar.

□

Theorem 1.9.8. *If $T \in B(X)$ then*

$$T \textit{ is polar} \iff T^n \textit{ is simply polar for some } n \in \mathbb{N}$$

Proof. (⇒) If T is polar then we can write $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$ with T_1 invertible and T_2 nilpotent. So $T^n = \begin{bmatrix} T_1^n & 0 \\ 0 & 0 \end{bmatrix}$, where n is the nilpotency of T_2. If we put $S = \begin{bmatrix} T_1^{-n} & 0 \\ 0 & I \end{bmatrix}$, then $T^n S T^n = T^n$ and $ST^n = T^n S$.

(⇐) If T^n is simply polar then $X = T^n(X) \oplus T^{-n}(0)$. Observe that since T^n is simply polar we have

$$\begin{aligned} T(T^n X) &= T^{n+1}(X) \supseteq T^{2n}(X) = T^n(X) \\ T(T^{-n}(0)) &\subseteq T^{-n+1}(0) \subseteq T^{-n}(0) \end{aligned}$$

Thus we see that $T|_{T^n(X)}$ is 1-1 and onto, so that invertible. Thus we may write

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : T^n(X) \oplus T^{-n}(0) \longrightarrow T^n(X) \oplus T^{-n}(0),$$

where $T_1 = T|_{T^n(X)}$ is invertible and $T_2 = T|_{T^{-n}(0)}$ is nilpotent with nilpotency n. Therefore T is polar. □

The following is an immediate result of Theorem 1.9.8 :

Corollary 1.9.9. *If* $T \in B(X)$ *then*

$$T \text{ is polar } \iff \text{ascent}\,(T) = \text{descent}\,(T) < \infty$$

Corollary 1.9.10. *If* $S, T \in B(X)$ *with* $ST = TS$*, then*

$$S \text{ and } T \text{ are polar } \implies ST \text{ is polar.}$$

Proof. Suppose $S^n(X) = S^{n+1}(X)$ and $T^n(X) = T^{n+1}(X)$. Then

$$\begin{aligned}(ST)^{mn+1}(X) &= S^{mn+1}T^{mn+1}(X) = S^{mn+1}T^{mn}(X) = T^{mn}S^{mn+1}(X)\\ &= T^{mn}S^{mn}(X) = (ST)^{mn}(X)\end{aligned}$$

Similarly,

$$(ST)^{-p-1}(0) = (ST)^{-p}(0).$$

□

Definition 1.9.11. An operator $T \in B(X)$ is called a *Browder* (or *Riesz-Schauder*) *operator* if T is Fredholm and quasipolar.

If T is Fredholm then by the remark above Definition 1.9.3,

$$T \text{ is quasipolar } \iff T \text{ is polar.}$$

Thus we have

$$T \text{ is Browder } \iff T \text{ is Fredholm and polar.}$$

Theorem 1.9.12. *If* $T \in B(X)$*, the following are equivalent:*
(a) T *is Browder, but not invertible;*
(b) T *is Fredholm and* $0 \in \text{iso}\,\sigma(T)$*;*
(c) T *is Weyl and* $0 \in \text{iso}\,\sigma(T)$*;*
(d) T *is Fredholm and* $\text{ascent}\,(T) = \text{descent}\,(T) < \infty$*.*

Proof. (a) ⇔ (b) : Theorem 1.9.6
(b) ⇔ (c) : From the continuity of the index
(b) ⇔ (d) : From Corollary 1.9.9. □

Theorem 1.9.13. *If* $K \in B(X)$*, then*

$$K \text{ is compact } \implies I + K \text{ is Browder.}$$

Proof. From the spectral theory of the compact operators,

$$-1 \in \operatorname{iso} \sigma(K) \quad (\text{in fact, } \lambda \neq 0 \Rightarrow \lambda \notin \operatorname{acc} \sigma(K)),$$

which gives

$$0 \in \operatorname{iso} \sigma(I+K).$$

From Corollary 1.4.6, $I+K$ is Fredholm. Now Theorem 1.9.12 says that $I+K$ is Browder.

□

Theorem 1.9.14. (Riesz-Schauder Theorem). *If* $T \in B(X)$ *then*

T is Browder $\iff$ $T = S+K$, where S is invertible and K is compact with $SK = KS$.

Proof. ($\Rightarrow$) If T is Browder then it is polar, so that we can write

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix},$$

where T_1 is invertible and T_2 is nilpotent. Since T is Fredholm, T_2 is also Fredholm. If we put

$$S = \begin{bmatrix} T_1 & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 0 \\ 0 & T_1 - I \end{bmatrix},$$

then evidently $T_2 - I$ is of finite rank. Thus S is invertible and K is of finite rank. Further,

$$T = S + K \quad \text{and} \quad SK = KS.$$

($\Leftarrow$) Suppose $T = S + K$ and $SK = KS$. Since, by Theorem 1.9.13, $I + S^{-1}K$ is Browder, so that $I + S^{-1}K$ is Fredholm and polar. Therefore, by Theorem 1.5.2 and Corollary 1.9.10, $T = S(I + S^{-1}K)$ is Fredholm and polar, and hence Browder. Here, note that S and $I + S^{-1}K$ commutes. □

Remark 1.9.15. If $S, T \in B(X)$ and $ST = TS$ then
(a) S, T are Browder $\iff$ ST is Browder;
(b) S is Browder and T is compact $\Longrightarrow$ $S + T$ is Browder.

Example 1.9.16. There exists a Weyl operator which is not Browder.

Proof. Put $T = \begin{bmatrix} U & 0 \\ 0 & U^* \end{bmatrix} : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$, where U is the unilateral shift. Evidently, T is Fredholm and $\operatorname{index} T = \operatorname{index} U + \operatorname{index} U^* = 0$, which says that T is Weyl. However, $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$; so that $0 \notin \operatorname{iso} \sigma(T)$, which implies that T is not Browder.

□

1.10 Essential Spectra

If $T \in B(X)$ we define:

(a) The *essential spectrum* of $T := \sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Fredolm}\}$

(b) The *Weyl spectrum* of $T := \omega(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\}$

(c) The *Browder spectrum* of $T := \sigma_b(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Browder}\}$

Evidently, $\sigma_e(T)$, $\omega(T)$ and $\sigma_b(T)$ are all compact;

$$\sigma_e(T) \subset \omega(T) \subset \sigma_b(T);$$

these are nonempty if $\dim X = \infty$.

Theorem 1.10.1. *If $T \in B(X)$ then*

(a) $\sigma(T) = \sigma_e(T) \cup \sigma_p(T) \cup \sigma_{com}(T)$;

(b) $\sigma(T) = \omega(T) \cup \big(\sigma_p(T) \cap \sigma_{com}(T)\big)$;

(c) $\sigma_b(T) = \sigma_e(T) \cup acc\, \sigma(T)$,

where $\sigma_{com}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not have dense range}\}$.

Proof. Immediate follow from definitions. □

Definition 1.10.2. We shall write

$$P_{00}(T) = \text{iso}\, \sigma(T) \backslash \sigma_e(T)$$

for the *Riesz points* of $\sigma(T)$. Evidently, $\lambda \in P_{00}(T)$ means that $T - \lambda I$ is Browder, but not invertible.

Lemma 1.10.3. *If Ω is locally connected and $H,\ K \subset \Omega$, then*

$$\partial K \subseteq H \cup \text{iso}\, K \implies K \subset \eta H \cup \text{iso}\, K$$

Proof. See [Har4]. □

Theorem 1.10.4. *If $T \in B(X)$ then*

(a) $\partial\sigma(T) \backslash \sigma_e(T) \subseteq iso\, \sigma(T)$;

(b) $\sigma(T) \subseteq \eta\sigma_e(T) \cup P_{00}(T)$

Proof. (a) This is an immediate consequence of the Punctured Neighborhood Theorem.

(b) From (a) and Lemma 1.10.3,

$$\begin{aligned} \sigma(T) &\subseteq \eta\sigma_e(T) \cup \text{iso}\, \sigma(T) \\ &= \eta\sigma_e(T) \cup P_{00}(T) \end{aligned}$$

by the fact that if $\lambda \notin \eta\sigma_e(T)$ and $\lambda \in \text{iso}\, \sigma(T)$, then $T - \lambda I$ is Fredholm and $\lambda \in \text{iso}\, \sigma(T)$ thus $T - \lambda I$ is Browder. □

1.11 Spectral Mapping Theorems

Recall the Calkin algebra $B(X)/K(X)$. The *Calkin homomorphism* π is defined by

$$\begin{aligned} \pi : B(X) &\longrightarrow B(X)/K(X) \\ \pi(T) &= T + K(X). \end{aligned}$$

Evidently, by the Atkinson's Theorem,

$$T \text{ is Fredholm } \iff \pi(T) \text{ is invertible.}$$

Theorem 1.11.1. *If $T \in B(X)$ and f is analytic in a neighborhood of $\sigma(T)$, then*

$$f(\sigma_e(T)) = \sigma_e(f(T))$$

Proof. Since $f(\pi(T)) = f(T + K(X)) = f(T) + K(X) = \pi(f(T))$ it follows that

$$f(\sigma_e(T)) = f(\sigma(\pi(T))) = \sigma(f(\pi(T))) = \sigma(\pi(f(T))) = \sigma_e(f(T)).$$

□

Theorem 1.11.2. *If $T \in B(X)$ and f is analytic in a neighborhood of $\sigma(T)$, then*

$$f(\sigma_b(T)) = \sigma_b(f(T))$$

Proof. Since by the analyticity of f, $f(\operatorname{acc} K) = \operatorname{acc} f(K)$, it follows that

$$\begin{aligned} f(\sigma_b(T)) &= f(\sigma_e(T) \cup \operatorname{acc}\sigma(T)) \\ &= f(\sigma_e(T)) \cup f(\operatorname{acc}\sigma(T)) \\ &= \sigma_e(f(T)) \cup \operatorname{acc}\sigma(f(T)) \\ &= \sigma_b(f(T)). \end{aligned}$$

□

Theorem 1.11.3. *If $T \in B(X)$ and p is a polynomial then*

$$\omega(p(T)) \subseteq p(\omega(T)).$$

Proof. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$; thus $p(z) = c_0(z - \alpha_1) \cdots (z - \alpha_n)$. Then

$$p(T) = c_0(T - \alpha_1 I) \cdots (T - \alpha_n I).$$

We now claim that

$$\begin{aligned} 0 \notin p(\omega(T)) &\Longrightarrow c_0(z - \alpha_1) \cdots (z - \alpha_n) \neq 0 \quad \text{for each } \lambda \in \omega(T) \\ &\Longrightarrow \lambda \neq \alpha_i \quad \text{for each } \lambda \in \omega(T) \\ &\Longrightarrow T - \alpha_i I \text{ is Weyl for each } i = 1, 2, \ldots n \\ &\Longrightarrow c_0(T - \alpha_1 I) \cdots (T - \alpha_n I) \text{ is Weyl} \\ &\Longrightarrow 0 \notin \omega(p(T)) \end{aligned}$$

□

In fact, we can show that $\omega(f(T)) \subseteq f(\omega(T))$ for any analytic function f in a neighborhood of $\sigma(T)$.

The inclusion of Theorem 1.11.3 may be proper. For example, if U is the unilateral shift, consider

$$T = \begin{bmatrix} U+I & 0 \\ 0 & U^*-I \end{bmatrix} : \ell^2 \oplus \ell^2 \longrightarrow \ell^2 \oplus \ell^2.$$

Then

$$\omega(T) = \sigma(T) = \big\{z \in \mathbb{C} : |1+z| \le 1\big\} \cup \big\{z \in \mathbb{C} : |1-z| \le 1\big\}.$$

Let $p(z) = (z+1)(z-1)$. Then

$$p(\omega(T)) \text{ is a cardioid containing } 0.$$

Therefore $0 \in p(\omega(T))$. However

$$p(T) = (T+I)(T-I) = \begin{bmatrix} U+2I & 0 \\ 0 & U^* \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U^*-2I \end{bmatrix},$$

so that $\operatorname{index}(p(T)) = \operatorname{index} U^* + \operatorname{index} U = 0$, which implies $0 \notin \omega(p(T))$. Therefore

$$p(\omega(T)) \nsubseteq \omega(p(T)).$$

1.12 The Continuity of Spectra

Let σ_n be a sequence of compact subsets of $\mathbb{C}$.

(a) The *limit inferior*, $\liminf \sigma_n$, is the set of all $\lambda \in \mathbb{C}$ such that every neighborhood of λ has a nonempty intersection with all but finitely many σ_n.

(b) The *limit superior*, $\limsup \sigma_n$, is the set of all $\lambda \in \mathbb{C}$ such that every neighborhood of λ intersects infinitely many σ_n.

(c) If $\liminf \sigma_n = \limsup \sigma_n$ then $\lim \sigma_n$ is said to be exist and is the common limit.

A mapping $\mathcal{T}$ on $B(X)$ whose values are compact subsets of $\mathbb{C}$ is said to be *upper semi-continuous* at T when

$$T_n \longrightarrow T \implies \limsup \mathcal{T}(T_n) \subset \mathcal{T}(T)$$

and to be *lower semi-continuous* at T when

$$T_n \longrightarrow T \implies \mathcal{T}(T) \subset \liminf \mathcal{T}(T_n).$$

If $\mathcal{T}$ is both upper and lower semi-continuous, then it is said to be continuous.

Example 1.12.1. The spectrum $\sigma : T \longmapsto \sigma(T)$ is not continuous in general: for example, if

$$T_n := \begin{bmatrix} U & \frac{1}{n}(I - UU^*) \\ 0 & U^* \end{bmatrix} \quad \text{and} \quad T := \begin{bmatrix} U & 0 \\ 0 & U^* \end{bmatrix}$$

then $\sigma(T_n) = \partial\mathbb{D}$, $\sigma(T) = \mathbb{D}$, and $T_n \longrightarrow T$.

Proposition 1.12.2. *σ is upper semi-continuous.*

Proof. Suppose $T^n \to T$ and $\lambda \in \limsup \sigma(T_n)$. Then there exists $\lambda_n \in \limsup \sigma(T_n)$ so that $\lambda_{n_k} \to \lambda$. Since $T_{n_k} - \lambda_{n_k} I$ is singular and $T_{n_k} - \lambda_{n_k} I \longrightarrow T - \lambda I$, it follows that $T - \lambda I$ is singular; therefore $\lambda \in \sigma(T)$. □

Theorem 1.12.3. *σ is continuous on the set of all hyponormal operators.*

Proof. Let T_n, T be hyponormal operators such that $T^n \to T$ in norm. We want to prove that

$$\sigma(T) \subset \liminf \sigma(T_n).$$

Assume $\lambda \notin \liminf \sigma(T_n)$. Then there exists a neighborhood $N(\lambda)$ of λ such that it does not intersect infinitely many $\sigma(T_n)$. Thus we can choose a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ such that for some $\varepsilon > 0$,

$$\operatorname{dist}\Big(\lambda, \sigma(T_{n_k})\Big) > \varepsilon.$$

Since T_{n_k} is hyponormal, it follows that

$$\operatorname{dist}(\lambda, \sigma(T_{n_k})) = \min_{\mu\in\sigma(T_{n_k}-\lambda)} |\mu| = \frac{1}{\max\limits_{\mu\in\sigma((T_{n_k}-\lambda)^{-1})} |\mu|} = \frac{1}{\|(T_{n_k}-\lambda)^{-1}\|},$$

where the second equality follows from the observation

$$\sigma(T^{-1}) = \left\{\frac{1}{z} : z \in \sigma(T)\right\}$$

because if $f(z) = \frac{1}{z}$ then $\sigma(T^{-1}) = \sigma(f(T)) = f(\sigma(T)) = \{\frac{1}{z} : z \in \sigma(T)\}$ and the last equality uses the fact that $(T_{n_k} - \lambda I)^{-1}$ is normaloid. So $\|(T_{n_k} - \lambda I)^{-1}\| < \frac{1}{\varepsilon}$. We thus have

$$\begin{aligned}\|(T_{n_k}-\lambda I)^{-1} - (T_{n_l}-\lambda I)^{-1}\| &= \|(T_{n_k}-\lambda I)^{-1}\Big\{(T_{n_k}-\lambda I) - (T_{n_l}-\lambda I)\Big\} - (T_{n_l}-\lambda I)^{-1}\| \\ &\le \|(T_{n_k}-\lambda I)^{-1}\| \cdot \|T_{n_l} - T_{n_k}\| \cdot \|(T_{n_l}-\lambda I)^{-1}\| \\ &< \frac{1}{\varepsilon^2}\|T_{n_l} - T_{n_k}\|.\end{aligned}$$

Since $T_{n_k} \to T$, it follows that $\{(T_{n_k} - \lambda I)^{-1}\}$ converges, to some operator B, say. Therefore

$$\begin{aligned}(T-\lambda I)B &= \lim(T_{n_k}-\lambda I)\cdot \lim(T_{n_k}-\lambda I)^{-1} \\ &= \lim(T_{n_k}-\lambda I)(T_{n_k}-\lambda I)^{-1} = 1.\end{aligned}$$

Similarly, $B(T - \lambda I) = 1$ and hence $\lambda \notin \sigma(T)$. ☐

Lemma 1.12.4. *Let A be a commutative Banach algebra. If $x \in A$ is not invertible and $\|y - x\| < \varepsilon$, then there exists λ such that $y - \lambda$ is not invertible and $|\lambda| < \varepsilon$.*

Proof. Since x is not invertible, it generates an ideal $\neq A$. Thus there exists a maximal ideal M containing x. So $z \in M \implies z$ is not invertible. Since $A/M \cong \mathbb{F}$, $\lambda\cdot 1 \in y+M$ for some y. Thus $y - \lambda \cdot 1 \in M$. Since $x \in M$ we have $y - x - \lambda \cdot 1 \in M$, so that $\lambda \in \sigma(y - x)$. Finally, $|\lambda| \le \|y - x\| < \varepsilon$. ☐

Theorem 1.12.5. *If in a Banach algebra A, $x_i \to x$ and $x_i x = x x_i$ for all i, then $\lim \sigma(x_i) = \sigma(x)$.*

Proof. Let B be the algebra generated by 1, x, and x_i. Then $(x-\mu)^{-1}$ and $(x_i-\mu)^{-1}$ are commutative whenever they exist. Let $\lambda \in \sigma(x)$, i.e., $x - \lambda$ is not invertible. By Lemma 1.12.4, there exists N such that

$$i > N \implies \sigma(x_i - \lambda) \cap N_\varepsilon(0) \neq \emptyset.$$

So $0 \in \liminf \sigma(x_i - \lambda)$, or $\lambda \in \liminf \sigma(x_i)$, so that

$$\sigma(x) \subseteq \liminf \sigma(x_i) \subseteq \limsup \sigma(x_i) \subseteq \sigma(x).$$

☐

Theorem 1.12.6. *ω is upper semi-continuous.*

Proof. We want to prove that

$$\limsup \omega(T_n) \subset \omega(T) \quad \text{if } T_n \to T.$$

Let $\lambda \notin \omega(T)$, so $T - \lambda I$ is Weyl. Since the set of Weyl operators forms an open set,

$$\exists\, \eta > 0 \text{ such that } \|T - \lambda I - S\| < \eta \implies S \text{ is Weyl.}$$

Let N be such that

$$\|(T - \lambda I) - (T_n - \lambda I)\| < \frac{\eta}{2} \quad \text{for } n \geq N.$$

Let $V = B(\lambda\,; \frac{\eta}{2})$. Then for $\mu \in V,\ n \geq N$,

$$\|(T - \lambda I) - (T_n - \mu I)\| < \eta,$$

so that $T_n - \mu I$ is Weyl, which implies that $\lambda \notin \limsup \omega(T_n)$. □

Theorem 1.12.7. *Let $T_n \to T$. If $T_nT = TT_n$ for all n, then $\lim \omega(T_n) = \omega(T)$.*

Proof. In view of Theorem 1.12.6, it suffices to show that

$$\omega(T) \subseteq \liminf \omega(T_n) \tag{1.15}$$

Observe that $\pi(T_n)\pi(T) = \pi(T)\pi(T_n)$ and hence by Theorem 1.12.5, $\lim \sigma_e(T_n) = \sigma_e(T)$. Towards (1.15), suppose $\lambda \notin \liminf \omega(T_n)$. So there exists a neighborhood $V(x)$ which does not intersect infinitely many $\omega(T_n)$. Since $\sigma_e(T_n) \subset \omega(T_n)$, V does not intersect infinitely many $\sigma_e(T_n)$, i.e., $\lambda \notin \lim \sigma_e(T_n) = \sigma_e(T)$. This shows that $T - \lambda I$ is Fredholm. By the continuity of index, $T - \lambda I$ is Weyl, i.e., $\lambda \notin \omega(T)$. □

Theorem 1.12.8. *If S and T are commuting hyponormal operators then*

$$S,\ T \text{ are Weyl} \iff ST \text{ is Weyl.}$$

Hence if f is analytic in a neighborhood of $\sigma(T)$, then

$$\omega(f(T)) = f(\omega(T)).$$

Proof. See [LeL2]. □

1.13 Comments and Problems

Let H be an infinite dimensional separable Hilbert space. An operator $T \in B(H)$ is called *a Riesz operator* if $\sigma_e(T) = 0$. If $T \in B(H)$ then the *West decomposition theorem* [Wes] says that

$$T \text{ is Riesz } \Longleftrightarrow T = K + Q \text{ with compact } K \text{ and quasinilpotent } Q:$$

this is equivalent to the following: if $Q_{B(H)}$ and $Q_{C(H)}$ denote the sets of quasinilpotents of $B(H)$ and $C(H)$, respectively, then

$$\pi\left(Q_{B(H)}\right) = Q_{C(H)}, \tag{1.16}$$

where $C(H) = B(H)/K(H)$ is the Calkin algebra and π denotes the Calkin homomorphism. It remains still open whether the West decomposition theorem survives in the Banach space setting.

Problem 1.1. *Is the equality (1.16) true if H is a Banach space ?*

Suppose A is a Banach algebra with identity 1: we shall write A^{-1} for the invertible group of A and A_0^{-1} for the connected components of the identity in A^{-1}. It was [Har3] known that

$$A_0^{-1} := \text{Exp}(A) = \{e^{c_1}e^{c_2}\cdots e^{c_k} : \ k \in \mathbb{N},\ c_i \in A\}.$$

Evidently, $\text{Exp}\,(A)$ is open, relatively closed in A^{-1}, connected and a normal subgroup. Write

$$\kappa(A) := A^{-1}/\text{Exp}\,(A)$$

for the *abstract index group*. The exponential spectrum $\epsilon(a)$ of $a \in A$ is defined by

$$\epsilon(a) := \{\lambda \in \mathbb{C} : \ a - \lambda \notin \text{Exp}\,(A)\}.$$

Clearly,

$$\partial\epsilon(a) \subset \sigma(a) \subset \epsilon(a).$$

If $A = B(H)$ then $\epsilon(a) = \sigma(a)$. We have known that $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$. However we were not able to answer to the following:

Problem 1.2. *If A is a Banach algebra and $a, b \in A$, does it follow that*

$$\epsilon(ab) \setminus \{0\} = \epsilon(ba) \setminus \{0\}\,?$$

Chapter 2

Weyl Theory

2.1 Introduction

In 1909, writing about differential equations, Hermann Weyl noticed something about the essential spectrum of a self adjoint operator on Hilbert space: when you take it away from the spectrum, you are left with the isolated eigenvalues of finite multiplicity. This was soon generalized to normal operators, and then to more and more classes of operators, bounded and unbounded, on Hilbert and on Banach spaces.

The *spectrum* $\sigma(T)$ of a bounded linear operator T on a complex Banach space X is of course the set of those complex numbers for which $T - \lambda I$ does not have an everywhere defined two-sided inverse: this concept extends at once to the spectrum $\sigma_A(a)$ of a Banach algebra element $a \in A$. Thus the *Fredholm essential spectrum* $\sigma_e(T)$ is the spectrum of the coset $T + K(X)$ of the operator $T \in B(X)$ in the Calkin algebra $B(X)/K(X)$. Equivalently $\lambda \in \mathbb{C}$ is excluded from the spectrum $\sigma(T)$ if and only if operator $T - \lambda I$ is one one and onto, and is excluded from the essential spectrum $\sigma_e(T)$ if and only if the operator $T - \lambda I$ has finite dimensional null space and range of finite co dimension.

The Fredholm essential spectrum is contained in the larger *Weyl spectrum*, which also includes points $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is Fredholm but with non *zero index*: the two finite dimensions involved are unequal. Equivalently, $T - \lambda I \notin B(X)^{-1} + K(X)$ cannot be expressed as the sum of an invertible and a compact operator. What is relevant here is that for self adjoint and more general normal operators the Weyl and the Fredholm spectra coincide: every normal Fredholm operator has index zero. Thus while the original Weyl observation of 1909 may have seemed to subtract the Fredholm essential spectrum from the spectrum, it can equally be interpreted as subtracting the Weyl essential spectrum. For non normal operators it is this modified version that seems to be the property that is of interest. For a linear operator on a Banach space the most obvious points of its spectrum are the *eigenvalues* $\pi_0(T)$, collecting $\lambda \in \mathbb{C}$ for which $T - \lambda I$ fails to be one-one. As is familiar from matrix theory, in finite dimensions this is all of the spectrum. In a sense therefore Weyl's theorem seems to be suggesting that for nice operators the spectrum splits into a finite dimensional component and a component modulo finite dimensions. Weyl's theorem asks not just that the spectrum split into Fredholm spectra and eigenvalues: it wants the spectrum to divide into Weyl spectrum and eigenvalues which are both topologically isolated in the spectrum, and geometrically of finite multiplicity, with finite dimensional eigenspaces.

2.2 Weyl's Theorem

If $T \in B(X)$ write $\pi_{0f}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0i}(T)$ for the eigenvalues of infinite multiplicity; $N(T)$ and $R(T)$ for the null space and the range of T, respectively. If we write $\operatorname{iso} K = K \setminus \operatorname{acc} K$, and ∂K for the topological boundary of K, and

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \dim N(T - \lambda I) < \infty\} \tag{2.1}$$

for the isolated eigenvalues of finite multiplicity, and ([Har4])

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T) \tag{2.2}$$

for the *Riesz points* of $\sigma(T)$, then by the punctured neighborhood theorem, i.e., $\partial\, \sigma(T) \setminus \sigma_e(T) \subseteq \operatorname{iso} \sigma(T)$ (cf. [Har4], [HaL1]),

$$\operatorname{iso} \sigma(T) \setminus \sigma_e(T) = \operatorname{iso} \sigma(T) \setminus \omega(T) = p_{00}(T) \subseteq \pi_{00}(T). \tag{2.3}$$

H. Weyl [We] examined the spectra of all compact perturbations $T+K$ of a single hermitian operator T and discovered that $\lambda \in \sigma(T+K)$ for every compact operator K if and only if λ is not an isolated eigenvalue of finite multiplicity in $\sigma(T)$. Today this result is known as Weyl's theorem: that is, we say that *Weyl's theorem holds for* $T \in B(X)$ if there is equality

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T). \tag{2.4}$$

In this section we explore the class of operators satisfying Weyl's theorem.

If $T \in B(X)$, write $r(T)$ for the spectral radius of T. It is familiar that $r(T) \le ||T||$. An operator T is called *normaloid* if $r(T) = ||T||$ and *isoloid* if $\operatorname{iso} \sigma(T) \subseteq \pi_0(T)$. If X is a Hilbert space, an operator $T \in B(X)$ is called *reduction-isoloid* if the restriction of T to any reducing subspace is isoloid.

Let X be a Hilbert space and suppose that $T \in B(X)$ is reduced by each of its finite-dimensional eigenspaces. If

$$\mathfrak{M} := \bigvee \{N(T - \lambda I) : \ \lambda \in \pi_{0f}(T)\},$$

then $\mathfrak{M}$ reduces T. Let $T_1 := T|\mathfrak{M}$ and $T_2 := T|\mathfrak{M}^\perp$. Then we have ([Be2, Proposition 4.1]) that

(i) T_1 is a normal operator with pure point spectrum;
(ii) $\pi_0(T_1) = \pi_{0f}(T)$;
(iii) $\sigma(T_1) = \operatorname{cl} \pi_0(T_1)$;
(iv) $\pi_0(T_2) = \pi_0(T) \setminus \pi_{0f}(T) = \pi_{0i}(T)$.

In this case, S.Berberian ([Be2, Definition 5.4]) defined

$$\tau(T) := \sigma(T_2) \cup \operatorname{acc} \pi_{0f}(T). \tag{2.5}$$

We shall call $\tau(T)$ the *Berberian spectrum* of T. S. Berberian has also shown that $\tau(T)$ is a nonempty compact subset of $\sigma(T)$. We can, however, show that Weyl spectra, Browder spectra, and Berberian spectra all coincide for operators reduced by each of its finite-dimensional eigenspaces:

Theorem 2.2.1. *If X is a Hilbert space and $T \in B(X)$ is reduced by each of its finite-dimensional eigenspaces then*

$$\tau(T) = \omega(T) = \sigma_b(T). \tag{2.6}$$

Proof. Let $\mathfrak{M}$ be the closed linear span of the eigenspaces $N(T-\lambda I)$ $(\lambda \in \pi_{0f}(T))$ and write

$$T_1 := T|\mathfrak{M} \quad \text{and} \quad T_2 := T|\mathfrak{M}^{\perp}.$$

From the preceding arguments it follows that T_1 is normal, $\pi_0(T_1) = \pi_{0f}(T)$ and $\pi_{0f}(T_2) = \emptyset$. For (2.6) it will be shown that

$$\omega(T) \subseteq \tau(T) \subseteq \sigma_b(T) \tag{2.7}$$

and

$$\sigma_b(T) \subseteq \omega(T). \tag{2.8}$$

For the first inclusion of (2.7) suppose $\lambda \in \sigma(T) \setminus \tau(T)$. Then $T_2 - \lambda I$ is invertible and $\lambda \in \operatorname{iso} \pi_0(T_1)$. Since also $\pi_0(T_1) = \pi_{0f}(T_1)$, we have that $\lambda \in \pi_{00}(T_1)$. But since T_1 is normal, it follows that $T_1 - \lambda I$ is Weyl and hence so is $T - \lambda I$. This proves the first inclusion. For the second inclusion of (2.7) suppose $\lambda \in \sigma(T) \setminus \sigma_b(T)$. Thus $T - \lambda I$ is Browder but not invertible. Observe that the following equality holds with no other restriction on either R or S:

$$\sigma_b(R \oplus S) = \sigma_b(R) \cup \sigma_b(S) \quad \text{for each } R \in B(X_1) \text{ and } S \in B(X_2). \tag{2.9}$$

Indeed if $\lambda \in \operatorname{iso} \sigma(R \oplus S)$ then λ is either an isolated point of the spectra of direct summands or a resolvent element of direct summands, so that if $R - \lambda I$ and $S - \lambda I$ are Fredholm then by (2.3), λ is either a Riesz point or a resolvent element of direct summands, which implies that $\sigma_b(R) \cup \sigma_b(S) \subseteq \sigma_b(R \oplus S)$, and the reverse inclusion is evident. From this we can see that $T_1 - \lambda I$ and $T_2 - \lambda I$ are both Browder. But since $\pi_{0f}(T_2) = \emptyset$, it follows that $T_2 - \lambda I$ is one-one and hence invertible. Therefore $\lambda \in \pi_{00}(T_1) \setminus \sigma(T_2)$, which implies that $\lambda \notin \tau(T)$. This proves the second inclusion of (2.7). For (2.8) suppose $\lambda \in \sigma(T) \setminus \omega(T)$ and hence $T - \lambda I$ is Weyl but not invertible. Observe that if X_1 is a Hilbert space and if an operator $R \in B(X_1)$ satisfies the equality $\omega(R) = \sigma_e(R)$, then

$$\omega(R \oplus S) = \omega(R) \cup \omega(S) \quad \text{for each Hilbert space } X_2 \text{ and } S \in B(X_2): \tag{2.10}$$

this follows from the fact that the index of a direct sum is the sum of the indices

$$\operatorname{index}(R \oplus S - \lambda(I \oplus I)) = \operatorname{index}(R - \lambda I) + \operatorname{index}(S - \lambda I)$$

whenever $\lambda \notin \sigma_e(R \oplus S) = \sigma_e(R) \cup \sigma_e(S)$. Since T_1 is normal, applying the equality (2.10) to T_1 in place of R gives that $T_1 - \lambda I$ and $T_2 - \lambda I$ are both Weyl. But since

$\pi_{0f}(T_2) = \emptyset$, we must have that $T_2 - \lambda I$ is invertible and therefore $\lambda \in \sigma(T_1) \setminus \omega(T_1)$. Thus from Weyl's theorem for normal operators we can see that $\lambda \in \pi_{00}(T_1)$ and hence $\lambda \in \operatorname{iso} \sigma(T_1) \cap \rho(T_2)$, which by (2.3), implies that $\lambda \notin \sigma_b(T)$. This proves (2.8) and completes the proof. ☐

As applications of Theorem 2.2.1 we will give several corollaries below.

Corollary 2.2.2. *If X is a Hilbert space and $T \in B(X)$ is reduced by each of its finite-dimensional eigenspaces then $\sigma(T) \setminus \omega(T) \subseteq \pi_{00}(T)$.*

Proof. This follows at once from Theorem 2.2.1. ☐

Weyl's theorem is not transmitted to dual operators: for example if $T : \ell^2 \to \ell^2$ is the unilateral weighted shift defined by

$$Te_n = \frac{1}{n+1} e_{n+1} \quad (n \geq 0), \tag{2.11}$$

then $\sigma(T) = \omega(T) = \{0\}$ and $\pi_{00}(T) = \emptyset$, and therefore Weyl's theorem holds for T, but fails for its adjoint T^*. We however have:

Corollary 2.2.3. *Let X be a Hilbert space. If $T \in B(X)$ is reduced by each of its finite-dimensional eigenspaces and $\operatorname{iso} \sigma(T) = \emptyset$, then Weyl's theorem holds for T and T^*. In this case, $\sigma(T) = \omega(T)$.*

Proof. If $\operatorname{iso} \sigma(T) = \emptyset$, then it follows from Corollary 2.2.2 that $\sigma(T) = \omega(T)$, which says that Weyl's theorem holds for T. The assertion that Weyl's theorem holds for T^* follows from noting that $\sigma(T)^* = \big(\sigma(T)\big)^-$, $\omega(T^*) = \big(\omega(T)\big)^-$ and $\pi_{00}(T^*) = \big(\pi_{00}(T)\big)^- = \emptyset$. ☐

In Corollary 2.2.3, the condition "$\operatorname{iso} \sigma(T) = \emptyset$" cannot be replaced by the condition "$\pi_{00}(T) = \emptyset$": for example consider the operator T defined by (2.11).

Corollary 2.2.4. ([Be1, Theorem]) *If X is a Hilbert space and $T \in B(X)$ is reduction-isoloid and is reduced by each of its finite-dimensional eigenspaces then Weyl's theorem holds for T.*

Proof. In view of Corollary 2.2.2, it suffices to show that $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$. Suppose $\lambda \in \pi_{00}(T)$. Then with the preceding notations, $\lambda \in \pi_{00}(T_1) \cap \big[\operatorname{iso} \sigma(T_2) \cup \rho(T_2)\big]$. If $\lambda \in \operatorname{iso} \sigma(T_2)$, then since by assumption T_2 is isoloid we have that $\lambda \in \pi_0(T_2)$ and hence $\lambda \in \pi_{0f}(T_2)$. But since $\pi_{0f}(T_2) = \emptyset$, we should have that $\lambda \notin \operatorname{iso} \sigma(T_2)$. Thus $\lambda \in \pi_{00}(T_1) \cap \rho(T_2)$. Since T_1 is normal it follows that $T_1 - \lambda I$ is Weyl and so is $T - \lambda I$; therefore $\lambda \in \sigma(T) \setminus \omega(T)$. ☐

Since hyponormal operators are isoloid and are reduced by each of its eigenspaces, it follows from Corollary 2.2.4 that Weyl's theorem holds for hyponormal operators.

If the condition "reduction-isoloid" is replaced by "isoloid" then Corollary 2.2.4 may fail: for example, consider the operator $T = T_1 \oplus T_2$, where T_1 is the one-dimensional zero operator and T_2 is an injective quasinilpotent compact operator.

If X is a Hilbert space, an operator $T \in B(X)$ is said to be *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$ (cf. [Al],[Ch3]). If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is semi-hyponormal.

Corollary 2.2.5. [CIO] *Weyl's theorem holds for every p-hyponormal operator.*

Proof. This follows from the fact that every p-hyponormal operator is isoloid and is reduced by each of its eigenspaces ([Ch3]). $\square$

L. Coburn [Co, Corollary 3.2] has shown that if $T \in B(X)$ is hyponormal and $\pi_{00}(T) = \emptyset$, then T is *extremally noncompact*, in the sense that

$$||T|| = ||\pi(T)||,$$

where π is the canonical map of $B(X)$ onto the Calkin algebra $B(X)/K(X)$. His proof relies upon the fact that Weyl's theorem holds for hyponormal operators, and hence $\sigma(T) = \omega(T)$ since $\pi_{00}(T) = \emptyset$. Now we can strengthen the Coburn's argument slightly:

Corollary 2.2.6. *If $T \in B(X)$ is normaloid and $\pi_{00}(T) = \emptyset$, then T is extremally noncompact.*

Proof. Since $\sigma(T) \subseteq \eta\, \omega(T) \cup p_{00}(T)$ for any $T \in B(X)$, we have that $\eta\, \sigma(T) \setminus \eta\, \omega(T) \subseteq \pi_{00}(T)$. Thus by our assumption, $\eta\, \sigma(T) = \eta\, \omega(T)$. Therefore we can argue that for each compact operator $K \in B(X)$,

$$||T|| = r(T) = r_\omega(T) = r_\omega(T + K) \leq r(T + K) \leq ||T + K||,$$

where $r_\omega(T)$ denotes the "Weyl spectral radius". This completes the proof. $\square$

Note that if $T \in B(X)$ is normaloid and $\pi_{00}(T) = \emptyset$, then Weyl's theorem may fail for T; for example take $X = \ell_2 \oplus \ell_2$ and $T = U \oplus U^*$, where U is the unilateral shift.

We next consider Weyl's theorem for Toeplitz operators.

The Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \dots\}$. An element $f \in L^2$ is referred to as analytic if $f \in H^2$ and coanalytic if $f \in L^2 \ominus H^2$. If P denotes the projection operator $L^2 \to H^2$, then for every $\varphi \in L^\infty(\mathbb{T})$, the operator T_φ on H^2 defined by

$$T_\varphi g = P(\varphi g) \quad \text{for all } g \in H^2 \tag{2.12}$$

is called the *Toeplitz operator with symbol* φ.

Theorem 2.2.7. [Co] *Weyl's theorem holds for every Toeplitz operator* T_φ.

Proof. It was known [Wi2] that $\sigma(T_\varphi)$ is always connected. Since there are no quasinilpotent Toeplitz operators except 0, $\sigma(T_\varphi)$ can have no isolated eigenvalues of finite multiplicity. Thus Weyl's theorem is equivalent to the fact that

$$\sigma(T_\varphi) = \omega(T_\varphi). \tag{2.13}$$

Since $T_\varphi - \lambda I = T_{\varphi-\lambda}$, it suffices to show that if T_φ is Weyl then T_φ is invertible. If T_φ is not invertible, but is Weyl then it is easy to see that both T_φ and $T_\varphi^* = T_{\overline{\varphi}}$ must have nontrivial kernels. Thus we want to show that this can not happen, unless $\varphi = 0$ and hence T_φ is the non-Weyl operator.

Suppose that there exist nonzero functions φ, f, and g ($\varphi \in L^\infty$ and $f, g \in H^2$) such that $T_\varphi f = 0$ and $T_{\overline{\varphi}} g = 0$. Then $P(\varphi f) = 0$ and $P(\overline{\varphi} g) = 0$, so that there exist functions $h, k \in H^2$ such that

$$\int h\, d\theta = \int k\, d\theta = 0 \quad \text{and} \quad \varphi f = \overline{h}, \quad \overline{\varphi} g = \overline{k}.$$

Thus by the F. and M. Riesz's theorem, φ, f, g, h, k are all nonzero except on a set of measure zero. We thus have that $\overline{f}/g = h/\overline{k}$ pointwise a.e., so that $\overline{fk} = gh$ a.e., which implies $gh = 0$ a.e. Again by the F. and M. Riesz's theorem, we can conclude that either $g = 0$ a.e. or $h = 0$ a.e. This contradiction completes the proof. □

We review here a few essential facts concerning Toeplitz operators with continuous symbols, using [Do1] as a general reference. The sets $C(\mathbb{T})$ of all continuous complex-valued functions on the unit circle $\mathbb{T}$ and $H^\infty(\mathbb{T}) = L^\infty \cap H^2$ are Banach algebras, and it is well-known that every Toeplitz operator with symbol $\varphi \in H^\infty$ is subnormal. The C^*-algebra $\mathfrak{A}$ generated by all Toeplitz operators T_φ with $\varphi \in C(\mathbb{T})$ has an important property which is very useful for spectral theory: the commutator ideal of $\mathfrak{A}$ is the ideal $K(H^2)$ of compact operators on H^2. As $C(\mathbb{T})$ and $\mathfrak{A}/K(H^2)$ are $*$-isomorphic C^*-algebras, then for every $\varphi \in C(\mathbb{T})$,

$$T_\varphi \text{ is a Fredholm operator if and only if } \varphi \text{ is invertible} \tag{2.14}$$

$$\operatorname{index} T_\varphi = -wn(\varphi)\ , \tag{2.15}$$

$$\sigma_e(T_\varphi) = \varphi(\mathbb{T})\ , \tag{2.16}$$

where $wn(\varphi)$ denotes the winding number of φ with respect to the origin. Finally, we make note that if $\varphi \in C(\mathbb{T})$ and if f is an analytic function defined on an open set containing $\sigma(T_\varphi)$, then $f \circ \varphi \in C(\mathbb{T})$ and $f(T_\varphi)$ is well-defined by the analytic functional calculus.

We require the use of certain closed subspaces and subalgebras of $L^\infty(\mathbb{T})$, which are described in further detail in [Do2] and Appendix 4 of [Ni]. Recall that the subspace $H^\infty(\mathbb{T}) + C(\mathbb{T})$ is a closed subalgebra of L^∞. The elements of the closed selfadjoint subalgebra QC, which is defined to be

$$QC = \big(H^\infty(\mathbb{T}) + C(\mathbb{T})\big) \cap \overline{\big(H^\infty(\mathbb{T}) + C(\mathbb{T})\big)}\,,$$

are called quasicontinuous functions. The subspace PC is the closure in $L^\infty(\mathbb{T})$ of the set of all piecewise continuous functions on $\mathbb{T}$. Thus $\varphi \in PC$ if and only if it is right continuous and has both a left- and right-hand limit at every point. There are certain algebraic relations among Toeplitz operators whose symbols come from these classes, including

$$T_\psi T_\varphi - T_{\psi\varphi} \in K(H^2) \ \text{ for every } \varphi \in H^\infty(\mathbb{T}) + C(\mathbb{T}) \text{ and } \psi \in L^\infty(\mathbb{T}) \,, \tag{2.17}$$

and

$$\text{the commutator } [T_\varphi, T_\psi] \ \text{ is compact for every } \varphi, \psi \in PC \,. \tag{2.18}$$

We add to these relations the following one.

Lemma 2.2.8. *If T_φ is a Toeplitz operator with quasicontinuous symbol φ, and if $f \in H(\sigma(T_\varphi))$, then $T_{f\circ\varphi} - f(T_\varphi)$ is a compact operator.*

Proof. Assume that $\varphi \in QC$. Recall from [Do1, p.188] that if $\psi \in H^\infty + C(\mathbb{T})$, then T_ψ is Fredholm if and only if ψ is invertible in $H^\infty + C(\mathbb{T})$. Therefore for every $\lambda \notin \sigma(T_\varphi)$, both $\varphi - \lambda$ and $\overline{\varphi - \lambda}$ are invertible in $H^\infty + C(\mathbb{T})$; hence, $(\varphi - \lambda)^{-1} \in QC$. Using this fact together with (2.17) we have that, for $\psi \in L^\infty$ and $\lambda, \mu \in \mathbb{C}$,

$$T_{\varphi-\mu} T_\psi T_{(\varphi-\lambda)^{-1}} - T_{(\varphi-\mu)\psi(\varphi-\lambda)^{-1}} \in K(H^2) \quad \text{whenever } \lambda \notin \sigma(T_\varphi) \,.$$

The arguments above extend to rational functions to yield: if r is any rational function with all of its poles outside of $\sigma(T_\varphi)$, then $r(T_\varphi) - T_{r\circ\varphi} \in K(H^2)$. Suppose that f is an analytic function on an open set containing $\sigma(T_\varphi)$. By Runge's theorem there exists a sequence of rational functions r_n such that the poles of each r_n lie outside of $\sigma(T_\varphi)$ and $r_n \to f$ uniformly on $\sigma(T_\varphi)$. Thus $r_n(T_\varphi) \to f(T_\varphi)$ in the norm-topology of $L(H^2)$. Furthermore, because $r_n \circ \varphi \to f \circ \varphi$ uniformly, we have $T_{r_n\circ\varphi} \to T_{f\circ\varphi}$ in the norm-topology. Hence, $T_{f\circ\varphi} - f(T_\varphi) = \lim \big(T_{r_n\circ\varphi} - r_n(T_\varphi)\big)$, which is compact. $\square$

Lemma 2.2.8 does not extend to piecewise continuous symbols $\varphi \in PC$, as we cannot guarantee that $T_\varphi^n - T_{\varphi^n} \in K(H^2)$ for each $n \in \mathbb{Z}^+$. For example, if $\varphi(e^{i\theta}) = \chi_{\mathbb{T}_+} - \chi_{\mathbb{T}_-}$, where $\chi_{\mathbb{T}_+}$ and $\chi_{\mathbb{T}_-}$ are characteristic functions of, respectively, the upper semicircle and the lower semicircle, then $T_\varphi^2 - I$ is not compact.

Corollary 2.2.9. *If T_φ is a Toeplitz operator with quasicontinuous symbol φ, then for every $f \in H(\sigma(T_\varphi))$,*

1. *$\omega(f(T_\varphi)) = \sigma(T_{f\circ\varphi})$, and*
2. *$f(T_\varphi)$ is essentially normal and is unitarily equivalent to a compact perturbation of $f(T_\varphi) \oplus M_{f\circ\varphi}$, where $M_{f\circ\varphi}$ is the operator of multiplication by $f\circ\varphi$ on $L^2(\mathbb{T})$.*

Proof. Because the Weyl spectrum is stable under the compact perturbations, it follows from Lemma 2.2.8 that $\omega(f(T_\varphi)) = \omega(T_{f\circ\varphi}) = \sigma(T_{f\circ\varphi})$, which proves statement (1). To prove (2), observe that because QC is a closed algebra, the composition of the analytic function f with $\varphi \in QC$ produces a quasicontinuous function $f \circ \varphi \in QC$.

Moreover, by (2.17), every Toeplitz operator with quasicontinuous symbol is essentially normal. The (normal) Laurent operator $M_{f\circ\varphi}$ on $L^2(\mathbb{T})$ has its spectrum contained within the spectrum of the (essentially normal) Toeplitz operator $T_{f\circ\varphi}$. Thus there is the following relationship involving the essentially normal operators $f(T_\varphi)$ and $M_{f\circ\varphi} \oplus f(T_\varphi)$:

$$\sigma_e\big(f(T_\varphi) \oplus M_{f\circ\varphi}\big) = \sigma_e(f(T_\varphi)) \quad \text{and} \quad \mathcal{SP}(f(T_\varphi)) = \mathcal{SP}\big(f(T_\varphi) \oplus M_{f\circ\varphi}\big),$$

where $\mathcal{SP}(T)$ denotes the spectral picture of an operator T. (The spectral picture $\mathcal{SP}(T)$ is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the Fredholm indices associated with these holes and pseudoholes.) Thus it follows from the Brown-Douglas-Fillmore theorem [Pe] that $f(T_\varphi)$ is compalent to $f(T_\varphi) \oplus M_{f\circ\varphi}$, in the sense that there exists a unitary operator W and a compact operator K such that $W\big(f(T_\varphi) \oplus M_{f\circ\varphi}\big)W^* + K = f(T_\varphi)$. $\square$

Corollary 2.2.9 (1) can be viewed as saying that $\sigma(f(T_\varphi)) \setminus \sigma(T_{f\circ\varphi})$ consists of holes with winding number zero.

We consider the following question ([Ob2]):

$$\textit{if } T_\varphi \textit{ is a Toeplitz operator, then does Weyl's theorem hold for } T_\varphi^2\,? \tag{2.19}$$

To answer the above question, we need a spectral property of Toeplitz operators with continuous symbols.

Lemma 2.2.10. *Suppose that φ is continuous and that $f \in H(\sigma(T_\varphi))$. Then*

$$\sigma(T_{f\circ\varphi}) \subseteq f(\sigma(T_\varphi))\,, \tag{2.20}$$

and equality occurs if and only if Weyl's theorem holds for $f(T_\varphi)$.

Proof. By Corollary 2.2.9, $\sigma(T_{f\circ\varphi}) = \omega(f(T_\varphi)) \subseteq \sigma(f(T_\varphi)) = f(\sigma(T_\varphi))$. Because $\sigma(T_\varphi)$ is connected, so is $f(\sigma(T_\varphi)) = \sigma(f(T_\varphi))$; therefore the set $\pi_{00}(f(T_\varphi))$ is empty. Again by Corollary 2.2.9, $\omega(f(T_\varphi)) = \sigma(T_{f\circ\varphi})$ and so $\omega(f(T_\varphi)) = \sigma(f(T_\varphi)) \setminus \pi_{00}(f(T_\varphi))$ if and only if $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$. $\square$

If φ is not continuous, it is possible for Weyl's theorem to hold for some $f(T_\varphi)$ without $\sigma(T_{f\circ\varphi})$ being equal to $f(\sigma(T_\varphi))$. One example is as follows. Let $\varphi(e^{i\theta}) = e^{\frac{i\theta}{3}}$ $(0 \le \theta < 2\pi)$, a piecewise continuous function. The operator T_φ is invertible but T_{φ^2} is not; hence $0 \in \sigma(T_{\varphi^2}) \setminus \{\sigma(T_\varphi)\}^2$. However $\omega(T_\varphi^2) = \sigma(T_\varphi^2)$, and $\pi_{00}(T_\varphi^2)$ is empty (see Figure 2); therefore Weyl's theorem holds for T_φ^2.

We can now answer the question (2.19): the answer is no.

Example 2.2.11. *There exists a continuous function $\varphi \in C(\mathbb{T})$ such that $\sigma(T_{\varphi^2}) \neq \{\sigma(T_\varphi)\}^2$.*

Proof. Let φ be defined by

$$\varphi(e^{i\,\theta}) = \begin{cases} -e^{2i\,\theta}+1 & (0 \le \theta \le \pi) \\ e^{-2i\,\theta}-1 & (\pi \le \theta \le 2\pi)\,. \end{cases}$$

The orientation of the graph of φ is shown in Figure 3. Evidently, φ is continuous and, in Figure 3, φ has winding number $+1$ with respect to the hole of C_1; the hole of C_2 has winding number -1. Thus we have $\sigma_e(T_\varphi) = \varphi(\mathbb{T})$ and $\sigma(T_\varphi) = \operatorname{conv}\varphi(\mathbb{T})$. On the other hand, a straightforward calculation shows that $\varphi^2(\mathbb{T})$ is the Cardioid $r = 2(1 + \cos\,\theta)$. In particular, $\varphi^2(\mathbb{T})$ traverses the Cardioid once in a counterclockwise direction and then traverses the Cardioid once in a clockwise direction. Thus $wn(\varphi^2 - \lambda) = 0$ for each λ in the hole of $\varphi^2(\mathbb{T})$. Hence $T_{\varphi^2-\lambda}$ is a Weyl operator and is, therefore, invertible for each λ in the hole of $\varphi^2(\mathbb{T})$. This implies that $\sigma(T_{\varphi^2})$ is the Cardioid $r = 2(1+\cos\,\theta)$. But because $\{\sigma(T_\varphi)\}^2 = \{\operatorname{conv}\varphi(\mathbb{T})\}^2 = \{(r,\theta) : r \le 2(1+\cos\,\theta)\}$, it follows that $\sigma(T_{\varphi^2}) \ne \{\sigma(T_\varphi)\}^2$. $\square$

We next consider Weyl's theorem through the local spectral theory. Local spectral theory is based on the existence of analytic solutions $f : U \to X$ to the equation $(T - \lambda I)f(\lambda) = x$ on an open subset $U \subset \mathbb{C}$, for a given operator $T \in B(X)$ and a given element $x \in X$. We define the spectral subspace as follows: for a closed set $F \subset \mathbb{C}$, let

$$\mathcal{X}_T(F) := \{x \in X : (T-\lambda I)f(\lambda) = x \text{ has an analytic solution } f : \mathbb{C}\setminus F \to X\}.$$

We say that $T \in B(X)$ has the *single valued extension property* (SVEP) at $\lambda_0 \in \mathbb{C}$ if for every neighborhood U of λ_0, $f = 0$ is the only analytic solution $f : U \to X$ satisfying $(T - \lambda I)f(\lambda) = 0$. We also say that T has the SVEP if T has this property at every $\lambda \in \mathbb{C}$. The local spectrum of T at x is defined by

$$\sigma_T(x) := \mathbb{C}\setminus\bigcup\Big\{(T-\lambda I)f(\lambda) = x \text{ has an analytic solution } f : U \to X \text{ on the open subset } U \subset \mathbb{C}\Big\}.$$

If T has the SVEP then $\mathcal{X}_T(F) = \{x \in X : \sigma_T(x) \subset F\}$.

The following lemma gives a connection of the SVEP with a finite ascent property.

Lemma 2.2.12. [Fin] *If $T \in B(X)$ is semi-Fredholm then*

$$T \text{ has the SVEP at } 0 \iff T \text{ has a finite ascent at } 0.$$

The finite dimensionality of $\mathcal{X}_T(\{\lambda\})$ is necessary ad sufficient for $T - \lambda I$ to be Fredholm whenever λ is an isolated point of the spectrum.

Lemma 2.2.13. [Ai] *Let $T \in B(X)$. If $\lambda \in iso\,\sigma(T)$ then*

$$\lambda \notin \sigma_e(T) \iff \mathcal{X}_T(\{\lambda\}) \text{ is finite dimensional.}$$

Theorem 2.2.14. *If $T \in B(X)$ has the SVEP then the following are equivalent:*
(a) *Weyl's theorem holds for T;*
(b) *$R(T - \lambda I)$ is closed for every $\lambda \in \pi_{00}(T)$;*
(c) *$\mathcal{X}_T(\{\lambda\})$ is finite dimensional for every $\lambda \in \pi_{00}(T)$.*

Proof. (a) $\Rightarrow$ (b): Evident.

(b) $\Rightarrow$ (a): If $\lambda \in \sigma(T) \setminus \omega(T)$ then by Lemma 2.2.12, $T - \lambda I$ has a finite ascent. Thus $T - \lambda I$ is Browder and hence $\lambda \in \pi_{00}(T)$. Conversely, if $\lambda \in \pi_{00}(T)$ then by assumption $T - \lambda I$ is Browder, so $\lambda \in \sigma(T) \setminus \omega(T)$.

(b) $\Leftrightarrow$ (c): Immediate from Lemma 2.2.13. □

An operator $T \in B(X)$ is called *reguloid* if each isolated point of spectrum is a regular point, in the sense that there is a generalized inverse:

$$\lambda \in \text{iso}\ \sigma(T) \Longrightarrow T - \lambda I = (T - \lambda I)S_\lambda(T - \lambda I) \text{ with } S_\lambda \in B(X).$$

It was known [Har4] that if T is reguloid then $R(T-\lambda I)$ is closed for each $\lambda \in \text{iso}\,\sigma(T)$. Also an operator $T \in B(X)$ is said to satisfy *the growth condition* (G_1), if for all $\lambda \in \mathbb{C} \setminus \sigma(T)$

$$||(T - \lambda I)^{-1}||\text{dist}(\lambda, \sigma(T)) \leq 1.$$

Lemma 2.2.15. *If $T \in B(X)$ then*

$$(G_1) \implies \textit{reguloid} \implies \textit{isoloid}. \tag{2.21}$$

Proof. Recall ([Har4, Theorem 7.3.4]) that if $T - \lambda I$ has a generalized inverse and if $\lambda \in \partial\sigma(T)$ is in the boundary of the spectrum then $T - \lambda I$ has an invertible generalized inverse. If therefore T is reguloid and $\lambda \in \text{iso}\,\sigma(T)$ then $T - \lambda I$ has an invertible generalized inverse, and hence ([Har4, (3.8.6.1)])

$$N(T - \lambda I) \cong X/R(T - \lambda I).$$

Thus if $N(T - \lambda I) = \{0\}$ then $T - \lambda I$ is invertible, a contradiction. Therefore λ is an eigenvalue of T, which proves the second implication of (2.21). Towards the first implication we may write T in place of $T - \lambda I$ and hence assume $\lambda = 0$: then using the spectral projection at $0 \in \mathbb{C}$ we can represent T as a 2×2 operator matrix:

$$T = \begin{bmatrix} T_0 & 0 \\ 0 & T_1 \end{bmatrix},$$

where $\sigma(T_0) = \{0\}$ and $\sigma(T_1) = \sigma(T) \setminus \{0\}$. Now we can borrow an argument of J. Stampfli ([Sta1, Theorem C]): take $0 < \epsilon \leq \frac{1}{2}\text{dist}(0, \sigma(T) \setminus \{0\})$ and argue

$$T_0 = \frac{1}{2\pi i}\int_{|z|=\epsilon} z(T - zI)^{-1}dz,$$

using the growth condition (G_1) to see that

$$||T_0|| \leq \frac{1}{2\pi}\int_{|z|=\epsilon} |z|\ ||(T - zI)^{-1}||\ |dz| \leq \frac{1}{2\pi}\epsilon\frac{1}{\epsilon}2\pi\epsilon = \epsilon, \tag{2.22}$$

which tends to 0 with ϵ. It follows that $T_0 = 0$ and hence that

$$T = \begin{bmatrix} 0 & 0 \\ 0 & T_1 \end{bmatrix} = TST \text{ with } S = \begin{bmatrix} 0 & 0 \\ 0 & T_1^{-1} \end{bmatrix}$$

has a generalized inverse. □

Corollary 2.2.16. *If $T \in B(X)$ is reguloid and has the SVEP then Weyl's theorem holds for T.*

Proof. Immediate from Theorem 2.2.14. □

Lemma 2.2.17. *Let $T \in B(X)$. If for any $\lambda \in \mathbb{C}$, $\mathcal{X}_T(\{\lambda\})$ is closed then T has the SVEP.*

Proof. This follows from [Ai, Theorem 2.31] together with the fact that

$$\mathcal{X}_T(\{\lambda\}) = \{x \in X : \lim_{n\to\infty} ||(T-\lambda I)^n x||^{\frac{1}{n}} = 0\}.$$

□

Corollary 2.2.18. *If $T \in B(X)$ satisfies*

$$\mathcal{X}_T(\{\lambda\}) = N(T-\lambda I) \quad \text{for every } \lambda \in \mathbb{C}, \tag{2.23}$$

then T has the SVEP and both T and T^* are reguloid. Thus in particular if T satisfies (2.23) then Weyl's theorem holds for T.

Proof. If T satisfies the condition (2.23) then by Lemma 2.2.17, T has the SVEP. The second assertion follows from [Ai, Theorem 3.96]. The last assertion follows at once from Corollary 2.2.16. □

An operator $T \in B(X)$ is said to be *paranormal* if

$$||Tx||^2 \le ||T^2x||\,||x|| \quad \text{for every } x \in X.$$

It was well known that if $T \in B(X)$ is paranormal then the following hold:

(a) T is normaloid;
(b) T has finite ascent;
(c) if x and y are nonzero eigenvectors corresponding to, respectively, distinct nonzero eigenvalues of T, then $||x|| \le ||x+y||$ ([ChR, Theorem 2,6])

In particular, p-hyponormal operators are paranormal (cf. [FIY]). An operator $T \in B(X)$ is said to be *totally paranormal* if $T - \lambda I$ is paranormal for every $\lambda \in \mathbb{C}$. Evidently, every hyponormal operator is totally paranormal. On the other hand, every totally paranormal operator satisfies (2.23): indeed, for every $x \in X$ and $\lambda \in \mathbb{C}$,

$$||(T-\lambda I)^n x||^{\frac{1}{n}} \ge ||(T-\lambda I)x|| \quad \text{for every } n \in \mathbb{N}.$$

So if $x \in \mathcal{X}_T(\{\lambda\})$ then $||(T-\lambda I)^n x||^{\frac{1}{n}} \to 0$ as $n \to \infty$, so that $x \in N(T-\lambda I)$, which gives $\mathcal{X}_T(\{\lambda\}) \subset N(T-\lambda I)$. The reverse inclusion is true for every operator. Therefore by Corollary 2.2.18 we can conclude that Weyl's theorem holds for totally paranormal operators. We can prove more:

Theorem 2.2.19. *Weyl's theorem holds for paranormal operators on a separable Banach space.*

Proof. It was known [ChR] that paranormal operators on a separable Banach space have the SVEP. So in view of Theorem 2.2.14 it suffices to show that $R(T-\lambda I)$ is closed for each $\lambda \in \pi_{00}(T)$. Suppose $\lambda \in \pi_{00}(T)$. Using the spectral projection $P = \frac{1}{2\pi i}\int_{\partial B}(\lambda I - T)^{-1}d\lambda$, where B is an open disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = T_1 \oplus T_2, \quad \text{where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

If $\lambda = 0$ then T_1 is a quasinilpotent paranormal operator, so that $T_1 = 0$. If $\lambda \neq 0$ write $T_A = \frac{1}{\lambda}T_1$. Then T_A is paranormal and $\sigma(T_A) = \{1\}$. Since T_A is invertible we have that T_A and T_A^{-1} are paranormal, and hence normaloid. So $||T_A|| = ||T_A^{-1}|| = 1$ and hence

$$||x|| = ||T_A^{-1}T_A x|| \le ||T_A x|| \le ||x||,$$

which implies that T_A and T_A^{-1} are isometries. Also since $T_A - 1$ is a quasinilpotent operator it follows that $T_A = I$, and hence $T_1 = \lambda I$. Thus we have that $T - \lambda I = 0 \oplus (T_2 - \lambda I)$ has closed range. This completes the proof. $\square$

Does Weyl's theorem hold for paranormal operators on an *arbitrary* Banach space? Paranormal operators on an arbitrary Banach space may not have the SVEP. So the proof of Theorem 2.2.19 does not work for arbitrary Banach spaces. In spite of it Weyl's theorem holds for paranormal operators on an arbitrary Banach space. To see this recall the *reduced minimum modulus* of T is defined by

$$\gamma(T) := \inf \frac{||Tx||}{\text{dist}\,(x,\, N(T)} \quad (x \notin N(T)).$$

It was known [Go] that $\gamma(T) > 0$ if and only if T has closed range.

Theorem 2.2.20. *Weyl's theorem holds for paranormal operators on a Banach space.*

Proof. The proof of Theorem 2.2.19 shows that with no restriction on X, $\pi_{00}(T) \subset \sigma(T) \setminus \omega(T)$ for every paranormal operator $T \in B(X)$. Thus we must show that $\sigma(T) \setminus \omega(T) \subset \text{iso}\,\sigma(T)$. Suppose $\lambda \in \sigma(T) \setminus \omega(T)$. If $\lambda = 0$ then T is Weyl and has finite ascent. Thus T is Browder, and hence $0 \in \text{iso}\,\sigma(T)$. If $\lambda \neq 0$ and $\lambda \notin \text{iso}\,\sigma(T)$ then we can find a sequence $\{\lambda_n\}$ of nonzero eigenvalues such that $\lambda_n \to \lambda$. By the property (c) above Theorem 2.2.19,

$$\text{dist}\left(x_{\lambda_n},\, N(T-\lambda I)\right) \ge 1 \quad \text{for each unit vector } x_{\lambda_n} \in N(T-\lambda_n I).$$

We thus have

$$\frac{||(T-\lambda I)x_n||}{\mathrm{dist}\,(x_{\lambda_n},\, N(T-\lambda I))} = \frac{|\lambda_n - \lambda|}{\mathrm{dist}\,(x_{\lambda_n},\, N(T-\lambda I))} \to 0,$$

which shows that $\gamma(T-\lambda I) = 0$ and hence $T-\lambda I$ does not have closed range, a contradiction. Therefore $\lambda \in \mathrm{iso}\,\sigma(T)$. This completes the proof. $\square$

2.3 Spectral Mapping Theorem for the Weyl spectrum

Let $\mathfrak{S}$ denote the set, equipped with the Hausdorff metric, of all compact subsets of $\mathbb{C}$. If $\mathfrak{A}$ is a unital Banach algebra then the spectrum can be viewed as a function $\sigma : \mathfrak{A} \to \mathfrak{S}$, mapping each $T \in \mathfrak{A}$ to its spectrum $\sigma(T)$. It is well-known that the function σ is upper semicontinuous, i.e., if $T_n \to T$ then $\limsup \sigma(T_n) \subset \sigma(T)$ and that in noncommutative algebras, σ does have points of discontinuity. The work of J. Newburgh [Ne] contains the fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel [CoM] have undertaken a detailed study of spectral continuity in the case where the Banach algebra is the C^*-algebra of all operators acting on a complex separable Hilbert space. Of interest is the identification of points of spectral continuity, and of classes $\mathfrak{C}$ of operators for which σ becomes continuous when restricted to $\mathfrak{C}$. In [BGS], the continuity of the spectrum was considered when restricted to certain subsets of the entire manifold of Toeplitz operators. The set of normal operators is perhaps the most immediate in the latter direction: σ is continuous on the set of normal operators. As noted in Solution 104 of [Ha3], Newburgh's argument uses the fact that the inverses of normal resolvents are normaloid. This argument can be easily extended to the set of hyponormal operators because the inverses of hyponormal resolvents are also hyponormal and hence normaloid. Although p-hyponormal operators are normaloid, it was shown [HwL1] that σ is continuous on the set of all p-hyponormal operators.

We now examine the continuity of the Weyl spectrum in pace of the spectrum. In general the Weyl spectrum is not continuous: indeed, it was in [BGS] that the spectrum is discontinuous on the entire manifold of Toeplitz operators. Since the spectra and the Weyl spectra coincide for Toeplitz operators, it follows at once that the weyl spectrum is discontinuous.

However the Weyl spectrum is upper semicontinuous.

Lemma 2.3.1. *The map $T \to \omega(T)$ is upper semicontinuous.*

Proof. Let $\lambda \in \omega(T)$. Since the set of Weyl operators forms an open set, there exists $\delta > 0$ such that if $S \in B(X)$ and $||T - \lambda I - S|| < \delta$ then S is Weyl. So there exists an integer N such that $||T - \lambda I - (T_n - \lambda I)|| < \frac{\delta}{2}$ for $n \geq N$. Let V be an open $(\delta/2)$-neighborhhod of λ. We have, for $\mu \in V$ and $n \geq N$,

$$||T - \lambda I - (T_n - \mu I)|| < \delta,$$

so that $T_n - \mu I$ is Weyl. This shows that $\lambda \notin \limsup \omega(T_n)$. Thus $\limsup \omega(T_n) \subset \omega(T)$. $\square$

Lemma 2.3.2. [Ne, Theorem 4] *If $\{T_n\}_n$ is a sequence of operators converging to an operator T and such that $[T_n, T]$ is compact for each n, then $\lim \sigma_e(T_n) = \sigma_e(T)$.*

Proof. Newburgh's theorem is stated as follows: if in a Banach algebra A, $\{a_i\}_i$ is a sequence of elements commuting with $a \in A$ and such that $a_i \to a$, then $\lim \sigma(a_i) = \sigma(a)$. If π denotes the canonical homomorphism of $B(X)$ onto the Calkin algebra $B(X)/K(X)$, then the assumptions give that $\pi(T_n) \to \pi(T)$ and $[\pi(T_n), \pi(T)] = 0$ for each n. Hence, $\lim \sigma(\pi(T_n)) = \sigma(\pi(T))$; that is, $\lim \sigma_e(T_n) = \sigma_e(T)$. $\square$

Theorem 2.3.3. *Suppose that $T, T_n \in B(X)$, for $n \in \mathbb{Z}^+$, are such that T_n converges to T. If $[T_n, T] \in K(X)$ for each n, then*

$$\lim \omega(f(T_n)) = \omega(f(T)) \quad \textit{for every } f \in H(\sigma(T)). \tag{2.24}$$

Remark. Because $T_n \to T$, by the upper-semicontinuity of the spectrum, there is a positive integer N such that $\sigma(T_n) \subseteq V$ whenever $n > N$. Thus, in the left-hand side of (2.24) it is to be understood that $n > N$.

Proof. If T_n and T commute modulo the compact operators then, whenever T_n^{-1} and T^{-1} exist, T_n, T, T_n^{-1} and T^{-1} all commute modulo the compact operators. Therefore $r(T_n)$ and $r(T)$ also commute modulo $K(X)$ whenever r is a rational function with no poles in $\sigma(T)$ and n is sufficiently large. Using Runge's theorem we can approximate f on compact subsets of V by rational functions r who poles lie off of V. So there exists a sequence of rational functions r_i whose poles lie outside of V and $r_i \to f$ uniformly on compact subsets of V. If $n > N$, then by the functional calculus,

$$f(T_n)f(T) - f(T)f(T_n) = \lim_i \big(r_i(T_n)r_i(T) - r_i(T)r_i(T_n)\big),$$

which is compact for each n. Furthermore,

$$\begin{aligned} ||f(T_n) - f(T)|| &= ||\frac{1}{2\pi i}\int_\Gamma f(\lambda)\big((\lambda - T_n)^{-1} - (\lambda - T)^{-1}\big)\, d\lambda|| \\ &\le \frac{1}{2\pi i}\,\ell(\Gamma) \max_{\lambda\in\Gamma} |f(\lambda)| \cdot \max_{\lambda\in\Gamma} ||(\lambda - T_n)^{-1} - (\lambda - T)^{-1}||\,, \end{aligned}$$

where Γ is the boundary of V and $\ell(\Gamma)$ denotes the arc length of Γ. The right-hand side of the above inequality converges to 0, and so $f(T_n) \to f(T)$. By Lemma 2.25, $\lim \sigma_e(f(T_n)) = \sigma_e(f(T))$. The arguments used by J.B. Conway and B.B. Morrel in Proposition 3.11 of [CoM] can now be used here to obtain the conclusion $\lim \omega(f(T_n)) = \omega(f(T))$. $\square$

In general there is only inclusion for the Weyl spectrum:

Theorem 2.3.4. *If $T \in B(X)$ then*

$$\omega(p(T)) \subseteq p(\omega(T)) \quad \textit{for every polynomial } p.$$

Proof. We can suppose p is nonconstant. Suppose $\lambda \notin p\omega(T)$. Writing $p(\mu) - \lambda = a(\mu - \mu_1)(\mu - \mu_2)\cdots(\mu - \mu_n)$, we have

$$p(T) - \lambda I = a(T - \mu_1 I)\cdots(T - \mu_n I). \tag{2.25}$$

For each i, $p(\mu_i) = \lambda \notin p\omega(T)$, so that $\mu_i \notin \omega(T)$, i.e., $T - \mu_i I$ is weyl. It thus follows from (2.25) that $p(T) - \lambda I$ is Weyl since the product of Weyl operators is Weyl. $\square$

In general the spectral mapping theorem is liable to fail for the Weyl spectrum:

Example 2.3.5. *Let $T = U \oplus (U^* + 2I)$, where U is the unilateral shift on ℓ_2, and let $p(\lambda) := \lambda(\lambda - 2)$. Then $0 \in p(\omega(T))$ but $0 \notin \omega(p(T))$.*

Proof. Observe $p(T) = T(T - 2I) = [U \oplus (U^* + 2I)][(U - 2I) \oplus U^*]$. Since U is Fredholm of index -1, and since $U^* + 2I$ and $U - 2I$ are invertible it follows that T and $T - 2I$ are Fredholm of indices -1 and $+1$, respectively. Therefore $p(T)$ is Weyl, so that $0 \notin \omega(p(T))$, while $0 = p(0) \in p(\omega(T))$. $\square$

Lemma 2.3.6. *If $T \in B(X)$ is isoloid then for every polynomial p,*

$$p(\sigma(T) \setminus \pi_{00}(T)) = \sigma(p(T)) \setminus \pi_{00}(p(T)).$$

Proof. We first claim that with no restriction on T,

$$\sigma(p(T)) \setminus \pi_{00}(p(T)) \subset p(\sigma(T) \setminus \pi_{00}(T)). \tag{2.26}$$

Let $\lambda \in \sigma(p(T)) \setminus \pi_{00}(p(T)) = p(\sigma(T)) \setminus \pi_{00}(p(T))$. There are two cases to consider.

Case 1. $\lambda \notin \operatorname{iso} p(\sigma(T))$. In this case, there exists a sequence (λ_n) in $p(\sigma(T))$ such that $\lambda_n \to \lambda$. So there exists a sequence (μ_n) in $\sigma(T)$ such that $p(\mu_n) = \lambda_n \to \lambda$. This implies that (μ_n) contains a convergent subsequence and we may assume that $\lim \mu_n = \mu_0$. Thus $\lambda = \lim p(\mu_n) = p(\mu_0)$. Since $\mu_0 \in \sigma(T) \setminus \pi_{00}(T)$, it follows that $\lambda \in p(\sigma(T) \setminus \pi_{00}(T))$.

Case 2. $\lambda \in \operatorname{iso} p(\sigma(T))$. In this case either λ is not an eigenvalue of $p(T)$ or it is an eigenvalue of infinite multiplicity. Let $p(T) - \lambda I = a_0(T - \mu_1 I) \cdots (T - \mu_n I)$. If λ is not an eigenvalue of $p(T)$ then none of $\mu_1, \cdots, \mu_n$ can be an eigenvalue of T and at least one of $\mu_1, \cdots, \mu_n$ is in $\sigma(T)$. Therefore $\lambda \in p(\sigma(T) \setminus \pi_{00}(T))$. If λ is an eigenvalue of $p(T)$ of infinite multiplicity then at least one of $\mu_1, \cdots, \mu_n$, say μ_1, is an eigenvalue of T of infinite multiplicity. Then $\mu_1 \in \sigma(T) \setminus \pi_{00}(T)$ and $p(\mu_1) = \lambda$, so that $\lambda \in p(\sigma(T) \setminus \pi_{00}(T))$. This proves (2.26). For the reverse inclusion of (2.26), we assume $\lambda \in p(\sigma(T) \setminus \pi_{00}(T))$. Since $p(\sigma(T)) = \sigma(p(T))$, we have $\lambda \in \sigma(p(T))$. If possible let $\lambda \in \pi_{00}(p(T))$. So $\lambda \in \operatorname{iso} \sigma(p(T))$. Let

$$p(T) - \lambda I = a_0(T - \mu_1 I) \cdots (T - \mu_n I). \tag{2.27}$$

The equality (2.27) shows that if any of $\mu_1, \cdots, \mu_n$ is in $\sigma(T)$ then it must be an isolated point of $\sigma(T)$ and hence an eigenvalue since T is isoloid. Since λ is an eigenvalue of finite multiplicity, any such μ must be an eigenvalue of finite multiplicity and hence belongs to $\pi_{00}(T)$. This contradicts the fact that $\lambda \in p(\sigma(T) \setminus \pi_{00}(T))$. Therefore $\lambda \notin \pi_{00}(T)$ and

$$p(\sigma(T) \setminus \pi_{00}(T)) \subset \sigma(p(T)) \setminus \pi_{00}(p(T)).$$

$\square$

Theorem 2.3.7. *If $T \in B(X)$ is isoloid and Weyl's theorem holds for T then for every polynomial p, Weyl's theorem holds for $p(T)$ if and only if $p(\omega(T)) = \omega(p(T))$.*

Proof. By Lemma 2.3.6, $p(\sigma(T)\setminus\pi_{00}(T))=\sigma(p(T))\setminus\pi_{00}(p(T))$. If Weyl's theorem holds for T then $\omega(T)=\sigma(T)\setminus\pi_{00}(T)$, so that

$$p(\omega(T))=p(\sigma(T)\setminus\pi_{00}(T))=\sigma(p(T))\setminus\pi_{00}(p(T)).$$

The result follows at once from this relationship. □

Example 2.3.8. Theorem 2.3.7 may fail if T is not isoloid. To see this define T_1 and T_2 on ℓ^2 by

$$T_1(x_1,x_2,\cdots)=(x_1,0,x_2/2,x_3/2,\cdots)$$

and

$$T_2(x_1,x_2,\cdots)=(0,x_1/2,x_2/3,x_3/4,\cdots).$$

Let $T:=T_1\oplus(T_2-I)$ on $X=\ell^2\oplus\ell^2$. Then

$$\sigma(T)=\{1\}\cup\{z:|z|\le 1/2\}\cup\{-1\},\quad \pi_{00}(T)=\{1\},$$

and

$$\omega(T)=\{z:|z|\le 1/2\}\cup\{-1\},$$

which shows that Weyl's theorem holds for T. Let $p(t)=t^2$. Then

$$\sigma(p(T))=\{z:|z|\le 1/4\}\cup\{1\},\quad \pi_{00}(p(T))=\{1\}$$

and

$$\omega(p(T))=\{z:|z|\le 1/4\}\cup\{1\}.$$

Thus $1\in p(\sigma(T)\setminus\pi_{00}(T))$, but $1\notin\sigma(p(T))\setminus\pi_{00}(p(T))$. Also $\omega(p(T))=p(\omega(T))$ but Weyl's theorem does not hold for $p(T)$.

Theorem 2.3.9. *If $p(\omega(T))=\omega(p(T))$ for every polynomial p, then $f(\omega(T))=\omega(f(T))$ for every $f\in H(\sigma(T))$.*

Proof. Let $(p_n(T))$ be a sequence of polynomials converging uniformly in a neighborhood of $\sigma(T)$ to $f(t)$ so that $p_n(T)\to f(T)$. Since $f(T)$ commutes with each $p_n(T)$, it follows from Theorem 2.3.3 that

$$\omega(f(T))=\lim\omega(p_n(T))=\lim p_n(\omega(T))=f(\omega(T)).$$

□

Theorem 2.3.10. *If $T\in B(X)$ then the following are equivalent:*

$$\text{index}(T-\lambda I)\,\text{index}(T-\mu I)\ge 0 \text{ for each pair } \lambda,\mu\in\mathbb{C}\setminus\sigma_e(T); \tag{2.28}$$

$$f(\omega(T))=\omega(f(T)) \text{ for every } f\in H(\sigma(T)). \tag{2.29}$$

Proof. The spectral mapping theorem for the Weyl spectrum may be rewritten as implication, for arbitrary $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}^n$,

$$(T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_n I) \text{ Weyl} \implies T-\lambda_j I \text{ Weyl for each } j=1,2,\cdots,n. \tag{2.30}$$

Now if $\text{index}(T-zI) \geq 0$ on $\mathbb{C}\setminus\sigma_e(T)$ then we have

$$\sum_{j=1}^{n}\text{index}(T-\lambda_j I) = \text{index}\prod_{j=1}^{n}(T-\lambda_j I) = 0 \Longrightarrow \text{index}(T-\lambda_j I) = 0 \ (j=1,2,\cdots,n),$$

and similarly if $\text{index}\,(T-zI) \leq 0$ off $\sigma_e(T)$. If conversely there exist λ, μ for which

$$\text{index}(T-\lambda I) = -m < 0 < k = \text{index}(T-\mu I) \tag{2.31}$$

then

$$(T-\lambda I)^k (T-\mu I)^m \tag{2.32}$$

is a Weyl operator whose factors are not Weyl. This together with Theorem 2.3.9 proves the equivalence of the conditions (2.28) and (2.29). $\square$

Corollary 2.3.11. *If X is a Hilbert space and $T \in B(X)$ is hyponormal then*

$$f(\omega(T)) = \omega(f(T)) \text{ for every } f \in H(\sigma(T)). \tag{2.33}$$

Proof. Immediate from Theorem 2.3.10 together with the fact that if T is hyponormal then $\text{index}\,(T-\lambda I) \leq 0$ for every $\lambda \in \mathbb{C}\setminus\sigma_e(T)$. $\square$

Corollary 2.3.12. *Let $T \in B(X)$. If*

(i) *Weyl's theorem holds for T;*
(ii) *T is isoloid;*
(iii) *T satisfies the spectral mapping theorem for the Weyl spectrum,*

then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Proof. A slight modification of the proof of Lemma 2.3.6 shows that if T is isoloid then

$$f\big(\sigma(T)\setminus\pi_{00}(T)\big) = \sigma(f(T))\setminus\pi_{00}(f(T)) \quad \text{for every } f \in H(\sigma(T)).$$

It thus follows from Theorem 1.7.8 and Corollary 2.3.11 that

$$\sigma(f(T))\setminus\pi_{00}(f(T)) = f\big(\sigma(T)\setminus\pi_{00}(T)\big) = f(\omega(T)) = \omega(f(T)),$$

which implies that Weyl's theorem holds for $f(T)$. $\square$

Corollary 2.3.13. *If $T \in B(X)$ has the SVEP then*

$$\omega(f(T)) = f(\omega(T)) \quad \text{for every } f \in H(\sigma(T)).$$

Proof. If $\lambda \notin \sigma_e(T)$ then by Lemma 2.2.12, $T - \lambda I$ has a finite ascent. Since if $S \in B(X)$ is Fredholm of finite ascent then $\text{index}\,(S) \leq 0$: indeed, either if S has finite descent then S is Browder and hence $\text{index}\,(S) = 0$, or if S does not have finite descent then

$$n\,\text{index}\,(S) = \dim N(S^n) - \dim R(S^n)^{\perp} \;\to\; -\infty \quad \text{as } n \to \infty,$$

which implies that $\text{index}\,(S) < 0$. Thus we have that $\text{index}\,(T - \lambda I) \leq 0$. Thus T satisfies the condition (2.28), which gives the result. □

Theorem 2.3.14. *If $T \in B(X)$ satisfies*

$$\mathcal{X}_T(\{\lambda\}) = N(T - \lambda I) \quad \textit{for every } \lambda \in \mathbb{C},$$

then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Proof. By Corollary 2.2.18, Weyl's theorem holds for T, T is isoloid, and T has the SVEP. In particular by Corollary 2.3.13, T satisfies the spectral mapping theorem for the Weyl spectrum. Thus the result follows from Corollary 2.3.12. □

2.4 Perturbation Theorems

In this section we consider how Weyl's theorem survives under "small" perturbations. Weyl's theorem is transmitted from $T \in B(X)$ to $T - K$ for commuting nilpotents $K \in B(X)$ To see this we need:

Lemma 2.4.1. *If $T \in B(X)$ and if N is a quasinilpotent operator commuting with T then $\omega(T+N) = \omega(T)$.*

Proof. It suffices to show that if $0 \notin \omega(T)$ then $0 \notin \omega(T+N)$. Let $0 \notin \omega(T)$ so that $0 \notin \sigma(\pi(T))$. For all $\lambda \in \mathbb{C}$ we have $\sigma(\pi(T+\lambda N)) = \sigma(\pi(T))$. Thus $0 \notin \sigma(\pi(T+\lambda N))$ for all $\lambda \in \mathbb{C}$, which implies $T + \lambda N$ is a Fredholm operator forall $\lambda \in \mathbb{C}$. But since T is Weyl, it follows that $T+N$ is also Weyl, that is, $0 \notin \omega(T+N)$. □

Theorem 2.4.2. *Let $T \in B(X)$ and let N be a nilpotent operator commuting with T. If Weyl's theorem holds for T then it holds for $T+N$.*

Proof. We first claim that

$$\pi_{00}(T+N) = \pi_{00}(T). \tag{2.34}$$

Let $0 \in \pi_{00}(T)$ so that $\ker(T)$ is finite dimensional. Let $(T+N)x = 0$ for some $x \neq 0$. Then $Tx = -Nx$. Since T commutes with N it follows that

$$T^m x = (-1)^m N^m x \quad \text{for every } m \in \mathbb{N}. \tag{2.35}$$

Let n be the nilpotency of N, i.e., n be the smallest positive integer such that $N^n = 0$. Then by (2.35) we have that for some r with $1 \le r \le n$, $T^r x = 0$ and then $T^{r-1}x \in N(T)$. Thus $N(T+N) \subset N(T^{n-1})$. Therefore $N(T+N)$ is finite dimensional. Also if for some x $(\neq 0)$ $Tx = 0$ then $(T+N)^n x = 0$, and hence 0 is an eigenvalue of $T+N$. Again since $\sigma(T+N) = \sigma(T)$ it follows that $0 \in \pi_{00}(T+N)$. By symmetry $0 \in \pi_{00}(T+N)$ implies $0 \in \pi_{00}(T)$, which proves (2.34). Thus we have

$$\begin{aligned}
\omega(T+N) &= \omega(T) \quad \text{(by Lemma 2.4.1)}\\
&= \sigma(T) \setminus \pi_{00}(T) \quad \text{(since Weyl's theorem holds for } T)\\
&= \sigma(T+N) \setminus \pi_{00}(T+N),
\end{aligned}$$

which shows that Weyl's theorem holds for $T+N$. □

Theorem 2.4.2 however does not extend to quasinilpotents: let

$$Q : (x_1, x_2, x_3, \cdots) \mapsto (\tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \tfrac{1}{4}x_4, \cdots) \text{ on } \ell^2$$

and set on $\ell^2 \oplus \ell^2$,

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}. \tag{2.36}$$

Evidently K is quasinilpotent commutes with T: but Weyl's Theorem holds for T because

$$\sigma(T) = \omega(T) = \{0, 1\} \text{ and } \pi_{00}(T) = \emptyset, \tag{2.37}$$

while Weyl's Theorem does not hold for $T + K$ because

$$\sigma(T + K) = \omega(T + K) = \{0, 1\} \text{ and } \pi_{00}(T + K) = \{0\}. \tag{2.38}$$

But if K is an *injective* quasinilpotent operator commuting with T then Weyl's theorem is transmitted from T to $T + K$.

Theorem 2.4.3. *If Weyl's theorem holds for $T \in B(X)$ then Weyl's theorem holds for $T + K$ if $K \in B(X)$ is an injective quasinilpotent operator commuting with T.*

Proof. First of all we prove that if there exists an injective quasinilpotent operator commuting with T, then

$$T \text{ is Weyl} \implies T \text{ is injective.} \tag{2.39}$$

To show this suppose K is an injective quasinilpotent operator commuting with T. Assume to the contrary that T is Weyl but not injective. Then there exists a nonzero vector $x \in X$ such that $Tx = 0$. Then by the commutativity assumption, $TK^n x = K^n T x = 0$ for every $n = 0, 1, 2, \cdots$, so that $K^n x \in N(T)$ for every $n = 0, 1, 2, \cdots$. We now claim that $\{K^n x\}_{n=0}^{\infty}$ is a sequence of linearly independent vectors in X. To see this suppose $c_0 x + c_1 K x + \cdots + c_n K^n x = 0$. We may then write $c_n(K - \lambda_1 I) \cdots (K - \lambda_n I)x = 0$. Since K is an injective quasinilpotent operator it follows that $(K - \lambda_1 I) \cdots (K - \lambda_n I)$ is injective. But since $x \neq 0$ we have that $c_n = 0$. By an induction we also have that $c_{n-1} = \cdots = c_1 = c_0 = 0$. This shows that $\{K^n x\}_{n=0}^{\infty}$ is a sequence of linearly independent vectors in X. From this we can see $N(T)$ is infinite-dimensional, which contradicts to the fact that T is Weyl. This proves (2.39). From (2.39) we can see that if Weyl's theorem holds for T then $\pi_{00}(T) = \emptyset$. We now claim that $\pi_{00}(T + K) = \emptyset$. Indeed if $\lambda \in \pi_{00}(T + K)$, then $0 < \dim N(T + K - \lambda I) < \infty$, so that there exists a nonzero vector $x \in X$ such that $(T + K - \lambda I)x = 0$. But since K commutes with $T + K - \lambda I$, the same argument as in the proof of (2.39) with $T + K - \lambda I$ in place of T shows that $N(T + K - \lambda I)$ is infinite-dimensional, a contradiction. Therefore $\pi_{00}(T + K) = \emptyset$ and hence Weyl's theorem holds for $T + K$ because $\varpi(T) = \varpi(T + K)$ with $\varpi = \sigma, \omega$. $\square$

In Theorem 2.4.3, "quasinilpotent" cannot be replaced by "compact". For example consider the following operators on $\ell^2 \oplus \ell^2$:

$$T = \begin{pmatrix} 0 & & & & \\ & \frac{1}{2} & & 0 & \\ & & \frac{1}{3} & & \\ & 0 & & \frac{1}{4} & \\ & & & & \ddots \end{pmatrix} \oplus I \quad \text{and} \quad K = \begin{pmatrix} 1 & & & & \\ & -\frac{1}{2} & & 0 & \\ & & -\frac{1}{3} & & \\ & 0 & & -\frac{1}{4} & \\ & & & & \ddots \end{pmatrix} \oplus Q,$$

where Q is an injective compact quasinilpotent operator on ℓ^2. Observe that Weyl's theorem holds for T, K is an injective compact operator, and $TK = KT$. But

$$\sigma(T+K) = \{0, 1\} = \omega(T+K) \quad \text{and} \quad \pi_{00}(T+K) = \{1\},$$

which says that Weyl's theorem does not hold for $T+K$.

On the other hand, Weyl's theorem for T is not sufficient for Weyl's theorem for $T+F$ with finite rank F. To see this, let $X = \ell^2$ and let $T, F \in B(X)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, x_1/2, x_2/3, \cdots)$$

and

$$F(x_1, x_2, x_3, \cdots) = (0, -x_1/2, 0, 0, \cdots).$$

since the point spectrum of T is empty it follows Weyl's theorem holds for T. Also F is a nilpotent operator. Since $0 \in \pi_{00}(T+F) \cap \omega(T+F)$, it follows that Weyl's theorem fails for $T+F$.

Lemma 2.4.4. *Let $T \in B(X)$. If $F \in B(X)$ is a finite rank operator then*

$$\dim N(T) < \infty \iff \dim N(T+F) < \infty.$$

Further if $TF = FT$ then

$$\operatorname{acc}\sigma(T) = \operatorname{acc}\sigma(T+F).$$

Proof. This follows from a straightforward calculation. □

Theorem 2.4.5. *Let $T \in B(X)$ be an isoloid operator and let $F \in B(X)$ be a finite rank operator commuting with T. If Weyl's theorem holds for T then it holds for $T+F$.*

Proof. We have to show that $\lambda \in \sigma(T+F) \setminus \omega(T+F)$ if and only if $\lambda \in \pi_{00}(T+F)$. Without loss of generality we may assume that $\lambda = 0$. We first suppose that $0 \in \sigma(T+F) \setminus \omega(T+F)$ and thus $T+F$ is Weyl but not invertible. It suffice to show that $0 \in \operatorname{iso}\sigma(T+F)$. Since T is Weyl and Weyl's theorem holds for T, it follows that $0 \in \rho(T)$ or $0 \in \operatorname{iso}\sigma(T)$. Thus by Lemma 2.4.4, $0 \notin \operatorname{acc}\sigma(T+F)$. But since $T+F$ is not invertible we have that $0 \in \operatorname{iso}\sigma(T+F)$.

Conversely, suppose that $0 \in \pi_{00}(T+F)$. We want to show that $T+F$ is Weyl. By our assumption, $0 \in \operatorname{iso}\sigma(T+F)$ and $0 < \dim N(T+F) < \infty$. By Lemma 2.4.4, we have

$$0 \notin \operatorname{acc}\sigma(T) \quad \text{and} \quad \dim N(T) < \infty. \tag{2.40}$$

If T is invertible then it is evident that $T+F$ is Weyl. If T is not invertible then by the first part of (2.40) we have $0 \in \operatorname{iso}\sigma(T)$. But since T is isoloid it follows that T is not one-one, which together with the second part of (2.40) gives $0 < \dim N(T) < \infty$. Since Weyl's theorem holds for T it follows that T is Weyl and so is $T+F$. □

Example 2.4.6. *There exists an operator $T \in B(X)$ and a finite rank operator $F \in B(X)$ commuting with T such that Weyl's theorem holds for T but it does not hold for $T + F$.*

Proof. Define on $\ell^2 \oplus \ell^2$, $T := I \oplus S$ and $F = K \oplus 0$, where $S : \ell^2 \to \ell^2$ is an injective quasinilpotent operator and $F : \ell^2 \to \ell^2$ is defined by

$$F(x_1, x_2, x_3, \cdots) = (-x_1, 0, 0, \cdots).$$

Then F is of finite rank and commutes with T. It is easy to see that

$$\sigma(T) = \omega(T) = \{0, 1\} \quad \text{and} \quad \pi_{00}(T) = \emptyset,$$

which implies that Weyl's theorem holds for T. We however have

$$\sigma(T + F) = \omega(T + F) = \{0, 1\} \quad \text{and} \quad \pi_{00}(T + F) = \{0\},$$

which implies that Weyl's theorem fails for $T + F$. □

Theorem 2.4.5 may fail if "finite rank" is replaced by "compact". In fact Weyl's theorem may fail even if K is both compact and quasinilpotent: for example, take $T = 0$ and K the operator on ℓ_2 defined by $K(x_1, x_2, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \cdots)$. We will however show that if "isoloid" condition is strengthened slightly then Weyl's theorem is transmitted from T to $T + K$ if K is either a compact or a quasinilpotent operator commuting with T. To see this we observe:

Lemma 2.4.7. *If $K \in B(X)$ is a compact operator commuting with $T \in B(X)$ then*

$$\pi_{00}(T + K) \subseteq iso\, \sigma(T) \cup \rho(T).$$

Proof. See [HanL2]. □

An operator $T \in B(X)$ will be said to be *finite-isoloid* if $\operatorname{iso} \sigma(T) \subseteq \pi_{0f}(T)$. Evidently finite-isoloid ⇒ isoloid. The converse is not true in general: for example, take $T = 0$. In particular if $\sigma(T)$ has no isolated points then T is finite-isoloid. We now have:

Theorem 2.4.8. *Suppose $T \in B(X)$ is finite-isoloid. If Weyl's theorem holds for T then Weyl's theorem holds for $T + K$ if $K \in B(X)$ commutes with T and is either compact or quasinilpotent.*

Proof. First we assume that K is a compact operator commuting with T. Suppose Weyl's theorem holds for T. We first claim that with no restriction on T,

$$\sigma(T + K) \setminus \omega(T + K) \subseteq \pi_{00}(T + K). \tag{2.41}$$

For (2.41), it suffices to show that if $\lambda \in \sigma(T + K) \setminus \omega(T + K)$ then $\lambda \in \operatorname{iso} \sigma(T + K)$. Assume to the contrary that $\lambda \in \operatorname{acc} \sigma(T + K)$. Then we have that $\lambda \in \sigma_b(T + K) = \sigma_b(T)$, so that $\lambda \in \sigma_e(T)$ or $\lambda \in \operatorname{acc} \sigma(T)$. Remember that the essential spectrum and the Weyl spectrum are invariant under compact perturbations. Thus if $\lambda \in \sigma_e(T)$ then $\lambda \in \sigma_e(T+K) \subseteq \omega(T+K)$, a contradiction. Therefore we should have that $\lambda \in \operatorname{acc}\sigma(T)$. But since Weyl's theorem holds for T and $\lambda \notin \omega(T+K) = \omega(T)$, it follows that $\lambda \in \pi_{00}(T)$, a contradiction. This proves (2.41). For the reverse inclusion suppose $\lambda \in \pi_{00}(T+K)$. Then by Lemma 2.4.7, either $\lambda \in \operatorname{iso}\sigma(T)$ or $\lambda \in \rho(T)$. If $\lambda \in \rho(T)$ then evidently $T+K-\lambda I$ is Weyl, i.e., $\lambda \notin \omega(T+K)$. If instead $\lambda \in \operatorname{iso}\sigma(T)$ then $\lambda \in \pi_{00}(T)$ whenever T is finite-isoloid. Since Weyl's theorem holds for T, it follows that $\lambda \notin \omega(T)$ and hence $\lambda \notin \omega(T+K)$. Therefore Weyl's theorem holds for $T+K$.

Next we assume that K is a quasinilpotent operator commuting with T. Then by Lemma 2.4.1, $\varpi(T) = \varpi(T+Q)$ with $\varpi = \sigma, \omega$. Suppose Weyl's theorem holds for T. Then

$$\sigma(T+K) \setminus \omega(T+K) = \sigma(T) \setminus \omega(T) = \pi_{00}(T) \subseteq \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T+K),$$

which implies that $\sigma(T+K) \setminus \omega(T+K) \subseteq \pi_{00}(T+K)$. Conversely, suppose $\lambda \in \pi_{00}(T+K)$. If T is finite-isoloid then $\lambda \in \operatorname{iso}\sigma(T+K) = \operatorname{iso}\sigma(T) \subseteq \pi_{0f}(T)$. Thus $\lambda \in \pi_{00}(T) = \sigma(T) \setminus \omega(T) = \sigma(T+K) \setminus \omega(T+K)$. This completes the proof. □

Corollary 2.4.9. *Suppose X is a Hilbert space and $T \in B(X)$ is p-hyponormal. If T satisfies one of the following:*

(i) $\operatorname{iso}\sigma(T) = \emptyset$;

(ii) *T has finite-dimensional eigenspaces,*

then Weyl's theorem holds for $T+K$ if $K \in B(X)$ is either compact or quasinilpotent and commutes with T.

Proof. Observe that each of the conditions (i) and (ii) forces p-hyponormal operators to be finite-isoloid. Since by Corollary 2.2.5 Weyl's theorem holds for p-hyponormal operators, the result follows at once from Theorem 2.4.8. □

In the perturbation theory the "commutative" condition looks so rigid. Without the commutativity, the spectrum can however undergo a large change under even rank one perturbations. In spite of it, Weyl's theorem may hold for (non-commutative) compact perturbations of "good" operators. We now give such a perturbation theorem. To do this we need:

Lemma 2.4.10. *If $N \in B(X)$ is a quasinilpotent operator commuting with $T \in B(X)$ modulo compact operators (i.e., $TN - NT \in K(X)$) then $\sigma_e(T+N) = \sigma_e(T)$ and $\omega(T+N) = \omega(T)$.*

Proof. Immediate from Lemma 2.4.1. □

Theorem 2.4.11. *Suppose $T \in B(X)$ satisfies the following:*

(i) *T is finite-isoloid;*

(ii) *$\sigma(T)$ has no "holes" (bounded components of the complement), i.e., $\sigma(T) = \eta\,\sigma(T)$;*

(iii) *$\sigma(T)$ has at most finitely many isolated points;*

(iv) *Weyl's theorem holds for T.*

If $K \in B(X)$ is either compact or quasinilpotent and commutes with T modulo compact operators then Weyl's theorem holds for $T + K$.

Proof. By Lemma 2.4.10, we have that $\sigma_e(T+K) = \sigma_e(T)$ and $\omega(T+K) = \omega(T)$. Suppose Weyl's theorem holds for T and $\lambda \in \sigma(T+K) \setminus \omega(T+K)$. We now claim that $\lambda \in \operatorname{iso}\sigma(T+K)$. Assume to the contrary that $\lambda \in \operatorname{acc}\sigma(T+K)$. Since $\lambda \notin \omega(T+K) = \omega(T)$, it follows from the punctured neighborhood theorem that $\lambda \notin \partial\,\sigma(T+K)$. Also since the set of all Weyl operators forms an open subset of $B(X)$, we have that $\lambda \in \operatorname{int}\big(\sigma(T+K) \setminus \omega(T+K)\big)$. Then there exists $\epsilon > 0$ such that $\{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\} \subseteq \operatorname{int}\big(\sigma(T+K) \setminus \omega(T+K)\big)$, and hence $\{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\} \cap \omega(T) = \emptyset$. But since

$$\partial\,\sigma(T+K) \setminus \operatorname{iso}\sigma(T+K) \subseteq \sigma_e(T+K) = \sigma_e(T),$$

it follows from our assumption that

$$\begin{aligned}\{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\} &\subseteq \operatorname{int}\big(\sigma(T+K) \setminus \omega(T+K)\big)\\ &\subseteq \eta\,\big(\partial\,\sigma(T+K) \setminus \operatorname{iso}\sigma(T+K)\big)\\ &\subseteq \eta\,\sigma_e(T) \subseteq \eta\,\sigma(T) = \sigma(T),\end{aligned}$$

which implies that $\{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\} \subseteq \sigma(T) \setminus \omega(T)$. This contradicts to Weyl's theorem for T. Therefore $\lambda \in \operatorname{iso}\sigma(T+K)$ and hence $\sigma(T+K) \setminus \omega(T+K) \subseteq \pi_{00}(T+K)$. For the reverse inclusion suppose $\lambda \in \pi_{00}(T+K)$. Assume to the contrary that $\lambda \in \omega(T+K)$ and hence $\lambda \in \omega(T)$. Then we claim $\lambda \notin \partial\,\sigma(T)$. Indeed if $\lambda \in \operatorname{iso}\sigma(T)$ then by assumption $\lambda \in \pi_{00}(T)$, which contradicts to Weyl's theorem for T. If instead $\lambda \in \operatorname{acc}\sigma(T) \cap \partial\,\sigma(T)$ then since $\operatorname{iso}\sigma(T)$ is finite it follows that

$$\lambda \in \operatorname{acc}\big(\partial\,\sigma(T)\big) \subseteq \operatorname{acc}\sigma_e(T) = \operatorname{acc}\sigma_e(T+K),$$

which contradicts to the fact that $\lambda \in \operatorname{iso}\sigma(T+K)$. Therefore $\lambda \notin \partial\,\sigma(T)$. Also since $\lambda \in \operatorname{iso}\sigma(T+K)$, there exists $\epsilon > 0$ such that

$$\{\mu \in \mathbb{C} : 0 < |\mu - \lambda| < \epsilon\} \subseteq \sigma(T) \cap \rho(T+K),$$

so that $\{\mu \in \mathbb{C} : 0 < |\mu - \lambda| < \epsilon\} \cap \omega(T) = \emptyset$, which contradicts to Weyl's theorem for T. Thus $\lambda \in \sigma(T+K) \setminus \omega(T+K)$ and therefore Weyl's theorem holds for $T+K$. $\square$

If, in Theorem 2.4.11, the condition "$\sigma(T)$ has no holes" is dropped then Theorem 2.4.11 may fail even though T is normal. For example, if on $\ell_2 \oplus \ell_2$

$$T = \left(\begin{smallmatrix} U & I-UU^* \\ 0 & U^* \end{smallmatrix}\right) \quad \text{and} \quad K = \left(\begin{smallmatrix} 0 & I-UU^* \\ 0 & 0 \end{smallmatrix}\right),$$

where U is the unilateral shift on ℓ_2, then T is unitary (essentially the bilateral shift) with $\sigma(T) = \mathbb{T}$, K is a rank one nilpotent, and Weyl's theorem does not hold for $T - K$.

Also in Theorem 2.4.11, the condition "iso $\sigma(T)$ is finite" is essential in the cases where K is compact. For example, if on ℓ_2

$$T(x_1, x_2, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots) \quad \text{and} \quad Q(x_1, x_2, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \cdots),$$

we define $K := -(T + Q)$. Then we have that (i) T is finite-isoloid; (ii) $\sigma(T)$ has no holes; (iii) Weyl's theorem holds for T; (iv) iso $\sigma(T)$ is infinite; (v) K is compact because T and Q are both compact; (vi) Weyl's theorem does not hold for $T + K$ $(= -Q)$.

Corollary 2.4.12. *If $\sigma(T)$ has no holes and at most finitely many isolated points and if K is a compact operator then Weyl's theorem is transmitted from T to $T + K$.*

Proof. Straightforward from Theorem 2.4.11. $\square$

Corollary 2.4.12 shows that if Weyl's theorem holds for T whose spectrum has no holes and at most finitely many isolated points then for every compact operator K, the passage from $\sigma(T)$ to $\sigma(T+K)$ is putting at most countably many isolated points outside $\sigma(T)$ which are Riesz points of $\sigma(T + K)$. Here we should note that this holds even if T is quasinilpotent because for every quasinilpotent operator T (more generally, "Riesz operators"), we have

$$\sigma(T + K) \subseteq \eta\, \sigma_e(T + K) \cup p_{00}(T + K) = \eta\, \sigma_e(T) \cup p_{00}(T + K) = \{0\} \cup p_{00}(T + K).$$

2.5 Comments and Problems

(a) Transaloid and SVEP. For an operator $T \in B(X)$ for a Hilbert space X, denote $W(T) = \{(Tx, x) : \ ||x|| = 1\}$ for the numerical range of T and $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$ for the numerical radius of T. An operator T is called *convexoid* if $\operatorname{conv}\sigma(T) = \operatorname{cl} W(T)$ and is called *spectraloid* if $w(T) = r(T) =$ the spectral radius. We call an operator $T \in B(X)$ *transaloid* if $T - \lambda I$ is normaloid for all $\lambda \in \mathbb{C}$. It was well known that

$$\begin{aligned}&\text{transaloid} \implies \text{convexoid} \implies \text{spectraloid},\\ &(G_1) \implies \text{convexoid} \quad \text{and} \quad (G_1) \implies \text{reguloid}.\end{aligned}$$

We would like to expect that Corollary 2.2.16 remains still true if "reguloid" is replaced by "transaloid"

Problem 2.1. *If $T \in B(X)$ is transaloid and has the SVEP, does Weyl's theorem hold for T ?*

The following question is a strategy to answer Problem 2.1.

Problem 2.2. *Does it follow that*

$$\textit{transaloid} \implies \textit{reguloid}\,?$$

If the answer to Problem 2.2 is affirmative then the answer to Problem 2.1 is affirmative by Corollary 2.2.16.

(b) $*$-paranormal operators. An operator $T \in B(X)$ for a Hilbert space X is said to be *$*$-paranormal* if

$$||T^*x||^2 \le ||T^2x||\,||x|| \quad \text{for every } x \in X.$$

It was [AT] known that if $T \in B(X)$ is $*$-paranormal then the following hold:

$$T \text{ is normaloid;} \tag{2.42}$$

$$N(T - \lambda I) \subset N((T - \lambda I)^*). \tag{2.43}$$

So if $T \in B(X)$ is $*$-paranormal then by (2.43), $T - \lambda I$ has finite ascent for every $\lambda \in \mathbb{C}$. Thus $*$-paranormal operators have the SVEP ([La]). On the other hand, by the same argument as the proof of Corollary **??** we can see that if $T \in B(X)$ is $*$-paranormal then

$$\sigma(T) \setminus \omega(T) \subset \pi_{00}(T). \tag{2.44}$$

However we were unable to decide:

Problem 2.3. *Does Weyl's theorem hold for $*$-paranormal operators ?*

The following question is a strategy to answer Problem 2.3.

Problem 2.4. *Is every $*$-paranormal operator isoloid?*

If the answer to Problem 2.4 is affirmative then the answer to Problem 2.3 is affirmative. To see this suppose $T \in B(X)$ is $*$-paranormal. In view of (2.44), it suffices to show that $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$. Assume $\lambda \in \pi_{00}(T)$. By (2.43), $T-\lambda I$ is reduced by its eigenspaces. Thus we can write

$$T - \lambda I = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} : \begin{bmatrix} N(T-\lambda I) \\ N(T-\lambda I)^{\perp} \end{bmatrix} \longrightarrow \begin{bmatrix} N(T-\lambda I) \\ N(T-\lambda I)^{\perp} \end{bmatrix}.$$

Thus $T = \left(\begin{smallmatrix} \lambda I & 0 \\ 0 & S+\lambda I \end{smallmatrix}\right)$. We now claim that S is invertible. Assume to the contrary that S is not invertible. Then $0 \in \operatorname{iso}\sigma(S)$ since $\lambda \in \operatorname{iso}\sigma(T)$. Thus $\lambda \in \operatorname{iso}\sigma(S+\lambda I)$. But since $S+\lambda I$ is also $*$-paranormal, it follows from our assumption that λ is an eigenvalue of $S+\lambda I$. Thus $0 \in \pi_0(S)$, which contradicts to the fact that S is one-one. Therefore S should be invertible. Note that $N(T-\lambda I)$ is finite-dimensional. Thus evidently $T-\lambda I$ is Weyl, so that $\lambda \in \sigma(T) \setminus \omega(T)$. This gives a proof.

(c) Subclasses of paranormal operators. An operator $T \in B(X)$ for a Hilbert space X is said to be *quasihyponormal* if $T^*(T^*T - TT^*)T \geq 0$ and is said to be *class* A-operator if $|T^2| \geq |T|^2$ (cf. [FIY]). Let $T = U\,|T|$ be the polar decomposition of T and $\widetilde{T} := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ be the Aluthge transformation of T (cf. [Al]). An operator $T \in B(X)$ for a Hilbert space X is called *w-hyponormal* if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$. It was well known that

$$\text{hyponormal} \Longrightarrow \text{quasihyponormal} \Longrightarrow \text{class } A \Longrightarrow \text{paranormal} \tag{2.45}$$

$$\text{hyponormal} \Longrightarrow p\text{-hyponormal} \Longrightarrow w\text{-hyponormal} \Longrightarrow \text{paranormal}. \tag{2.46}$$

Since by Theorem 2.2.20, Weyl's theorem holds for paranormal operators on an arbitrary Banach space, all classes of operators in (2.45) and (2.46) enjoy Weyl's theorem.

Chapter 3

Hyponormal and Subnormal Theory

3.1 Hyponormal Operators

An operator $A \in B(H)$ is called *hyponormal* if

$$[A^*, A] \equiv A^*A - AA^* \geq 0.$$

Thus if $A \in B(H)$ then

$$A \text{ is hyponormal } \iff \|Ah\| \geq \|A^*h\| \text{ for all } h \in H.$$

If $A^*A \leq AA^*$, or equivalently, $\|A^*h\| \geq \|Ah\|$ for all h, then A is called a *cohyponormal* operator. Operators that are either hyponormal or cohyponormal are called *seminormal.*

Proposition 3.1.1. *Let $A \in B(H)$ be a hyponormal operator. Then we have:*

(a) *If A is invertible then A^{-1} is hyponormal.*

(b) *$A - \lambda$ is hyponormal for every $\lambda \in \mathbb{C}$.*

(c) *If $\lambda \in \pi_0(A)$ and $Af = \lambda f$ then $A^*f = \overline{\lambda} f$, i.e., $\ker(A-\lambda) \subseteq \ker(A-\lambda)^*$.*

(d) *If f and g are eigenvectors corresponding to distinct eigenvalues of A then $f \perp g$.*

(e) *If $\mathcal{M} \in \operatorname{Lat} A$ then $A|_{\mathcal{M}}$ is hyponormal.*

Proof. (a) Recall that if T is positive and invertible then

$$T \geq 1 \Longrightarrow T^{-1} \leq 1:$$

because if $T \in C^*(T) \equiv C(X)$ then $T = f \geq 1 \Rightarrow T^{-1} = \frac{1}{f} \leq 1$. Since $A^*A \geq AA^*$ and A is invertible,

$$\begin{aligned} &A^{-1}(A^*A)(A^*)^{-1} \geq A^{-1}(AA^*)(A^*)^{-1} = 1 \\ &\Longrightarrow A^*A^{-1}(A^*)^{-1}A \leq 1 \\ &\Longrightarrow A^{-1}(A^*)^{-1} = (A^*)^{-1}(A^*A^{-1}A^{*-1}A)A^{-1} \leq (A^*)^{-1}A^{-1} \\ &\Longrightarrow A^{-1} \text{ is hyponormal.} \end{aligned}$$

(b) $(A-\lambda)(A^*-\overline{\lambda}) = AA^*-\lambda A^*-\overline{\lambda}A+|\lambda|^2 \leq A^*A-\lambda A^*-\overline{\lambda}A+|\lambda|^2 = (A^*-\overline{\lambda})(A-\lambda)$.

(c) Immediate from the fact that $\|(A^*-\overline{\lambda})f\| \leq \|(A-\lambda)f\|$.

(d) $Af = \lambda f,\ Ag = \mu g \Rightarrow \lambda\langle f, g\rangle = \langle Af, g\rangle = \langle f, A^*g\rangle = \langle f, \overline{\mu}g\rangle = \mu\langle f, g\rangle$.

(e) If $\mathcal{M} \in \operatorname{Lat} A$ then

$$\begin{aligned} &A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^\perp \end{matrix} \text{ is hyponormal} \\ &\Longrightarrow 0 \leq [A^*, A] = \begin{bmatrix} [B^*, B] - CC^* & * \\ * & * \end{bmatrix} \\ &\Longrightarrow [B^*, B] \geq CC^* \geq 0 \\ &\Longrightarrow B \text{ is hyponormal.} \end{aligned}$$

□

Corollary 3.1.2. *If A is hyponormal and $\lambda \in \pi_0(A)$ then* $\ker(A-\lambda)$ *reduces A. Hence if A is a pure hyponormal then $\pi_0(A) = \emptyset$.*

Proof. From Proposition 3.1.1(c), if $f \in \ker(A-\lambda)$ then $Af = \lambda f \in \ker(A-\lambda)$ and $A^*f = \overline{\lambda}f \in \ker(A-\lambda)$. □

Proposition 3.1.3. [Sta1] *If A is hyponormal then $\|A^n\| = \|A\|^n$, so*

$$\|A\| = \gamma(A), \ \textit{where } r(\cdot) \textit{ denoted the spectral radius},$$

in other words, A is normaloid.

Proof. Observe

$$\|A^n f\|^2 =< A^n f, A^n f >=< A^*A^n f, A^{n-1}f > \leq \|A^*A^n f\| \cdot \|A^{n-1}f\| \leq \|A^{n+1}f\| \cdot \|A^{n-1}f\|.$$

Hence $\|A^n\|^2 \leq \|A^{n+1}\| \cdot \|A^{n-1}\|$. We use an induction. Clearly, it is true for $n = 1$. Suppose $\|A^k\| = \|A\|^k$ for $1 \leq k \leq n$. Then $\|A\|^{2n} = \|A^n\|^2 \leq \|A^{n+1}\| \cdot \|A^{n-1}\| = \|A^{n+1}\| \cdot \|A\|^{n-1}$, so $\|A\|^{n+1} \leq \|A^{n+1}\|$. Also $r(A) = \lim \|A^n\|^{\frac{1}{n}} = \|A\|$. □

Corollary 3.1.4. *If A is hyponormal and $\lambda \notin \sigma(A)$ then*

$$\frac{1}{\|(\lambda - A)^{-1}\|} = \text{dist}\,\big(\lambda, \sigma(A)\big).$$

Proof. Observe

$$||\frac{1}{(\lambda - A)^{-1}}|| = \frac{1}{\max_{\mu \in \sigma(\lambda - A)^{-1}} |\mu|} = \min_{\mu \in \sigma(\lambda - A)} |\mu| = \text{dist}\big(\lambda, \sigma(A)\big).$$

□

Proposition 3.1.5. [Sta1] *If A is hyponormal then A is isoloid, i.e.,* $\text{iso}\,\sigma(A) \subseteq \pi_0(A)$. *The pure hyponormal operators have no isolated points in their spectrum.*

Proof. Replacing A by $A - \lambda$ we may assume that $\lambda = 0$. Observe that the only quasinilpotent hyponormal operator is zero. Consider the spectral decomposition of A:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \text{ where } \sigma(A_1) = \{0\},\ \sigma(A_2) = \sigma(A)\backslash\{0\}.$$

Then $A_1 = 0$, so $0 \in \pi_0(A)$.

The second assertion comes from the fact that $\ker(A - \lambda)$ is a reducing subspaces of a hyponormal operator A. □

Corollary 3.1.6. *The only compact hyponormal operator is normal.*

Proof. Recall that if K is compact then every nonzero point of $\sigma(K)$ is isolated. So if K is hyponormal then every eigenspaces reduces K and the restriction of K to each eigenspace is normal. Consider the restriction of K to the orthogonal complement of the span of all the eigenvectors. The resulting operator is hyponormal and quasinilpotent, and hence 0. Therefore K is normal. □

Proposition 3.1.7. *Let A be a hyponormal operator. Then we have:*

(a) *A is invertible $\Longleftrightarrow$ A is right invertible.*
(b) *A is Fredholm $\Longleftrightarrow$ A is right Fredholm.*
(c) *$\sigma(A) = \sigma_r(A)$ and $\sigma_e(A) = \sigma_{re}(A)$.*
(d) *A is pure, $\lambda \in \sigma(A)\backslash\sigma_e(A) \Longrightarrow \text{index}\,(A - \lambda) \leq -1$.*

Proof. (a) Observe that

$$\begin{aligned} A \text{ is right invertible} &\Longrightarrow \exists\ B \text{ such that } AB = 1 \\ &\Longrightarrow A \text{ is onto and hence } \ker A^* = (\text{ran} A)^\perp = \{0\} \\ &\Longrightarrow \ker A = \{0\} \\ &\Longrightarrow A \text{ is invertible.} \end{aligned}$$

(b) Similar to (a).

(c) From (a) and (b).

(d) Observe that

$$\begin{aligned} A \text{ is pure hyponormal} &\Longrightarrow A-\lambda \text{ is pure hyponormal} \\ &\Longrightarrow \ker(A-\lambda)=\{0\} \text{ (by Proposition 3.1.5)} \\ &\Longrightarrow A-\lambda \text{ is not onto since } \lambda\in\sigma(A) \\ &\Longrightarrow \operatorname{index}(A-\lambda)=\dim(\ker(A-\lambda))-\dim\left(\operatorname{ran}(A-\lambda)^{\perp}\right) \\ &\qquad\qquad = -\dim\left(\operatorname{ran}(A-\lambda)^{\perp}\right)\le -1. \end{aligned}$$

□

Write $\mathcal{F}$ denotes the set of Fredholm operators. We here give a direct proof showing that Weyl's theorem holds for hyponormal operators.

Proposition 3.1.8. *If $A \in B(H)$ is hyponormal then*

$$\sigma(A)\backslash\omega(A)=\pi_{00}(A),$$

where $\pi_{00}(A)$ = the set of isolated eigenvalues of finite multiplicity.

Proof. ($\Leftarrow$) If $\lambda\in\pi_{00}(A)$ then $\ker(A-\lambda)$ reduces A. So

$$A=\lambda I\bigoplus B,$$

where I is the identity on a finite dimensional space, B is hyponormal and $\lambda\notin\sigma(B)$. So $\lambda\notin\omega(A)$.

($\Rightarrow$) Suppose $\lambda\in\sigma(A)\backslash\omega(A)$, and so $A-\lambda$ not invertible, Fredholm with index $(A-\lambda)=0$. We may assume $\lambda=0$. Since $A\in\mathcal{F}$ and index $A=0$, it follows that 0 is an eigenvalue of finite multiplicity.

It remains to show that $0\in\operatorname{iso}\sigma(A)$. Observe that

$$\ker(A)\subseteq\ker(A^*)=(\operatorname{ran}A)^{\perp} \text{ and } 0=\operatorname{index}(A)=\dim(\ker(A))-\dim\left(\operatorname{ran}A\right)^{\perp}\right),$$

so that $\ker(A)=(\operatorname{ran}A)^{\perp}$. So

$$A=0\bigoplus B,$$

where B is invertible. Since $\sigma(A)=\{0\}\cup\sigma(B)$, 0 must be an isolated point of $\sigma(A)$. □

Corollary 3.1.9. *If $A\in B(H)$ is a pure hyponormal then*

$$\|A\|\le\|A+K\| \text{ for every compact operator } K.$$

Proof. Since A is pure, $\pi_0(A)=\emptyset$. So $\sigma(A)=\omega(A)=\bigcap_{K\in K(H)}\sigma(A+K)$. Thus for every compact operator K, $\sigma(A)\subseteq\sigma(A+K)$. Therefore, $\|A\|=r(A)\le r(A+K)\le\|A+K\|$. □

3.2 The Berger-Shaw Theorem

If A is a selfadjoint operator then A is said to be *absolutely continuous* if its scalar-valued spectral measure is absolutely continuous with respect to the Lebesgue measure on the line.

Let $N = \int z dE(z)$ be the spectral decomposition of N. A scalar-valued spectral measure for N is a positive Borel measure μ on $\sigma(N)$ such that

$$\mu(\triangle) = 0 \Longleftrightarrow E(\triangle) = 0.$$

Since $W^*(N)$ is an abelian von Neumann algebra, $W^*(N)$ has a separating vector e_0, i.e.,

$$Ae_0 = 0 \Longrightarrow A = 0 \text{ for } A \in W^*(N).$$

Define μ on $\sigma(N)$ by

$$\mu(\triangle) = \|E(\triangle)e_0\|^2.$$

In fact, this μ is the unique scalar-valued spectral measure for N.

Theorem 3.2.1. (Putnam, 1963) *If S is a pure hyponormal operator and $S = A+iB$, where A and B are selfadjoint then A and B are absolutely continuous.*

Proof. See [Con2, p.150]. $\square$

Definition 3.2.2. An operator $T \in B(H)$ is said to be *finitely multicyclic* if there exist a finite number of vectors $g_1, \cdots, g_m \in H$ such that

$$H = \bigvee \big\{ f(T)g_j : 1 \leq j \leq m \text{ and } f \in \operatorname{Rat} \sigma(T) \big\}.$$

The vectors $g_1, \cdots, g_m$ are called *generating vectors.* If T is finitely multicyclic and m is the smallest number of generating vectors then T is said to be *m-multicyclic.*

Theorem 3.2.3. (The Berger-Shaw Theorem) *If T is an m-multicyclic hyponormal operator then $[T^*, T]$ is a trace class operator and*

$$\operatorname{tr} [T^*, T] \leq \frac{m}{\pi} \operatorname{Area} (\sigma(T)).$$

This inequality is sharp: indeed, consider the unilateral shift T:

$$[T^*, T] = \begin{bmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{bmatrix}, \quad \sigma(T) = \operatorname{cl} \mathbb{D}, \quad m = 1,$$

so

$$\mathrm{tr}\,[T^*, T] = 1,\ \frac{m}{\pi}\mathrm{Area}(\sigma(T)) = \frac{1}{\pi}\cdot\pi = 1.$$

To prove Theorem 3.2.3 we need auxiliary lemmas. Recall the Hilbert-Schmidt norm of X:

$$\begin{aligned}\|X\|_2 &\equiv \left[\sum \langle |X|^2 e_n, e_n\ \rangle\right]^{\frac{1}{2}}\\ &= \left[\sum \langle X^*Xe_n, e_n\rangle\right]^{\frac{1}{2}}\\ &= \left[\mathrm{tr}\,(X^*X)\right]^{\frac{1}{2}}.\end{aligned}$$

Lemma 3.2.4. *If $T \in B(H)$ and P is a finite rank projection then*

$$\mathrm{tr}\,\big(P[T^*, T]P\big) \le \|P^\perp TP\|_2^2.$$

Proof. Write

$$T = \begin{bmatrix} A & B \\ C & P\end{bmatrix} : \begin{bmatrix}\mathrm{ran}P \\ \mathrm{ran}P^\perp\end{bmatrix} \to \begin{bmatrix}\mathrm{ran}P \\ \mathrm{ran}P^\perp\end{bmatrix}.$$

Since $P = \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}$,

$$P[T^*, T]P = [A^*, A] + C^*C - BB^*.$$

So by the above remark, $\mathrm{tr}\,(P[T^*, T]P) = \mathrm{tr}[A^*, A] + \|C\|_2^2 - \|B\|_2^2$. But since A is a finite-dimensional operator,

$$\mathrm{tr}[A^*, A] = 0.$$

Hence $\mathrm{tr}\,(P[T^*, T]P) \le \|C\|_2^2 = ||P^\perp TP||_2^2$. □

Lemma 3.2.5. *If $T \in B(H)$ is an m-multicyclic operator then there exists a sequence $\{P_k\}$ of finite rank projections such that $P_k \uparrow 1$(SOT) and*

$$\mathrm{rank}\ \big(P_k^\perp TP_k\big) \le m \text{ for all } k \ge 1.$$

Proof. Let $g_1, \cdots, g_m$ be the generating vectors for T and let $\{\lambda_j\}$ be a countable dense subset of $\mathbb{C} \setminus \sigma(T)$; for convenience, arrange $\{\lambda_j\}$ so that each point is repeated infinitely often. Let P_k be the projection of H onto

$$\bigvee \left\{T^j(T-\lambda_1)^{-1}\cdots(T-\lambda_k)^{-1}g_i : 0 \le j \le 2k,\ 1 \le i \le m\right\}.$$

Thus P_k is finite rank, $P_k \le P_{k+1}$, and

$$\mathrm{rank}\,[P_k^\perp TP_k] \le m \text{ for all } k \ge 1.$$

We should prove that $P_k \to 1(\mathrm{SOT})$. Since $\{P_k\}$ is increasing, $\mathcal{L} = \mathrm{cl}\bigcup_k \mathrm{ran}P_k$ is a closed linear space. To show that $P_k \to 1(\mathrm{SOT})$ it suffices to show that $\mathcal{L} = H$. To do this, it suffices to show that $f(T)\mathcal{L} \subseteq \mathcal{L}$ for all $f \in \mathrm{Rat}(\sigma(T))$. Since $\{\lambda_j\}$ is dense in $\sigma(T)^c$, it is only necessary to show that $f(T)\mathcal{L} \subseteq \mathcal{L}$ when f is a rational function with poles in $\{\lambda_j\}$. Hence we must show that

$$T\mathcal{L} \subseteq \mathcal{L} \quad \text{and} \quad (T-\lambda_j)^{-1}\mathcal{L} \subseteq \mathcal{L}.$$

From the definition of $\mathcal{L}$ we see that these two conditions are equivalent, respectively, to show that for all $\beta \geq 1$:

$$T\Big(T^j(T-\lambda_1)^{-1}\cdots(T-\lambda_k)^{-1}g_i\Big) \in \mathcal{L} \text{ for } 0 \leq j \leq 2k\ ; \tag{3.1}$$

$$(T-\lambda_m)^{-1}\Big(T^j(T-\lambda_1)^{-1}\cdots(T-\lambda_k)^{-1}g_i\Big) \in \mathcal{L} \text{ for } 0 \leq j \leq 2k \text{ and all } m. \tag{3.2}$$

To prove (3.1) we need only consider the case where $j = 2k$. Now

$$T^{2k+1}(T-\lambda_1)^{-1}\cdots(T-\lambda_{2k})^{-1}g_i \in \mathrm{ran}P_{2k}$$

and $A = (T-\lambda_{k+1})\cdots(T-\lambda_{2k})$ is a polynomial in T of degree $2k-k$. Hence

$$T^{2k+1}(T-\lambda_1)^{-1}\cdots(T-\lambda_k)^{-1}g_i = AT^{2k+1}(T-\lambda_1)^{-1}\cdots(T-\lambda_{2k})^{-1}g_i \in \mathrm{ran}P_{2k} \subseteq \mathcal{L},$$

which proves (3.1).

Since (3.1) implies that $\mathcal{L}$ is an invariant subspace for T, to show (3.2) it suffices to show that

$$(T-\lambda_m)^{-1}\Big((T-\lambda_1)^{-1}\cdots(T-\lambda_k)^{-1}g_i\Big) \in \mathcal{L} \text{ for all } m.$$

Since λ_m is repeated infinitely often, we may assume $m \geq k+2$. If $B = (T-\lambda_{k+1})\cdots(T-\lambda_{m-1})$, then B is a polynomial in T of degree $m+k-1$. Hence

$$(T-\lambda_m)^{-1}\Big((T-\lambda_1)^{-1}\cdots(T-\lambda_k)^{-1}g_i\Big) = B(T-\lambda_1)^{-1}\cdots(T-\lambda_m)^{-1}g_i \in \mathrm{ran}P_m \subseteq \mathcal{L},$$

which proves (3.2). □

Lemma 3.2.6. *If $T \in B(H)$ is an m-multicyclic hyponormal operator then*

$$\mathrm{tr}\,[T^*, T] \leq m\|T\|^2.$$

Proof. By Lemma 3.2.5, there exists an increasing sequence $\{P_k\}$ of finite rank projections such that $P_k \uparrow 1(\mathrm{SOT})$ and $\mathrm{rank}\,[P_k^\perp T P_k] \leq m$ for all $k \geq 1$. Note that

$$\|P_k^\perp T P_k\|_2^2 \leq m\|P_k^\perp T P_k\|^2.$$

Since $\{P_k\}$ is an increasing sequence,

$$\operatorname{tr}[T^*, T] = \lim_k \operatorname{tr}\big(P_k[T^*, T]P_k\big).$$

By Lemma 3.2.4 we get

$$\operatorname{tr}\,[T^*, T] \le \operatorname{limsup}\|P_k^{\perp}TP_k\|_2^2 \le \operatorname{limsup}\left(m\|P_k^{\perp}TP_k\|^2\right) \le m\|T\|^2.$$

$\square$

We are ready for:

Proof of the Berger-Shaw Theorem. Let $R = \|T\|$ and put $D = \overline{B}(0; R)$. If $\varepsilon > 0$, let $D_1, \cdots, D_n$ be pairwise disjoint closed disks contained in $D \setminus \sigma(T)$ such that

$$\operatorname{Area}(D) < \operatorname{Area}\sigma(T) + \sum_j \operatorname{Area}(D_j) + \varepsilon.$$

If $D_j = \overline{B}(a_j; r_j)$, this inequality says

$$\pi R^2 - \pi\sum_j r_j^2 < \operatorname{Area}\sigma(T) + \varepsilon.$$

If S is the unilateral shift of multiplicity 1, let $S_j = (a_j + r_jS)^{(m)}$. Now that each S_j is m-multicyclic. Thus

$$A = \begin{bmatrix} T & & & \\ & S_1 & 0 & \\ & 0 & \ddots & \\ & & & S_n \end{bmatrix}$$

is an m-multicyclic hyponormal operator since the spectra of the operator summands are pairwise disjoint. Also $\|A\| = R$. By Lemma 3.2.6, $\operatorname{tr}\,[A^*, A] \le mR^2$. But

$$\operatorname{tr}\,[A^*, A] = \operatorname{tr}\,[T^*, T] + \sum_{j=1}^n \operatorname{tr}\,[S_j^*, S_j] = \operatorname{tr}\,[T^*, T] + m\sum_{j=1}^n r_j^2.$$

Therefore

$$\pi\operatorname{tr}\,[T^*, T] \le m\left(\pi R^2 - \pi\sum_{j=1}^n r_j^2\right) \le m\big(\operatorname{Area}\sigma(T) + \varepsilon\big).$$

Since ε was arbitrary, the proof is complete.

Theorem 3.2.7. (Putnam's inequality) *If $S \in B(H)$ is a hyponormal operator then*

$$\|[S^*, S]\| \le \frac{1}{\pi}\operatorname{Area}(\sigma(S)).$$

Proof. Fix $\|f\| \leq 1$ and let $\mathcal{K} \equiv \bigvee\{r(s)f : r \in \operatorname{Rat}(\sigma(S))\}$. If $T = S|_{\mathcal{K}}$ then T is an 1-multicyclic hyponormal operator. By the Berger-Shaw theorem and the fact that $\|T^*f\| \leq \|S^*f\|$, we get

$$\begin{aligned} \langle [S^*, S]f, f\rangle &= \|Sf\|^2 - \|S^*f\|^2 \\ &\leq \|Tf\|^2 - \|T^*f\|^2 \\ &= \langle [T^*, T]f, f\rangle \\ &\leq \operatorname{tr}[T^*, T] \\ &\leq \frac{1}{\pi}\text{Area}(\sigma(T)) \\ &\leq \frac{1}{\pi}\text{Area}(\sigma(S)). \end{aligned}$$

Since f was arbitrary, the result follows. □

Corollary 3.2.8. *If S is a hyponormal operator such that* Area$(\sigma(S)) = 0$ *then S is normal.*

3.3 Subnormal Operators

Definition 3.3.1. An operator S on a Hilbert space H is called *subnormal* if there exists a Hilbert space $K \supseteq H$ and a normal operator N on K such that

$$NH \subseteq H \text{ and } N|_H = S.$$

The concept of subnormality was introduced in P. Halmos in 1950. Loosely speaking, a subnormal operator is one that has a normal extension. Every isometry is subnormal (by Wold-von Neumann decomposition).

Proposition 3.3.2. *Every subnormal operator is hyponormal.*

Proof. If S is subnormal then

$$\exists \text{ a normal operator } \ N = \begin{bmatrix} S & A \\ 0 & B \end{bmatrix}.$$

So

$$0 = N^*N - NN^* = \begin{bmatrix} [S^*, S] - AA^* & S^*A \\ A^*S & A^*A + [B^*, B] \end{bmatrix},$$

which implies that $[S^*, S] = AA^* \geq 0$. $\square$

An example of a hyponormal operator that is not subnormal:

$$A \equiv U^* + 2U;$$

then A is hyponormal, but A^2 is not; so A is not subnormal (To see this use Theorem 3.3.7 below).

Example 3.3.3. Let μ be a compactly supported measure on $\mathbb{C}$ and define N_μ on $L^2(\mu)$ by

$$N_\mu f = zf.$$

Then N_μ is normal since $N_\mu^* f = \overline{z} f$. If $P^2(\mu)$ is the closure in $L^2(\mu)$ of analytic polynomials, define S_μ on $P^2(\mu)$ by

$$S_\mu f = zf.$$

Then S_μ is subnormal and N_μ is a normal extension of S_μ.

Definition 3.3.4. An operator S is called *quasinormal* if S and S^*S commute.

Proposition 3.3.5. *If $S = UA$ is the polar decomposition of S then*

$$S \text{ is quasinormal} \Longleftrightarrow UA = AU.$$

Proof. ($\Leftarrow$) $UA = AU \Longrightarrow SA^2 = UA^3 = A^2UA = A^2S \Longrightarrow S$ is quasinormal.

($\Rightarrow$) If S is quasinormal then $SA^2 = A^2S$ ($A^2 = S^*S$). Thus $SA = AS$, so $(UA - AU)A = SA - AS = 0$. Thus $UA - AU = 0$ on $\operatorname{ran} A$. But if $f \in (\operatorname{ran} A)^{\perp} = \ker A$ then since $\ker A = \ker U$, we have $Uf = 0$. Therefore $UA = AU$. $\square$

Proposition 3.3.6. *Every quasinormal operator is subnormal.*

Proof. Suppose S is quasinormal.

(Case 1: $\ker S = \{0\}$) If $S = UA$ is the polar decomposition of S then U must be an isometry. If $E = UU^*$ then E is the projection onto $\operatorname{ran} U$. Thus $(I - E)U = U^*(I - E) = 0$. Define $V, B \in B(H \oplus H)$ by

$$V = \begin{bmatrix} U & I - E \\ 0 & U^* \end{bmatrix}, \quad V = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}.$$

Let $N = VB$. Since $UA = AU$ and $U^*A = AU^*$ it follows that N is normal. Since

$$N = \begin{bmatrix} S & (I - E)A \\ 0 & U^*A \end{bmatrix} = \begin{bmatrix} S & (I - E)A \\ 0 & S^* \end{bmatrix},$$

we have $NH \subseteq H$ and $N|_H = S$.

(Case 2: $\ker S \neq \{0\}$) Here $\ker S = \mathcal{L} \subseteq \ker S^*$ since $S^* = AU^* = U^*A$. Let $S_1 := \left(S|_{\mathcal{L}}\right)^{\perp}$. So $S = S_1 \oplus 0$ on $\mathcal{L}^{\perp} \oplus \mathcal{L} = H$. Now $S^*S = S_1^*S_1 \oplus 0$. Observe S_1 is quasinormal. By Case 1, S_1 is subnormal and therefore S is subnormal. $\square$

Remember [Con2, p.44] that

S is pure quasinormal $\Longleftrightarrow S = U \otimes A$, where A is a positive operator with $\ker A = \{0\}$.

If X is a locally compact space, a positive operator-valued measure(POM) on X is defined by a function Q such that

Q: a Borel set $\triangle \subseteq X \mapsto Q(\triangle)$, a positive operator, $\in \mathcal{B}(\mathcal{H})$;
$Q(X) = 1$;
$\langle Q(\cdot)f, f\rangle$ is a regular Borel measure on X.

Every spectral measure is a POM. But the converse is false. Let E be a spectral measure on X with values in $B(K)$, H be a subspace of K and let P be the orthogonal projection of K onto H. Define

$$Q(\triangle) := PE(\triangle)|_H.$$

Then Q is a POM with $\|Q(\triangle)\| \leq 1$ for all $\triangle$. But Q is a spectral measure if and only if P commutes with $E(\triangle)$ for any $\triangle$.

If Q is a POM and ϕ is a bounded Borel function on X then $\int \phi dQ$ denotes the unique operator T defined by the bounded quadratic form

$$\langle Tf, f\rangle = \int \phi(x) d\langle Q(x)f, f\rangle.$$

Theorem 3.3.7. *If $S \in B(H)$, the following are equivalent:*

(a) *S is subnormal.*

(b) (Bram-Halmos, 1955/1950) *If $f_0, \cdots, f_n \in H$ then*

$$\sum_{j,k} \langle S^j f_k, S^k f_j\rangle \geq 0. \tag{3.3}$$

(c) (Embry, 1973) *For any $f_0, \cdots, f_n \in H$*

$$\sum_{j,k} \langle S^{j+k} f_j, S^{j+k} f_k\rangle \geq 0. \tag{3.4}$$

(d) (Bunce and Deddens, 1977) *If $B_0, \cdots, B_n \in C^*(S)$ then*

$$\sum_{j,k} B_j^* S^{*k} S^j B_k \geq 0$$

(e) (Bram, 1955) *There is a POM Q supported on a compact subset of $\mathbb{C}$ such that*

$$S^{*n} S^m = \int \overline{z}^n z^m dQ(z) \quad \textit{for all } m, n \geq 0. \tag{3.5}$$

(f) (Embry, 1973) *There is a POM Q on some interval $[0, a] \subseteq \mathbb{R}$ such that*

$$S^{*n} S^n = \int t^{2n} dQ(t) \quad \textit{for all } n \geq 0.$$

Proof. (a) $\Rightarrow$ (b): Let $N = \begin{bmatrix} S & * \\ 0 & * \end{bmatrix} \begin{matrix} H \\ H\prime \end{matrix}$ be a normal operator on K. If P is the projection of K onto H, then $S^{*n}f = PN^{*n}f,\ f \in H$. If $f_0, \cdots, f_n \in H$ then

$$\begin{aligned}\sum_{j,k} \langle S^j f_k, S^k f_j \rangle &= \sum_{j,k} \langle N^j f_k, N^k f_j \rangle \\ &= \sum_{j,k} \langle N^{*k} N^j f_k, f_j \rangle \\ &= \sum_{j,k} \langle N^j N^{*k} f_k, f_j \rangle \\ &= \sum_{j,k} \langle N^{*k} f_k, N^{*j} f_j \rangle \\ &= \left\| \sum_k N^{*k} f_k \right\|^2 .\end{aligned}$$

So (3.3) holds.

(b) ⇒ (c): Put $g_k = S^k f_k$. Then (3.3) implies

$$\sum_{j,k} \langle S^j g_k, S^k g_j \rangle = \sum_{j,k} \langle S^{j+k} f_k, S^{j+k} f_j \rangle .$$

So (3.4) holds.

(c) ⇒ (a): See [Con2].

(b) ⇒ (d): If $B_0, \cdots, B_n \in C^*(S)$, let $f_k = B_k f$. Then

$$(3.3) \Longleftrightarrow \left\langle \sum_{j,k} B_j^* S^{*k} S^j B_k f, f \right\rangle \geq 0.$$

(d) ⇒ (b): By Zorn's lemma,

$$\text{any operator} = \bigoplus \text{ star-cyclic operator.}$$

So we may assume that S has a star-cyclic vector e_0, i.e., assume $H = \text{cl}\,[C^*(S)e_0]$. If $B_0, \cdots, B_n \in C^*(S)$ then (3.3) holds for $f_k = B_k e_0$. Since (3.3) holds for a dense set of vector, (3.3) holds for all vectors.

(a) ⇒ (e): Let $N = \int z dE(z)$ be the spectral decomposition of N, a normal extension of S acting on $K \supseteq H$. Let P be the orthogonal projection of K onto H. Define

$$Q(\triangle) := PE(\triangle)|_H \text{ for every Borel subset } \triangle \text{ of } \mathbb{C}.$$

Then Q is a POM and is supported on $\sigma(N)$. Also for all $h \in H$,

$$\begin{aligned}\langle S^{*n}S^m h, h\rangle &= \langle N^{*n}N^m h, h\rangle \\ &= \int \overline{z}^n z^m d\langle E(z)h, h\rangle \\ &= \int \overline{z}^n z^m d\langle E(z)h, Ph\rangle \\ &= \int \overline{z}^n z^m d\langle PE(z)h, h\rangle \\ &= \int \overline{z}^n z^m d\langle Q(z)h, h\rangle \\ &= \left\langle \left(\int \overline{z}^n z^m dQ(z)\right) h, h\right\rangle,\end{aligned}$$

so that

$$S^{*n}S^m = \int \overline{z}^n z^m dQ(z).$$

(e) $\Rightarrow$ (f): Let Q be the POM hypothesized in (i) and $K = \operatorname{supp} Q$. For a Borel set $\triangle \subseteq [0, \infty)$, define

$$Q_+(\triangle) := Q\{z \in \mathbb{C} : |z| \in \triangle\}.$$

In fact, $Q_+(\triangle) = Q(\tau^{-1}\triangle)$, where $\tau(z) = |z|$. Then Q_+ is a POM whose support $\subseteq [0, a]$ with $a = \max_{z\in K} |z|$. For any $f \in H$,

$$\int t^{2n} d\langle Q_+(t)f, f\rangle = \int |z|^{2n} d\langle Q(z)f, f\rangle.$$

(f) $\Rightarrow$ (c): Fix $f_0, \cdots, f_n \in H$ and define scalar-valued measures μ_{jk} by

$$\mu_{jk}(\triangle) = \langle Q(\triangle)f_j, f_k\rangle.$$

Let μ be a positive measure on $[0, a]$ such that $\mu_{jk} \ll \mu$ for any j, k. Let $h_{jk} = \frac{d\mu_{jk}}{d\mu}$ (Radon-Nikodym derivative). For each $u \in C[0, a]$, $\rho(u) = \int u dQ$ defines a bounded operator and $\rho : C[0, a] \to B(H)$ is a positive linear map. Note that for all u, $\langle \rho(u)f_j, f_k\rangle = \int u d\mu_{jk} = \int u h_{j,k} d\mu$. Moreover if $\lambda_0, \cdots, \lambda_n \in \mathbb{C}$ and $u \geq 0$ then

$$\sum_{j,k} \left(\int u h_{jk} d\mu\right) \lambda_j \overline{\lambda}_k = \left\| \sum_j \rho(u)^{\frac{1}{2}} \lambda_j f_j \right\|^2 \geq 0.$$

It follows that $(h_{jk}(t))_{j,k}$ is positive $(n+1) \times (n+1)$ matrix for $[\mu]$ almost every t. This implies that

$$\sum_{j,k} h_{jk}(t) t^{2j} t^{2k} \geq 0 \text{ a.e. } [\mu].$$

Therefore

$$\begin{aligned}0 &\leq \int \sum_{j,k} h_{jk}(t) t^{2(j+k)} d\mu(t)\\ &= \sum_{j,k} \int t^{2(j+k)} d\mu_{jk}(t)\\ &= \sum_{j,k} \left\langle S^{j+k} f_j, S^{j+k} f_k \right\rangle,\end{aligned}$$

so (3.4) holds. $\square$

Remark. Without loss of generality we may assume that $||S|| < 1$. Let $K = H^\infty$ and let $K_0 =$ the finitely nonzero sequences in K. Let

$$M = \begin{bmatrix} 1 & S^* & S^{*2} & \cdots \\ S & S^*S & S^{*2}S & \cdots \\ S^2 & S^*S^2 & S^{*2}S^2 & \cdots \\ S^3 & S^*S^3 & S^{*2}S^3 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \text{on } K_0.$$

If $f = (f_0, \cdots, f_n, \cdots) \in K_0$ then

$$\begin{aligned}\sum_j \|(Mf)_j\|^2 &= \sum_j \left\| \sum_k S^{*k} S^j f_k \right\|^2\\ &\leq \sum_j \left[\sum_k \|S\|^{k+j} \|f_k\|^2 \right]\\ &\leq \sum_j \left[\sum_k \|S\|^{2k+2j} \right] \left[\sum_k \|f_k\| \right]^2\\ &\leq (1 - \|S\|^2)^{-2} \|f\|^2.\end{aligned}$$

Since $\|S\| < 1$, $Mf \in K$ and M extends to a bounded operator on K. Clearly, M is hermitian. Note

$$\langle Mf, f \rangle_K = \sum_{j,k} \langle S^j f_k, S^k f_j \rangle.$$

So

$$(3.3) \text{ holds} \iff M \text{ is positive.}$$

Recall the Smul'jan theorem – if $M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ $(A, C$ hermitian, A invertible), then

$$M \geq 0 \iff B^* A^{-1} B \leq C.$$

We thus have that if

$$M=\begin{bmatrix} 1 & S^* & \cdots & S^{*k} & \cdots \\ S & S^*S & \cdots & S^{*k}S & \cdots \\ S^2 & S^*S^2 & \cdots & S^{*k}S^2 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \end{bmatrix}$$

then

$$\begin{aligned} M\ge 0 &\Longleftrightarrow \begin{bmatrix} S \\ S^2 \\ S^3 \\ \vdots \end{bmatrix}\begin{bmatrix} S^* & S^{*2} & S^{*3} & \cdots \end{bmatrix} \le \begin{bmatrix} S^*S & S^{*2}S & S^{*3}S & \cdots \\ S^*S^2 & S^{*2}S^2 & S^{*3}S^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\ &\Longleftrightarrow \begin{bmatrix} S^*S-SS^* & S^{*2}S-SS^{*2} & \cdots \\ S^*S^2-S^2S^* & S^{*2}S^2-S^2S^{*2} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}\ge 0 \\ &\Longleftrightarrow \begin{bmatrix} [S^*,S] & [S^{*2},S] & [S^{*3},S] & \cdots \\ [S^*,S^2] & [S^{*2},S^2] & [S^{*3},S^2] & \cdots \\ [S^*,S^3] & [S^{*2},S^3] & [S^{*3},S^3] & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}\ge 0 \end{aligned}$$

Definition 3.3.8. If $\mathcal{A}$ is a $\mathcal{C}^*-$algebra, define $s\in\mathcal{A}$ to be subnormal if $\sum_{j,k} a_j^* s^{*k} s^j a_k \ge 0$ for any choice $a_0,\cdots,a_n\in\mathcal{C}^*(s)$.

It is easy to see that if $\mathcal{A},\mathcal{B}$ are $\mathcal{C}^*-$algebras and $\rho:\mathcal{A}\longrightarrow\mathcal{B}$ is a $*-$homomorphism then ρ maps subnormal elements of $\mathcal{A}$ onto subnormal elements of $\mathcal{B}$. In particular, if (ρ,H) is a representation of $\mathcal{A}$, $\rho(s)$ is a subnormal operator on H whenever S is a subnormal element of $\mathcal{A}$.

Remark. (Agler [Ag2, 1985]'s characterization of subnormal operators) If S is a contraction then

$$S \text{ is subnormal} \Longleftrightarrow \sum_{k=0}^{n}(-1)^k\binom{n}{k}S^{*k}S^k\ge 0 \text{ for all } n\ge 1.$$

Note that if N is a normal extension of $S\in B(H)$ to K then

$$K\supseteq \bigvee\left\{N^{*k}h : h\in H, k=0,1,2,\cdots\right\}.$$

If $\mathcal{L}:=\bigvee\left\{N^{*k}h : h\in H, k=0,1,2,\cdots\right\}$ then

$$\mathcal{L} \text{ is a reducing subspace for } N \text{ that contains } H.$$

Thus $N|_{\mathcal{L}}$ is also a normal extension of S. Moreover if $\mathcal{R}$ is any reducing subspace for N that contains H then $\mathcal{R}$ must contain $\mathcal{L}$.

Definition 3.3.9. If S is a subnormal operator on H and N is a normal extension of S to K then N is called a *minimal normal extension* of S if

$$K = \bigvee \left\{ N^{*k} h : h \in H, k \geq 0 \right\}.$$

Proposition 3.3.10. *If S is a subnormal operator then any two minimal normal extensions are unitarily equivalent.*

Proof. For $p = 1, 2$ let N_p be a minimal normal extension of S acting on $K_p \supseteq H$. Define $U : K_1 \longrightarrow K_2$ by

$$U\left(N_1^{*n} h\right) = N_2^{*n} h \ (h \in H).$$

We want to show that U is an isomorphism. If $h_1, \cdots, h_m \in H$ and $n_1, \cdots, n_m \geq 0$ then

$$\begin{aligned}
\left\| \sum_k N_2^{*n_k} h_k \right\|^2 &= \left\langle \sum_k N_2^{*n_k} h_k, \sum_j N_2^{*n_j} h_j \right\rangle \\
&= \sum_{j,k} \langle {N_2}^{n_j} h_k, {N_2}^{n_k} h_j \rangle \\
&= \sum_{j,k} \langle S^{n_j} h_k, S^{n_k} h_j \rangle \\
&= \sum_{j,k} \langle {N_1}^{n_j} h_k, {N_1}^{n_k} h_j \rangle \\
&= \left\| \sum_k N_1^{*n_k} h_k \right\|^2,
\end{aligned}$$

which shows that

$$U\left[\sum_k N_1^{*n_k} h_k\right] = \sum_k N_2^{*n_k} h_k$$

is a well defined linear operator from a dense linear manifold in K_1 onto a dense linear manifold in K_2 and U is an isometry. Also for all $h \in H$, $Uh = h$. Thus for $h \in H$ and $n \geq 0$,

$$U N_1 N_1^{*n} h = U N_1^{*n} S h = N_2^{*n} S h = N_2 N_2^{*n} h = N_2 U N_1^{*n} h,$$

i.e., $U N_1 = N_2 U$, so that N_1 and N_2 are unitarily equivalent. □

Now it is legitimate to speak of *the* minimal normal extension of a subnormal operator. Therefore it is unambiguous to define the *normal spectrum* of a subnormal operator S, $\sigma_n(S)$, as the spectrum of its minimal normal extension.

Proposition 3.3.11. *If S is a subnormal operator then the following hold:*

(a) (Halmos, 1952) $\sigma_n(S) \subseteq \sigma(S)$.

(b) $\sigma_{ap}(S) \subseteq \sigma_n(S)$ *and* $\partial\sigma(S) \subseteq \partial\sigma_n(S)$.

(c) (Bram, 1955) *If* U *is a bounded component of* $\mathbb{C}\backslash\sigma_n(S)$*, then either* $U \cap \sigma(S) = \emptyset$ *or* $U \subseteq \sigma(S)$.

Proof. (a) We want to show that S is invertible $\Rightarrow N$ is invertible.

If $N = \int z dE(z)$ is the spectral decomposition of N, $\varepsilon > 0$, and $M = E(B(0;\varepsilon))K$ then we claim that

$$||N^k f|| \le \varepsilon^k \|f\| \quad \text{for } k = 1, 2, 3, \cdots \text{ and } f \in M.$$

To see this let $\triangle := B(0;\varepsilon)$. Then

$$NE(\triangle) = \int z\chi_\triangle(z) dE(z) = \phi(N), \text{ where } \phi = z\chi_\triangle.$$

We thus have

$$\|NE(\triangle)\| = \|\phi(N)\| \le \|\phi\| = \sup|\phi(z)| = \sup\{|z| : z \in \triangle\} \le \varepsilon.$$

So if $f \in M$ then $E(\triangle)f = f$. Therefore

$$\|Nf\| = \|NE(\triangle)f\| \le \|NE(\triangle)\|\|f\| \le \varepsilon\|f\|.$$

So if $f \in M$ and $h \in \mathcal{H}$,

$$\begin{aligned} |\langle f, h\rangle| &= \left|\langle f, S^k S^{-k} h\rangle\right| = \left|\langle f, N^k S^{-k} h\rangle\right| = \left|\langle N^{*k} f, S^{-k} h\rangle\right| \\ &\le \left\|N^{*k} f\right\| \cdot \left\|S^{-k} h\right\| \le \varepsilon^k \|f\| \left\|S^{-k}\right\| \le \varepsilon^k \left\|S^{-1}\right\|^k \|f\| \, \|h\|. \end{aligned}$$

Letting $k \to \infty$ shows that

$$\varepsilon < \frac{1}{\|S^{-1}\|} \implies \langle f, h\rangle = 0,$$

so that $H \subseteq M^\perp$. Since M is a reducing subspace for N, $N|_{M^\perp}$ is a normal extension of S. By the minimality of N, $M = \{0\}$ and so N is invertible because $N = N_\varphi$ and $|\varphi(x) \ge \varepsilon$ a.e.

(b) Observe that

$$\begin{aligned} \lambda \in \sigma_{ap}(S) &\Longrightarrow \exists \text{ unit vectors } h_n \in \mathcal{H} \text{ such that } \|(\lambda - S)h_n\| \longrightarrow 0. \\ &\qquad \text{But } (\lambda - S)h_n = (\lambda - N)h_n. \\ &\Longrightarrow \sigma_{ap}(S) \subseteq \sigma_{ap}(N) = \sigma(N) = \sigma_n(S). \end{aligned}$$

$$\lambda \in \partial\sigma(S) \Longrightarrow \lambda \in \sigma_{ap}(S) \Longrightarrow \lambda \in \sigma_n(S) \Longrightarrow \lambda \notin \operatorname{int}\sigma_n(S) \Longrightarrow \lambda \in \partial\sigma_n(S).$$

(c) (Due to S. Parrot) Let U be a bounded component of $\sigma_n(S)^c$ and put

$$U_+ = U \setminus \sigma(S) \text{ and } U_- = U \cap \sigma(S).$$

So $U = U_- \cup U_+$, $U_+ \cap U_- = \emptyset$ and U_+ is open. By (b), $U_- = U \cap \operatorname{int}\sigma(S)$, so that U_- is open. By the connectedness of U, either $U_+ = \emptyset$ or $U_- = \emptyset$. □

Corollary 3.3.12. *If S is a subnormal operator whose minimal normal extension is N then*

$$r(S) = \|S\| = \|N\| = r(N).$$

Proof. Since $r(S) \leq \|S\| \leq \|N\| = r(N)$, the result follows from Proposition 3.3.11. $\square$

Definition 3.3.13. If $A \in B(H), e_0 \in H$, K is a compact subset of $\mathbb{C}$ containing $\sigma(A)$ then e_0 is called a *Rat*(K) *cyclic vector* for A if

$$\left\{ u(A)e_0 : u \in \text{Rat}\,(K) \right\} \text{ is dense in } H.$$

An operator is called Rat (K) *cyclic* if it has a Rat (K) cyclic vector. In the cases that $K = \sigma(S)$, A is called a *rationally cyclic operator.*

Recall that e_0 is a cyclic vector for A (A is a cyclic operator) if $\{p(A)e_0 :$ p is a polynomial$\}$ is dense in H. By Runge's theorem, e_0 is cyclic for $A \Leftrightarrow e_0$ is Rat$\widehat{\sigma(A)}$ cyclic for A.

Note that if S is subnormal and $N = \text{mme}\,(S)$ then since $\sigma(N) \subseteq \sigma(S)$, it follows that if K contains $\sigma(S)$ then $u(N)$ is well defined for any $U \in \text{Rat}\,(K)$.

Theorem 3.3.14. *If S is subnormal and has a* Rat (K) *cyclic vector e_0 then there exists a unique compactly supported measure μ on K and an isomorphism $U : K \to L^2(\mu)$ such that*

(a) *$UH = R^2(K, \mu)$;*

(b) *$Ue_0 = 1$;*

(c) *$UNU^{-1} = N_\mu$;*

(d) *if $V = U|_H$, then V is an isomorphism of H onto $R^2(K, \mu)$ and $VSV^{-1} = N_\mu|_{R^2(K,\mu)}$.*

Proof. If $N = \text{mme}\,(S)$ then $K = \bigvee \{N^{*n}u(N)e_0 : n \geq 0, u \in \text{Rat}\,(K)\}$. We claim that e_0 is a $*-$cyclic vector for N. Indeed, let $\mathcal{L} = \bigvee \{N^{*n}u(N)e_0 : n, k \geq 0\}$. Evidently, $\mathcal{L}$ is a reducing subspace for N. By the Stone-Weierstrass theorem, $C(K)$ is the uniformly closed linear span of $\left\{\overline{z}^n z^k : n, k \geq 0\right\}$. Since $\text{Rat}\,(K) \subseteq C(K)$, we have that $u(N)e_0 \in \mathcal{L}$ for every $U \in \text{Rat}\,(K)$. Thus $H \subseteq \mathcal{L}$. By the minimality of N we have $H = K$. Hence e_0 is a $*-$cyclic vector for N. Therefore there exists a compactly supported measure μ and an isomorphism

$$U : K \to L^2(\mu) \text{ such that } Ue_0 = 1 \text{ and } UNU^{-1} = N_\mu.$$

So (b) and (c) hold. Observe $U\phi(N) = \phi(N_\mu)U$ for every bounded Borel function ϕ. In particular, for $u \in \text{Rat}\,(K)$, $Uu(S)e_0 = Uu(N)e_0 = u(N_\mu)Ue_0 = u(N_\mu)1 = u$. Taking limits gives (a). The assertion (d) is immediate. The proof of the uniqueness of μ comes from the Stone-Weierstrass theorem. $\square$

Corollary 3.3.15. *An operator S is a cyclic subnormal operator if and only if $S \cong S_\mu$ for some compactly supported measure μ.*

For any compact K, define

$$R(K) := \text{ the uniform closure in } C(K) \text{ of } \operatorname{Rat}(K).$$

Define $\|f\|_K = \sup_{z\in K}|f(z)|$. For a subnormal operator S, we may define $f(S)$ for all functions $f \in R(\sigma(S))$. If $f \in R(\sigma(S))$, $f(S) = f(N)|_H$. So $f(S)$ is subnormal, so that

$$\sigma(f(S)) = f(\sigma(S))$$

$$\|f\|_{\sigma(S)} = \|f(S)\| \le \|f(N)\| = \|f\|_{\sigma(N)} \le \|f\|_{\sigma(S)},$$

i.e., the map $f \mapsto f(S)$ is an isometry from $R(\sigma(S))$ into $B(H)$. Define, for $f \in R(\sigma(S))$,

$$f(S) := \text{ the image of } f \text{ under this isomorphism.}$$

Then

$$f(N)H \subseteq H, \quad f(N)|_H = f(S).$$

Theorem 3.3.16. *If S is subnormal and $N = \operatorname{mne}(S)$, and for each $f \in R(\sigma(S))$, $f(S) = f(N)|_H$ then the map $f \mapsto f(S)$ is a multiplicative linear isometry from $R(\sigma(S))$ into $B(H)$ that extends the Riesz functional calculus for S. Moreover,*

$$\sigma(f(S)) = f(\sigma(S)) \quad \textit{for } f \in R(\sigma(S)).$$

Proof. See [Con2]. □

Lemma 3.3.17. *If S is subnormal and $\sigma(S) \subseteq \mathbb{R}$ then S is hermitian.*

Proof. Let $N = \operatorname{mne}(S)$, acting on K. By Proposition 3.3.11, $\sigma(N) \subseteq \sigma(S) \subseteq \mathbb{R}$. Hence $N = N^*$. Then every invariant subspace for N reduces N. In particular, H reduces N. By the minimality of N, $K = H$. So $S = N$ and hence S is hermitian. □

Proposition 3.3.18. *If S is subnormal and $R(\sigma(S)) = C(\sigma(S))$ then S is normal.*

Proof. Let $\phi(z) = \operatorname{Re} z$ and $\psi(z) = \operatorname{Im} z$. By hypothesis, $\phi, \psi \in R(\sigma(S))$. By Theorem 3.3.16, $\phi(S)$ is subnormal and

$$\sigma(\phi(S)) = \phi(\sigma(S)) \subseteq \mathbb{R}.$$

Therefore $\phi(S)$ is hermitian. Similarly, $\psi(S)$ is hermitian. Since $\phi + i\psi = z$, $S = \phi(S) + i\psi(S)$. Since $\phi(S)\psi(S) = \psi(S)\phi(S)$, it follows S is normal. □

Remark. If σ is compact and $R(\sigma) = C(\sigma)$ then σ is called *thin*. It was known [Wer] that

(i) σ is thin $\Longrightarrow \operatorname{int}\sigma = \emptyset$;
(ii) The converse of (i) fails;
(iii) $m(\sigma) = 0 \Longrightarrow \sigma$ thin.

3.4 p-Hyponormal Operators

Recall that the *numerical range* of $T \in B(H)$ is defined by

$$W(T) := \Big\{ \langle Tx, x\rangle : \ ||x|| = 1 \Big\}$$

and the *numerical radius* of T is defined by

$$w(T) := \sup \Big\{ |\lambda| : \ \lambda \in W(T) \Big\}.$$

It was well-known (cf. [Ha3]) that

(a) $W(T)$ is convex (Toeplitz-Haussdorff theorem);
(b) $\operatorname{conv}\sigma(T) \subset \operatorname{cl} W(T)$;
(c) $r(T) \le w(T) \le ||T||$;
(d) $\frac{1}{\operatorname{dist}(\lambda, \sigma(T))} \le ||(T-\lambda)^{-1}|| \le \frac{1}{\operatorname{dist}(\lambda, \operatorname{cl} W(T))}$.

Definition 3.4.1. (a) T is called *normaloid* if $||T|| = r(T)$;
(b) T is called *spectraloid* if $w(T) = r(T)$;
(c) T is called *convexoid* if $\operatorname{conv}\sigma(T) = \operatorname{cl} W(T)$;
(d) T is called *transaloid* if $T - \lambda$ is normaloid for any λ;
(e) T is siad to satisfy (G_1)-condition if

$$||(T-\lambda)^{-1}|| \le \frac{1}{\operatorname{dist}(\lambda, \ \sigma(T))}, \quad \text{in fact, } ||(T-\lambda)^{-1}|| = \frac{1}{\operatorname{dist}(\lambda, \ \sigma(T))}.$$

(f) T is called *paranormal* if $||T^2 x|| \ge ||Tx||^2$ for any x with $||x|| = 1$.

It was well-known that it T is paranormal then
(i) T^n is paranormal for any n;
(ii) T is normaloid;
(iii) T^{-1} is paranormal if it exists;
and that

$$\text{hyponormal} \subset \text{paranormal} \subset \text{normaloid} \subset \text{spectraloid}.$$

Theorem 3.4.2. *If $T \in B(H)$ then*

(a) *T is convexoid $\Longleftrightarrow$ $T - \lambda$ is spectraloid for any λ, i.e., $w(T-\lambda) = r(T-\lambda)$;*

(b) *T is convexoid $\Longleftrightarrow$ $||(T-\lambda)^{-1}|| \le \frac{1}{\operatorname{dist}(\lambda, \ \operatorname{conv}\sigma(T))}$ for any $\lambda \notin \operatorname{conv}\sigma(T)$.*

Proof. (a) Note that

$$\begin{aligned}\operatorname{conv} X &= \text{the intersection of all disks containing } X\\ &= \bigcap_{\mu}\Big\{\lambda:\ |\lambda-\mu|\le \sup_{x\in X}|x-\mu|\Big\}.\end{aligned}$$

Since $\operatorname{cl} W(T)$ is convex,

$$\begin{aligned}\operatorname{cl} W(T) &= \bigcap_{\mu}\Big\{\lambda:\ |\lambda-\mu|\le w(T-\mu)\Big\};\\ \operatorname{conv}\sigma(T) &= \bigcap_{\mu}\Big\{\lambda:\ |\lambda-\mu|\le r(T-\mu)\Big\}.\end{aligned}$$

so the result immediately follows.

(b) ($\Rightarrow$) Clear from the preceding remark.

($\Leftarrow$) Suppose

$$||(T-\lambda)^{-1}||\le \frac{1}{\operatorname{dist}(\lambda,\ \operatorname{conv}\sigma(T))}\ \text{ for any } \lambda\notin\operatorname{conv}\sigma(T),$$

or equivalently,

$$||(T-\lambda)\,x||\ge \frac{1}{\operatorname{dist}(\lambda,\ \operatorname{conv}\sigma(T))}\ \text{ for any } \lambda\notin\operatorname{conv}\sigma(T) \text{ and } ||x||=1.$$

Thus

$$||Tx||^2-2\operatorname{Re}\langle Tx,x\rangle\overline{\lambda}+|\lambda|^2\ge \inf_{s\in\operatorname{conv}\sigma(T)}\Big(|s|^2-2\operatorname{Re} s\overline{\lambda}+|\lambda|^2\Big).$$

Taking $\lambda=|\lambda|e^{-i(\theta+\pi)}$, dividing by $|\lambda|$ and letting $\lambda\to\infty$, we have

$$\operatorname{Re}\langle Tx,x\rangle e^{i\theta}\ \ge \inf_{s\in\operatorname{conv}\sigma(T)}\operatorname{Re}\left(se^{i\theta}\right)\quad \text{for } ||x||=1,$$

which implies $\operatorname{cl} W(T)\subset\operatorname{conv}\sigma(T)$. Therefore $\operatorname{cl} W(T)=\operatorname{conv}\sigma(T)$. □

Corollary 3.4.3. *We have:*

(a) transaloid ⇒ convexoid;

(b) (G_1) ⇒ convexoid.

Proof. (a) Clear.

(b) $||(T-\lambda)^{-1}||=\frac{1}{\operatorname{dist}(\lambda,\ \sigma(T))}\le\frac{1}{\operatorname{dist}(\lambda,\ \operatorname{conv}\sigma(T))}$ □

Definition 3.4.4. An operator $T\in B(H)$ is said to satisfy the *projection property* if $\operatorname{Re}\sigma(T)=\sigma(\operatorname{Re}T)$, where $\operatorname{Re}T:=\frac{1}{2}(T+T^*)$.

Theorem 3.4.5. *An operator $T \in B(H)$ is convexoid if and only if*

$$\text{Re conv}\,\sigma\left(e^{i\theta}T\right) = \text{conv}\,\sigma\left(\text{Re}\,(e^{i\theta}T)\right) \quad \textit{for any } \theta \in [0, 2\pi).$$

Proof. Observe that

$$\begin{aligned}\text{Re}\left(e^{i\theta}\text{conv}\,\sigma(T)\right) &= \text{conv}\,\sigma\left(\text{Re}\,(e^{i\theta}T)\right)\\ &= \text{cl}\,W\left(\text{Re}\,(e^{i\theta}T)\right)\\ &= \text{Re cl}\,W(e^{i\theta}T)\\ &= \text{Re }\left(e^{i\theta}\text{cl}\,W(T)\right).\end{aligned}$$

which implies that $\text{conv}\,\sigma(T) = \text{cl}\,W(T)$ and this argument is reversible. □

Example 3.4.6. There exist convexoid operators which are not normaloid and vice versa. (see [Ha2, Problem 219]).

Example 3.4.7. (An example of a non-convexoid and papranormal operator) Let U be the unilateral shift on ℓ^2, $P = \text{diag}(1, 0, 0, \ldots)$ and put

$$T = \begin{bmatrix} U+I & P \\ 0 & 0 \end{bmatrix}.$$

Then $\sigma(T) = \sigma(U+I) \cup \{0\} = \{\lambda : |\lambda - 1| \le 1\}$. But if $x = (-\frac{1}{2}, 0, 0, \ldots)$ and $y = (\frac{\sqrt{3}}{2}, 0, 0, \ldots)$ then $\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| = 1$ and

$$W(T) \ni \langle T(x \oplus y),\ x \oplus y\rangle = \frac{1}{4} - \frac{\sqrt{3}}{4} < 0.$$

Therefore T is not convexoid, but T is papranormal (see [T. Furuta, Invitation to Linear operators]).

Definition 3.4.8. An operator $T \in B(H)$ is called *p-hyponormal* if

$$(T^*T)^p \ge (TT^*)^p.$$

If $p = 1$, T is hyponormal ad if $p = \frac{1}{2}$, T is called *seminormal.* It was known that q-hyponormal $\Rightarrow$ p-hyponormal for $p \le q$ by Löner-Heinz inequality.

Theorem 3.4.9. *p-hyponormal* $\Longrightarrow$ *paranormal.*

Proof. See [An]. □

It was also well-known that if T is p-hyponormal then
(i) T is normaloid;
(ii) T is reduced by its eigenspaces;
(iii) T^{-1} is paranormal if it exists.

However p-hyponormal operators need not be transaloid. In fact, p-hyponormality is not translation-invariant. To see this we first recall:

Lemma 3.4.10. *If T is p-hyponormal then T^n is $\frac{p}{n}$-hyponormal for $0 < p \leq 1$.*

Proof. See [AW]. □

Theorem 3.4.11. *There exists an operator T satisfying*
(i) *T is semi-hyponormal;*
(ii) *$T - \lambda$ is not p-hyponormal for any $p > 0$ and some $\lambda \in \mathbb{C}$.*

Proof. Let

$$S \equiv 4U^2 + U^{*2} + 2UU^* + 2 \quad (U = \text{the unilateral shift on } \ell^2).$$

Then we claim that

(a) S is semi-hyponormal;
(b) $S - 4$ is not p-hyponormal for any $p > 0$, in fact $S - 4$ is not paranormal.

Indeed, if we put $\varphi(z) = 2z + z^{-1}$ the T_φ is hyponormal but T_φ^2 is not because Since $T_\varphi^2 = S$, so S is semi-hyponormal. On the other hand, observe that

$$||(S-4)e_0||^2 = 20 \quad \text{and} \quad ||(S-4)^2 e_0|| = \sqrt{384},$$

so

$$||(S-4)e_0||^2 > ||(S-4)^2 e_0||,$$

which is not paranormal. □

3.5 Comments and Problems

The following problem on p-hyponormal operators remains still open:

Problem 3.1.

(a) *Is every p-hyponormal operator convexoid ?*
(b) *Does every p-hyponormal operator satisfy the* (G_1)*-condition ?*
(c) *Does every p-hyponormal operator satisfy the projection properry ?*

In fact,

$$\text{Yes to (b)} \implies \text{Yes to (a)} \implies \text{Yes to (c)}.$$

It was known that the projection property holds for every hyponormal operator. For a proof, see [Put2].

For a partial answer see [M. Cho, T. Huruya, Y. Kim, J. Lee, A note on real parts of some semi-hyponormal operator.]

It is easily check that every p-hyponormal weighted shift is hyponormal. However we were unable to answer the following:

Problem 3.2. *Is every p-hyponormal Toeplitz operator hyponormal ?*

We conclude with a problem of hyponormal operators with finite rank self-commutators. In general it is quite difficult to determine the subnormality of an operator by definition. An alternative description of subnormality is given by the Bram-Halmos criterion, which states that an operator T is subnormal if and only if

$$\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([Bra], [Con1, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{bmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{bmatrix} \geq 0 \qquad (\text{all } k \geq 1 \text{ }). \tag{3.6}$$

Condition (3.6) provides a measure of the gap between hyponormality and subnormality. An operator $T \in B(H)$ is called *k-hyponormal* if the $(k+1) \times (k+1)$ operator matrix in (3.6) is positive; the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([CMX]). It now seems to be interesting to consider the following problem:

$$\text{Which 2-hyponormal operators are subnormal?} \tag{3.7}$$

The first inquiry involves the self-commutator. The self-commutator of an operator plays an important role in the study of subnormality. B. Morrel [Mor] showed that a pure subnormal operator with rank-one self-commutator (pure means having no normal summand) is unitarily equivalent to a linear function of the unilateral shift. Morrel's theorem can be essentially stated (also see [Con2, p.162]) that if

$$\begin{cases} \text{(i) } T \text{ is hyponormal;} \\ \text{(ii) } [T^*,T] \text{ is of rank-one; and} \\ \text{(iii) } \ker[T^*,T] \text{ is invariant for } T, \end{cases} \tag{3.8}$$

then $T-\beta$ is quasinormal for some $\beta\in\mathbb{C}$. Now remember that every pure quasinormal operator is unitarily equivalent to $U\otimes P$, where U is the unilateral shift and P is a positive operator with trivial kernel. Thus if $[T^*,T]$ is of rank-one (and hence so is $[(T-\beta)^*,(T-\beta)]$), we must have $P\cong\alpha\ (\neq 0)\in\mathbb{C}$, so that $T-\beta\cong\alpha U$, or $T\cong\alpha U+\beta$. It would be interesting (in the sense of giving a simple sufficiency for the subnormality) to note that Morrel's theorem gives that if T satisfies the condition (3.8) then T is subnormal. On the other hand, it was shown ([CuL2, Lemma 2.2]) that if T is 2-hyponormal then $T(\ker[T^*,T])\subseteq\ker[T^*,T]$. Therefore by Morrel's theorem, we can see that

every 2-hyponormal operator with rank-one self-commutator is subnormal. (3.9)

On the other hand, M. Putinar [Pu4] gave a matricial model for the hyponormal operator $T\in B(H)$ with finite rank self-commutator, in the cases where

$$H_0 := \bigvee_{k=0}^{\infty} T^{*k}\big(\operatorname{ran}[T^*,T]\big) \text{ has finite dimension } d \quad\text{and}\quad H=\bigvee_{n=0}^{\infty} T^n H_0.$$

In this case, if we write

$$H_n := G_n\ominus G_{n-1}\ \ (n\geq 1)\quad\text{and}\quad G_n:=\bigvee_{k=0}^{n} T^k H_0\ \ (n\geq 0),$$

then T has the following two-diagonal structure relative to the decomposition $H=H_0\oplus H_1\oplus\cdots$:

$$T=\begin{bmatrix} B_0 & 0 & 0 & 0 & \cdots \\ A_0 & B_1 & 0 & 0 & \cdots \\ 0 & A_1 & B_2 & 0 & \cdots \\ 0 & 0 & A_2 & B_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.10}$$

where

$$\begin{cases} \dim(H_n)=\dim(H_{n+1})=d\ \ (n\geq 0); \\ [T^*,T]=([B_0^*,B_0]+A_0^*A_0)\oplus 0_\infty; \\ [B_{n+1}^*,B_{n+1}]+A_{n+1}^*A_{n+1}=A_nA_n^*\ \ (n\geq 0); \\ A_n^*B_{n+1}=B_nA_n^*\ \ (n\geq 0). \end{cases} \tag{3.11}$$

We will refer the operator (3.10) to the *Putinar's matricial model of rank d.* This model was also introduced in [GuP, Pu1, Xi3, Ya1], and etc.

We here review a few essential facts concerning weak subnormality. Note that the operator T is subnormal if and only if there exist operators A and B such that $\widehat{T} := \begin{bmatrix} T & A \\ 0 & B \end{bmatrix}$ is normal, i.e.,

$$\begin{cases} [T^*, T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*, B] + A^*A = 0. \end{cases} \tag{3.12}$$

The operator $\widehat{T}$ is called a *normal extension* of T. We also say that $\widehat{T}$ in $B(K)$ is a *minimal normal extension* (briefly, m.n.e.) of T if K has no proper subspace containing H to which the restriction of $\widehat{T}$ is also a normal extension of T. It is known that

$$\widehat{T} = \text{ m.n.e.}(T) \iff K = \bigvee \{\widehat{T}^{*n} h : \ h \in H, \ n \geq 0\},$$

and the m.n.e.(T) is unique.

An operator $T \in B(H)$ is said to be *weakly subnormal* if there exist operators $A \in B(H', H)$ and $B \in B(H')$ such that the first two conditions in (3.12) hold:

$$[T^*, T] = AA^* \quad \text{ and } \quad A^*T = BA^*, \tag{3.13}$$

or equivalently, there is an extension $\widehat{T}$ of T such that $\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f \quad$ for all $f \in H$. The operator $\widehat{T}$ is called a *partially normal extension* (briefly, p.n.e.) of T. We also say that $\widehat{T}$ in $B(K)$ is a *minimal partially normal extension* (briefly, m.p.n.e.) of T if K has no proper subspace containing H to which the restriction of $\widehat{T}$ is also a partially normal extension of T. It is known ([CuL2, Lemma 2.5 and Corollary 2.7]) that

$$\widehat{T} = \text{ m.p.n.e.}(T) \iff K = \bigvee \{\widehat{T}^{*n} h : \ h \in H, \ n = 0, 1\},$$

and the m.p.n.e.(T) is unique. For convenience, if $\widehat{T} =$ m.p.n.e.(T) is also weakly subnormal then we write $\widehat{T}^{(2)} := \widehat{\widehat{T}}$ and more generally, $\widehat{T}^{(n)} := \widehat{\widehat{T}^{(n-1)}}$, which will be called the *n-th minimal partially normal extension of* T. It was ([CuL2], [CJP]) shown that

$$\text{2-hyponormal} \implies \text{ weakly subnormal} \implies \text{ hyponormal} \tag{3.14}$$

and the converses of both implications in (3.14) are not true in general. It was ([CuL2]) known that

$$T \text{is weakly subnormal} \Longrightarrow T(\ker [T^*, T]) \subseteq \ker [T^*, T] \tag{3.15}$$

and it was ([CJP]) known that if $\widehat{T} :=$ m.p.n.e.(T) then for any $k \geq 1$,

$$T \text{ is } (k+1)\text{-hyponormal} \Longleftrightarrow T \text{ is weakly subnormal and } \widehat{T} \text{ is } k\text{-hyponormal.} \tag{3.16}$$

So, in particular, one can see that if T is subnormal then $\widehat{T}$ is subnormal. It is worth to noticing that in view of (3.14) and (3.15), Morrel's theorem gives that every weakly subnormal operator with rank-one self-commutator is subnormal.

We now have

Theorem 3.5.1. *Let* $T \in B(H)$. *If*

(i) T *is a pure hyponormal operator;*
(ii) $[T^*, T]$ *is of rank-two; and*
(iii) $\ker [T^*, T]$ *is invariant for* T,

then the following hold:

1. *If* $T|_{\ker [T^*,T]}$ *has the rank-one self-commutator then* T *is subnormal;*
2. *If* $T|_{\ker [T^*,T]}$ *has the rank-two self-commutator then* T *is either a subnormal operator or the Putinar's matricial model (3.10) of rank two.*

Proof. See [LeL3]. $\square$

Since the operator (3.10) can be constructed from the pair of matrices $\{A_0, B_0\}$, we know that the pair $\{A_0, B_0\}$ is a complete set of unitary invariants for the operator (3.10). Many authors used the following Xia's unitary invariants $\{\Lambda, C\}$ to describe pure subnormal operators with finite rank self-commutators:

$$\Lambda := \left(T^*|_{\operatorname{ran}[T^*,T]}\right)^* \quad \text{and} \quad C := [T^*, T]|_{\operatorname{ran}[T^*,T]}.$$

Consequently,

$$\Lambda = B_0 \quad \text{and} \quad C = [B_0^*, B_0] + A_0^2.$$

We know that given Λ and C (or equivalently, A_0 and B_0) corresponding to a pure subnormal operator we can reconstruct T. Now the following question naturally arises: "what are the restrictions on matrices A_0 and B_0 such that they represent a subnormal operator ?" In the cases where A_0 and B_0 operate on a finite dimensional Hilbert space, D. Yakubovich [Ya1] showed that such a description can be given in terms of a topological property of a certain algebraic curve, associated with A_0 and B_0. However there is a subtle difference between Yakubovich's criterion and the Putinar's model operator (3.10). In fact, in some sense, Yakubovich gave conditions on A_0 and B_0 such that the operator (3.10) can be constructed so that the condition (3.11) is satisfied. By comparison, the Putinar's model operator (3.10) was already constructed so that it satisfies the condition (3.11). Thus we would guess that if the operator (3.10) can be constructed so that the condition (3.11) is satisfied then two matrices $\{A_0, B_0\}$ in (2.8) must satisfy the Yakubovich's criterion. In this viewpoint, we have the following:

Conjecture 3.3. The Putinar's matricial model (3.10) of rank two is subnormal.

An affirmative answer to the conjecture would show that if T is a hyponormal operator with rank-two self-commutator and satisfying that $\ker [T^*, T]$ is invariant for T then T is subnormal. Hence, in particular, one could obtain: *Every weakly subnormal operator with rank-two self-commutator is subnormal.*

Chapter 4

Weighted Shifts

4.1 Berger's theorem

Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \alpha_2, \cdots$ (called weights), the (unilateral) weighted shift W_α associated with α is the operator $\ell^2(\mathbb{Z}_+)$ defined by

$$W_\alpha e_n = \alpha_n e_{n+1} \ (n \geq 0),$$

where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that

$$W_\alpha \text{ is compact } \iff \alpha_n \to 0.$$

Indeed, $W_\alpha = UD$, where U is the unilateral shift and D is the diagonal operator whose diagonal entries are α_n.

We observe:

Proposition 4.1.1. *If $T \equiv W_\alpha$ is a weighted shift and $\omega \in \partial\mathbb{D}$ then $T \cong \omega T$.*

Proof. If $Ve_n := \omega^n e_n$ for all n then $VTV^* = \omega T$. □

As a consequence of Proposition 4.1.1, we can see that the spectrum of a weighted shift must be a circular symmetry:

$$\sigma(W_\alpha) = \sigma(\omega W_\alpha) = \omega\sigma(W_\alpha).$$

Indeed we have:

Theorem 4.1.2. *If $T \equiv W_\alpha$ is a weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ such that $\alpha_n \to \alpha_+$ then*

(i) $\sigma_p(T) = \emptyset$;
(ii) $\sigma(T) = \{\lambda : |\lambda| \leq \alpha_+\}$;
(iii) $\sigma_e(T) = \{\lambda : |\lambda| = \alpha_+\}$;
(iv) $|\lambda| < \alpha_+ \Rightarrow \operatorname{ind}(T - \lambda) = -1$.

Proof. The assertion (i) is straightforward. For the other assertions, observe that if $\alpha_+ = 0$ then T is compact and quasinilpotent. If instead $\alpha_+ > 0$ then $T - \alpha_+ U$ (U :=the unilateral shift) is a weighted shift whose weight sequence converges to 0. Hence $T - \alpha_+ U$ is a compact and hence

$$\sigma_e(T) = \sigma_e(\alpha_+ U) = \alpha_+ \sigma_e(U) = \{\lambda : |\lambda| = \alpha_+\}.$$

If $|\lambda| < \alpha_+$ then $T - \lambda$ is Fredholm and

$$\text{index}\,(T - \lambda) = \text{index}\,(\alpha_+ U - \lambda) = -1.$$

In particular, $\{\lambda : |\lambda| \le \alpha_+\} \subset \sigma(T)$. By the assertion (i), we can conclude that $\sigma(T) = \{\lambda : |\lambda| \le \alpha_+\}$. ☐

Theorem 4.1.3. *If $T \equiv W_\alpha$ is a weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ then*

$$[T^*, T] = \begin{bmatrix} \alpha_0^2 & & & \\ & \alpha_1^2 - \alpha_0^2 & & \\ & & \alpha_2^2 - \alpha_1^2 & \\ & & & \ddots \end{bmatrix}$$

Proof. From a straightforward calculation. ☐

The moments of W_α are defined by

$$\beta_0 := 1, \quad \beta_{n+1} = \alpha_0 \cdots \alpha_n,$$

but we reserve this term for the sequence $\gamma_n := \beta_n^2$.

Theorem 4.1.4. (Berger's theorem) *Let $T \equiv W_\alpha$ be a weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}$ and define the moment of T by*

$$\gamma_0 := 1 \text{ and } \gamma_n := \alpha_0^2 \alpha_1^2 \cdots \alpha_{n-1}^2 \ (n \ge 1).$$

Then T is subnormal if and only if there exists a probability measure ν on $\big[0, \|T\|^2\big]$ such that

$$\gamma_n = \int_{[0,\|T\|^2]} t^n d\nu(t) \ (t \ge 1). \tag{4.1}$$

Proof. ($\Rightarrow$) Note that T is cyclic. So if T is subnormal then $T \cong S_\mu$, i.e., there is an isomorphism $U : L^2(\mu) \longrightarrow P^2(\mu)$ such that

$$Ue_0 = 1 \text{ and } UTU^{-1} = S_\mu.$$

Observe $T^n e_0 = \sqrt{\gamma_n} e_n$ for all n. Also $U(T^n e_0) = S_\mu^n U e_0 = S_\mu^n 1 = z^n$. So

$$\int |z|^{2n} d\mu = \int |UT^n e_0|^2 d\mu = \int |U(\sqrt{\gamma_n} e_n)|^2 d\mu = \gamma_n \int |Ue_n|^2 d\mu = \gamma_n \|Ue_n\| = \gamma_n.$$

If ν is defined on $[0, ||T||^2]$ by

$$\nu(\Delta) := \mu\big(\{z : \ |z|^2 \in \Delta\}\big),$$

then ν is a probability measure and $\gamma_n = \int t^n d\nu(t)$.

($\Leftarrow$) If ν is the measure satisfying (4.1), define the measure μ by $d\mu(re^{i\theta}) = \frac{1}{2\pi} d\theta d\mu(r)$. Then we can see that $T \cong S_\mu$. $\square$

Example 4.1.5. (a) The Bergman shift B_α is the weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}$ given by

$$\alpha_n = \sqrt{\frac{n+1}{n+2}} \ (n \geq 0).$$

Then B_α is subnormal: indeed,

$$\gamma_n := \alpha_0^2 \alpha_1^2 \cdots \alpha_{n-1}^2 = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} = \frac{1}{n+1}$$

and if we define $\mu(t) = t$, i.e., $d\mu = dt$ then

$$\int_0^1 t^n d\mu(t) = \frac{1}{n+1} = \gamma_n.$$

(b) If $\alpha_n : \beta, 1, 1, 1, \cdots$ then W_α is subnormal: indeed $\gamma_n = \beta^2$ and if we define $d\mu = \beta^2 \delta_1 + (1-\beta^2)\delta_0$ then $\int_0^1 t^n d\mu = \beta^2 = \gamma_n$.

Remark. Recall that the Bergman space $A(\mathbb{D})$ for $\mathbb{D}$ is defined by

$$A(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} : \ f \text{ is analytic with } \int_{\mathbb{D}} |f|^2 d\mu < \infty \right\}.$$

Then the orthonormal basis for $A(\mathbb{D})$ is given by $\{e_n \equiv \sqrt{n+1}\, z^n : \ n = 0, 1, 2, \cdots\}$ with $d\mu = \frac{1}{\pi} dA$. The Bergman operator $T : A(\mathbb{D}) \to A(\mathbb{D})$ is defined by

$$Tf = zf.$$

In this case the matrix (α_{ij}) of the Bergman operator T with respect to the basis $\{e_n \equiv \sqrt{n+1}\, z^n : \ n = 0, 1, 2, \cdots\}$ is given by

$$\begin{aligned}
\alpha_{ij} &= \langle Te_j, \, e_i \rangle \\
&= \langle T\sqrt{j+1}\, z^j, \, \sqrt{i+1}\, z^i \rangle \\
&= \langle \sqrt{j+1}\, z^{j+1}, \, \sqrt{i+1}\, z^i \rangle \\
&= \sqrt{(j+1)(i+1)} \int_{\mathbb{D}} z^{j+1} \overline{z}^i d\mu \\
&= \sqrt{(j+1)(i+1)} \, \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r^{j+1+i} e^{i(j+1-i)\theta} \cdot r dr \, d\theta \\
&= \begin{cases} \sqrt{\frac{j+1}{j+2}} & (i = j+1) \\ 0 & (i \neq j+1) : \end{cases}
\end{aligned}$$

therefore

$$T = \begin{bmatrix} 0 & & & & \\ \sqrt{\frac{1}{2}} & 0 & & & \\ & \sqrt{\frac{2}{3}} & 0 & & \\ & & \sqrt{\frac{3}{4}} & 0 & \\ & & & \ddots & \ddots \end{bmatrix}.$$

4.2 k-Hyponormality

Given an n-tuple $\mathbf{T} = (T_1, \dots, T_n)$ of operators acting on H, we let

$$[\mathbf{T}^*, \mathbf{T}] \equiv \begin{bmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \vdots & \vdots \\ [T_1^*, T_k] & [T_2^*, T_k] & \cdots & [T_n^*, T_n] \end{bmatrix}.$$

By analogy with the case $n = 1$, we shall say that $\mathbf{T}$ is (jointly) hyponormal if $[\mathbf{T}^*, \mathbf{T}] \geq 0$.

An operator $T \in B(H)$ is called *k-hyponormal* if $(1, T, T^2, \cdots, T^k)$ is jointly hyponormal, i.e.,

$$\begin{aligned} M_k(T) &\equiv \Big([T^{*j}, T^i] \Big)_{i,j=1}^{k} \\ &= \begin{bmatrix} [T^*, T] & [T^{*2}, T] & \cdots & [T^{*k}, T] \\ [T^*, T^2] & [T^{*2}, T^2] & \cdots & [T^{*k}, T^2] \\ \vdots & \vdots & \vdots & \vdots \\ [T^*, T^k] & [T^{*2}, T^k] & \cdots & [T^{*k}, T^k] \end{bmatrix} \geq 0 \end{aligned}$$

An application of Choleski algorithm for operator matrices shows that $M_k(T) \geq 0$ is equivalent to the positivity of the following matrix

$$\begin{bmatrix} 1 & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \vdots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{bmatrix}.$$

The Bram-Halmos criterion can be then rephrased as saying that

$$T \text{ is subnormal} \Longleftrightarrow T \text{ is } k\text{-hyponormal for every } k \geq 1.$$

Recall ([Ath],[CMX],[CoS]) that T is called *weakly k-hyponormal* if

$$LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^{k} \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is weakly positive, i.e.,

$$\left\langle M_k(T) \begin{bmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{bmatrix}, \begin{bmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{bmatrix} \right\rangle \geq 0 \quad {}^\forall \lambda_1, \cdots, \lambda_k \in \mathbb{C}.$$

Observe that

$$\left\langle \left[(\overline{\lambda_1} T + \cdots + \overline{\lambda_k} T^k)^*, (\overline{\lambda_1} T + \cdots + \overline{\lambda_k} T^k) \right] x, x \right\rangle$$

$$= \left\langle \begin{bmatrix} [T^*, T] & [T^{*2}, T] & \cdots & [T^{*k}, T] \\ [T^*, T^2] & [T^{*2}, T^2] & \cdots & [T^{*k}, T^2] \\ \vdots & \vdots & \vdots & \vdots \\ [T^*, T^k] & [T^{*2}, T^k] & \cdots & [T^{*k}, T^k] \end{bmatrix} \begin{bmatrix} \lambda_1 x \\ \lambda_2 x \\ \vdots \\ \lambda_k x \end{bmatrix}, \begin{bmatrix} \lambda_1 x \\ \lambda_2 x \\ \vdots \\ \lambda_k x \end{bmatrix} \right\rangle \tag{4.2}$$

If $k = 2$ then T is said to be *quadratically hyponormal.* If $k = 3$ then T is said to be *cubically hyponormal.* Also T is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$.

Evidently, by (4.2)

$$k\text{-hyponormal} \Longrightarrow \text{ weakly } k\text{-hyponormal.}$$

The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([Cu1, Cu2, CuF1, CuF2, CuF3, CLL, CuL1, CuL2, CuL3, CMX, DPY, McCP]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle. For weighted shifts, positive results appear in [Cu1] and [CuF3], although no concrete example of a weighted shift which is polynomially hyponormal but not subnormal has yet been found (the existence of such weighted shifts was established in [CP1] and [CP2]).

Theorem 4.2.1. *Let $T \equiv W_\alpha$ be a weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_0^\infty$. The following are equivalent:*

(a) *T is k-hyponormal;*

(b) *For every $n \geq 0$, the Hankel matrix*

$$(\gamma_{n+i+j})_{i,j=0}^k \equiv \begin{bmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k+1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+2} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+k+1} & \gamma_{n+k+2} & \cdots & \gamma_{n+2k+2} \end{bmatrix} \quad \textit{is positive.}$$

Proof. [Cu1, Theorem 4] □

Lemma 4.2.2. *Let $\mathbf{T} = (T_1, T_2)$ be a pair of operators on H. Then $\mathbf{T}$ is (jointly) hyponormal if and only if*

(i) *T_1 is hyponormal*
(ii) *T_2 is hyponormal*
(iii) $|\langle [T_2^*, T_1] y, x \rangle|^2 \leq \langle [T_1^*, T_1] x, x \rangle \langle [T_2^*, T_2] y, y \rangle$ *(for any $x, y \in H$).*

Proof. $[\mathbf{T}^*, \mathbf{T}] \geq 0 \Longleftrightarrow \left\langle [\mathbf{T}^*, \mathbf{T}^*] \begin{pmatrix} x \\ ty \end{pmatrix}, \begin{pmatrix} x \\ ty \end{pmatrix} \right\rangle \geq 0$ for any $x, y \in H$ and $t \in \mathbb{R}$. Thus

$$\begin{aligned}
[\mathbf{T}^*, \mathbf{T}^*] \geq 0 \Longleftrightarrow & \left\langle \begin{bmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{bmatrix} \begin{pmatrix} x \\ ty \end{pmatrix}, \begin{pmatrix} x \\ ty \end{pmatrix} \right\rangle \geq 0 \\
\Longleftrightarrow & \langle [T_1^*, T_1]x, x \rangle + t^2 \langle [T_2^*, T_2]y, y \rangle + 2t \mathrm{Re} \, \langle [T_2^*, T_1]y, x \rangle \geq 0 \qquad (\dagger) \\
\Longrightarrow & \text{If } T_1 \text{ and } T_2 \text{ are hyponormal then} \\
& \quad t^2 \langle [T_2^*, T_2]y, y \rangle + 2t \big| \langle [T_2^*, T_1]y, x \rangle \big| + \langle [T_1^*, T_1]x, x \rangle \geq 0 \\
\Longrightarrow & D/4 \equiv |\langle [T_2^*, T_1]y, x \rangle|^2 - \langle [T_1^*, T_1]x, x \rangle \langle [T_2^*, T_2]y, y \rangle \leq 0 \\
\Longrightarrow & \big| \langle [T_2^*, T_1]y, x \rangle \big|^2 \leq \langle [T_1^*, T_1]x, x \rangle \langle [T_2^*, T_2]y, y \rangle \qquad (*)
\end{aligned}$$

Conversely if (*) holds then

$$\mathrm{Re} \, \langle [T_2^*, T_1]y, x \rangle^2 \leq \langle [T_1^*, T_1]x, x \rangle \langle [T_2^*, T_2]y, y \rangle,$$

which implies (†) holds. □

Corollary 4.2.3. *Let $\mathbf{T} = (T_1, T_2)$ be a pair of operators on H. Then $\mathbf{T}$ is hyponormal if and only if T_1 and T_2 are hyponormal and*

$$[T_2^*, T_1] = [T_1^*, T_1]^{\frac{1}{2}} D [T_2^*, T_2]^{\frac{1}{2}}$$

for some contraction D.

Proof. This follows from a theorem of Smul'jan [Smu]:

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \Longleftrightarrow A \geq 0, \; C \geq 0 \text{ and } B = \sqrt{A} D \sqrt{C} \text{ for some contraction } D.$$

□

Corollary 4.2.4. *Let $T \equiv W_\alpha$ be a weighted shift with weight sequence $\alpha : \ \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots$. Then the following are equivalent:*

(i) *T is 2-hyponormal;*
(ii) $\alpha_{n+1}^2 \left(\alpha_{n+2}^2 - \alpha_n^2 \right)^2 \leq \left(\alpha_{n+1}^2 - \alpha_n^2 \right) \left(\alpha_{n+2}^2 \alpha_{n+3}^2 - \alpha_n^2 \alpha_{n+1}^2 \right)$ *for any $n \geq 0$;*
(iii) $\alpha_n^2 \left(\alpha_{n+2}^2 - \alpha_{n+1}^2 \right)^2 \leq \alpha_{n+2}^2 \left(\alpha_{n+1}^2 - \alpha_n^2 \right) \left(\alpha_{n+3}^2 - \alpha_{n+2}^2 \right)$ *for any $n \geq 0$.*

Proof. By Corollary 4.2.3,

$$(T, T^2) \text{ hyponormal} \Longleftrightarrow [T^{*2}, T] = [T^*, T]^{\frac{1}{2}} E [T^{*2}, T^2]^{\frac{1}{2}} \text{ for some contraction } E.$$

Observe that $[T^*, T]$ and $[T^{*2}, T^2]$ are diagonal and that $[T^{*2}, T]$ is a backward weighted shift. It follows that E is a backward weighted shift. So it suffices to check that $(n, n+1)$-entries of E. Now,

$$\begin{aligned}\left\langle [T^{*2}, T]e_{n+1}, e_n \right\rangle &= \left\langle [T^*, T]^{\frac{1}{2}} E [T^{*2}, T^2]^{\frac{1}{2}} e_{n+1}, e_n \right\rangle \\ &= \left\langle E[T^{*2}, T^2]^{\frac{1}{2}} e_{n+1}, [T^*, T]^{\frac{1}{2}} e_n \right\rangle \\ &= \left\langle \left\langle E[T^{*2}, T^2]^{\frac{1}{2}} e_{n+1}, e_{n+1} \right\rangle e_{n+1}, \left\langle [T^*, T]^{\frac{1}{2}} e_n, e_n \right\rangle e_n \right\rangle \\ &= \left\langle [T^{*2}, T^2]^{\frac{1}{2}} e_{n+1}, e_{n+1} \right\rangle \left\langle [T^*, T]^{\frac{1}{2}} e_n, e_n \right\rangle \langle E e_{n+1}, e_{n+1} \rangle .\end{aligned}$$

Thus we can see that such a contraction E exists if and only if

$$\left| \left\langle [T^{*2}, T]e_{n+1}, e_n \right\rangle \right|^2 \le \langle [T^*, T]e_n, e_n \rangle \left\langle [T^{*2}, T^2]e_{n+1}, e_{n+1} \right\rangle \quad {}^\forall n \ge 0$$

which gives

$$\begin{aligned}\alpha_{n+1}^2 \left(\alpha_{n+2}^2 - \alpha_n^2\right)^2 &\le \left(\alpha_{n+1}^2 - \alpha_n^2\right)\left(\alpha_{n+2}^2 \alpha_{n+3}^2 - \alpha_n^2 \alpha_{n+1}^2\right) \\ (\alpha_1^2 \alpha_0)^2 &\le \alpha_0^2 \alpha_1^2 \alpha_2^2 \ \ \text{(this holds automatically since } \alpha_1 \le \alpha_2\text{)},\end{aligned}$$

which gives (i) ⇔ (ii).

Finally, (iii) is just (ii) suitably rewritten. □

4.3 The Propagation

We introduce:

Definition 4.3.1. If $\alpha_0 < \alpha_1 = \alpha_2 = \alpha_3 = \cdots$ then (α_n) is said to be *flat*.

Proposition 4.3.2. *If $T \equiv W_\alpha$ is a weighted shift with flat weights then T is subnormal.*

Proof. Without loss of generality we may assume that

$$(\alpha_n) : \alpha, 1, 1, 1, \cdots .$$

Then $\gamma_n = \alpha^2$ for any $n = 0, 1, 2, \cdots$. Put $d\mu = \alpha^2\delta_1 + (1-\alpha^2)\delta_0$, where δ_k = the point mass at k. Then $\int_0^1 t^n d\mu = (1-\alpha^2)\cdot 0 + \alpha^2 \cdot 1 = \alpha^2 = \gamma_n$. Therefore, T is subnormal. □

Theorem 4.3.3. *Let T be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$.*

(i) [Sta3] *Let T be subnormal. Then*

$$\alpha_n = \alpha_{n+1} \text{ for some } n \geq 0 \Longrightarrow \alpha \text{ is flat}$$

(ii) [Cu2] *Let W_α be 2-hyponormal. Then*

$$\alpha_n = \alpha_{n+1} \text{ for some } n \geq 0 \Longrightarrow \alpha \text{ is flat.}$$

Proof. (ii) ⇒ (i): Obvious.

(ii) Immediate from Corollary 4.2.4(ii). □

Lemma 4.3.4. *Let T be a weighted shift whose restriction to $\bigvee\{e_1, e_2, \cdots\}$ is subnormal with associated measure μ. Then T is subnormal if and only if*

(i) $\frac{1}{t} \in L^1(\mu)$, *i.e.*, $\int \frac{1}{t} d\mu < \infty$;

(ii) $\alpha_0^2 \leq \left(\left\|\frac{1}{t}\right\|_{L^1}\right)^{-1}$.

In particular, T is never subnormal when $\mu(\alpha) > 0$.

Proof. Let $S := T|_{\bigvee\{e_1,e_2,\cdots\}}$. Then S has weights $\alpha_k(S) := \alpha_{k+1}$ $(k \geq 0)$. So the corresponding "β numbers" are related by the equation

$$\beta_k(S) = \alpha_1 \cdots \alpha_k = \frac{\beta_{k+1}}{\alpha_0} \ (k = 1, 2, \cdots).$$

Since $\beta_k(S)^2 = \int t^{2k} d\mu$, we see that

T is subnormal $\Longleftrightarrow \exists$ a probability measure ν on $[0, \|T\|]$ such that

$$\frac{1}{\alpha_0^2}\int t^{2(k+1)} d\nu(t) = \frac{\beta_{k+1}^2}{\alpha_0^2} = \beta_k(S)^2 = \int t^{2k} d\mu \ (k \geq 0).$$

So $t^2 d\nu = \alpha_0^2 d\mu$. Thus

$$T \text{ is subnormal} \Longleftrightarrow d\nu = \lambda\delta_0 + \frac{\alpha_0^2}{t} d\mu \text{ for some } \lambda \geq 0.$$

Thus

$$T \text{ is subnormal} \Longleftrightarrow \begin{cases} \alpha_0^2 \int \frac{1}{t} d\mu \leq 1 \text{ or } \frac{1}{t} \in L^1(\mu) \\ \alpha_0^2 \left\|\frac{1}{t}\right\| \leq 1. \end{cases}$$

$\square$

Theorem 4.3.5. *For $x > 0$, let T_x be the weighted shift whose weight sequence is given by*

$$x, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \cdots$$

(a) *T_x is subnormal* $\Longleftrightarrow 0 < x \leq \sqrt{\frac{1}{2}}$

(b) *T_x is $k-$hyponormal* $\Longleftrightarrow 0 < x \leq \frac{k+1}{\sqrt{2k(k+2)}}$

In particular, T_x is 2-hyponormal $\Longleftrightarrow 0 < x \leq \frac{3}{4}$

(c) *T_x is quadratically hyponormal* $\Longleftrightarrow 0 < x \leq \sqrt{\frac{2}{3}}$.

Proof. (a) $T_x|_{\bigvee\{e_1,e_2,\cdots\}}$ has measure $d\mu = 2tdt$. So, $\frac{1}{t} \in L^1(\mu)$ and

$$\left\|\frac{1}{t}\right\|_{L^1(\mu)} = 2 \Longrightarrow x^2 \leq \frac{1}{2} \Longrightarrow x \leq \sqrt{\frac{1}{2}}.$$

(b) It is sufficient to show that

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} \geq 0$$

Since

$$\gamma_0 = 1,\ \gamma_1 = x^2,\ \gamma_2 = \frac{2}{3}x^2,\ \gamma_3 = \frac{1}{2}x^2,\ \gamma_4 = \frac{2}{5}x^2,$$

we have

$$\det\begin{bmatrix} 1 & x^2 & \frac{2}{3}x^2 \\ x^2 & \frac{2}{3}x^2 & \frac{1}{2}x^2 \\ \frac{2}{3}x^2 & \frac{1}{2}x^2 & \frac{2}{5}x^2 \end{bmatrix} = x^2 \det\begin{bmatrix} \frac{1}{x^2} & 1 & \frac{2}{3} \\ 1 & \frac{2}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & \frac{2}{5} \end{bmatrix}$$
$$= x^2\left(\frac{1}{60x^2} - \frac{4}{135}\right) \geq 0 \Longrightarrow x \leq \frac{3}{4}.$$

(c) See [Cu2] □

Let W_α be a weighted shift with weights $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$. For $s \in \mathbb{C}$, write

$$D(s) := \left[\left(W_\alpha + sW_\alpha^2\right)^*, W_\alpha + sW_\alpha^2\right]$$

and let

$$D_n(s) := P_n D(s) P_n = \begin{bmatrix} q_0 & \overline{\gamma_0} & 0 & \cdots & 0 & 0 \\ \gamma_0 & q_1 & \overline{\gamma_1} & \cdots & 0 & 0 \\ 0 & \gamma_1 & q_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{n-1} & \overline{\gamma_{n-1}} \\ 0 & 0 & 0 & \cdots & \gamma_{n-1} & q_n \end{bmatrix},$$

where $P_n :=$ the orthogonal projection onto the subspace spanned by $\{e_0, \cdots, e_n\}$,

$$\begin{cases} q_n := u_n + |s|^2 v_n \\ \gamma_n := s\sqrt{w_n}, \end{cases}$$

where

$$\begin{cases} u_n := \alpha_n^2 - \alpha_{n-1}^2 \\ v_n := \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2 \\ w_n = \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2, \end{cases}$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$.

Clearly,

$$W_\alpha \text{ is quadratically hyponormal } \Longleftrightarrow D_n(s) \geq 0 \text{ for any } s \in \mathbb{C}, \text{ for any } n \geq 0.$$

Let $d_n(\cdot) = \det D_n(\cdot)$. Then d_n satisfies the following 2-step recursive formula:

$$d_0 = q_0,\ d_1 = q_0 q_1 - |\gamma_0|^2,\ d_{n+2} = q_{n+2}d_{n+1} - |\gamma_{n+1}|^2 d_n.$$

If we let $t := |s|^2$, we observe that d_n is a polynomial in t of degree $n+1$. If we write

$$d_n \equiv \sum_{i=0}^{n+1} c(n,i)t^i,$$

then $c(n, i)$ satisfy a double-indexed recursive formula, i.e.,

$$\begin{cases} c(1,1) = u_1 v_0 + v_1 u_0 - w_0 \\ c(n,0) = u_0 \cdots u_n \\ c(n,n+1) = v_0 \cdots v_n \\ c(n+2,i) = u_{n+2} c(n+1,i) + v_{n+2} c(n+1,i-1) - w_{n+1} c(n,i-1). \end{cases}$$

Theorem 4.3.6. (Outer propagation) *Let T be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$. If T is quadratically hyponormal then*

$$\alpha_n = \alpha_{n+1} = \alpha_{n+2} \text{ for some } n \implies \alpha_n = \alpha_{n+1} = \alpha_{n+2} = \alpha_{n+3} = \cdots .$$

Proof. We may assume that $n = 0$ and $\alpha_0 = \alpha_1 = \alpha_2 = 1$. We want to show that $\alpha_3 = 1$. A straightforward calculation shows that

$$\begin{aligned} d_0 &= 1 + t \\ d_1 &= t^2 \\ d_2 &= \left(\alpha_3^2 - 1\right) t^3 \\ d_3 &= \left(\alpha_3^2 - 1\right)\left(\alpha_3^2 \alpha_4^2 - 1\right) t^4 \\ d_4 &= q_4 d_3 - \gamma_3^2 d_2 \\ &= \left[\left(\alpha_4^2 - \alpha_3^2\right) + t\left(\alpha_4^2 \alpha_5^2 - \alpha_3^2\right)\right]\left(\alpha_3^2 - 1\right)\left(\alpha_3^2 \alpha_4^2 - 1\right) t^4 - t\alpha_3^2\left(\alpha_4^2 - 1\right)^2\left(\alpha_3^2 - 1\right) t^3. \end{aligned}$$

So

$$\lim_{t \to 0^+} \frac{d_4}{t^4} = -\alpha_4^2\left(\alpha_3^2 - 1\right)^3 \geq 0,$$

which implies that $\alpha_3 = 1$. □

Theorem 4.3.7. (Inner Propagation) *Let T be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$. If T is quadratically hyponormal then*

$$\alpha_n = \alpha_{n+1} = \alpha_{n+2} \text{ for some } n \implies \alpha_1 = \cdots = \alpha_n.$$

Proof. Withou loss of generality we may assume $n = 2$, i.e., $\alpha_2 = \alpha_3 = \alpha_4 = 1$. We want to show that $\alpha_1 = 1$. We consider d_3. Now,

$$\begin{aligned} &d_3(0) = q_3(0) d_2(0) = 0 \text{ since } q_3(0) = \alpha_3^2 - \alpha_2^2 = 0 \\ &d_3'(0) = q_3'(0) d_2(0) - \alpha_2^2(\alpha_3^2 - \alpha_1^2)^2 \alpha_1(0) = \cdots = 0 \\ &d_3''(0) = 2q_3'(0) d_2'(0) - 2(1 - \alpha_1^2)\alpha_1'(0) = \cdots = -2\alpha_0^4(1 - \alpha_1^2)^3. \end{aligned}$$

Therefore

$$d_3(t) = -\alpha_0^4(1 - \alpha_1^2)^3 t^2 + \cdots .$$

Since $d_3 \geq 0$ (all $t \geq 0$), it follows $\alpha_1 = 1$. □

Theorem 4.3.8. (Propagation of quadratic hyponormality) *Let T be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$. If T is quadratically hyponormal then*

$$\alpha_n = \alpha_{n+1} \text{ for some } n \geq 1 \implies \alpha \text{ is flat, i.e., } \alpha_1 = \alpha_2 = \cdots .$$

Proof. Without loss of generality we may assume $n = 1$ and $\alpha_1 = \alpha_2 = 1$. We want to show that $\alpha_0 = 1$ or $\alpha_3 = 1$. Then we have

$$d_4(t) = \alpha_0^2\alpha_4^2(\alpha_0^2 - 1)(\alpha_3^2 - 1)^3t^2 + c(4,3)t^3 + c(4,4)t^4 + c(4,5)t^5,$$

so

$$\lim_{t\to 0+} \frac{d_4(t)}{t^2} = \alpha_0^2\alpha_4^2(\alpha_0^2 - 1)(\alpha_3^2 - 1)^3 \geq 0.$$

Thus $\alpha_0 = 1$ or $\alpha_3 = 1$, so that three equal weights are present. □

Remark. However the condition "$n \geq 1$" cannot be relaxed to "$n \geq 0$". For example, in view of Theorem 4.3.5, if

$$\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \cdots$$

then W_α is quadratically hyponormal but not subnormal. In fact, W_α is not cubically hyponormal : if we let

$$c_5(t) := \det\left(P_5[(W_\alpha + tW_\alpha^2 + t^2W_\alpha^3)^*, (W_\alpha + tW_\alpha^2 + t^2W_\alpha^3)]P_5\right)$$

then

$$\lim_{t\to 0+} \frac{c_5(t)}{t^8} = -\frac{1}{2041200} < 0.$$

We have a related problem (see Problems 4.1 and 4.2).

Theorem 4.3.9. *If W_α is a polynomial hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$ such that $\alpha_0 = \alpha_1$ then α is flat.*

Proof. Without loss of generality we may assume $\alpha_0 = \alpha_1 = 1$. We claim that if $\alpha_0 = \alpha_1 = 1$ and W_α is weakly k-hyponormal then

$$(2 - \alpha_{k-1}^2)\alpha_k^2 \geq 1 \text{ for all } k \geq 3. \tag{4.3}$$

For (4.3) suppose W_α is weakly k-hyponormal. Then $T_k := W_\alpha + sW_\alpha^k$ is hyponormal for every $s \in \mathbb{R}$. For $k \geq 3$,

$$D_k = P_k[T_k^*, T_k]P_k = \begin{bmatrix} q_{k,0} & 0 & 0 & \cdots & \gamma_{k,0} & 0 \\ 0 & q_{k,1} & 0 & \cdots & 0 & \gamma_{k,1} \\ 0 & 0 & q_{k,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_{k,0} & 0 & 0 & \cdots & q_{k,k-1} & 0 \\ 0 & \gamma_{k,1} & 0 & \cdots & 0 & q_{k,k} \end{bmatrix},$$

where

$$q_{k,j} := \begin{cases} (\alpha_j^2 - \alpha_{j-1}^2) + s^2(\alpha_{k+j-1}^2\alpha_{k+j-2}^2 \cdots \alpha_j^2) & (0 \le j \le k-1) \\ (\alpha_k^2 - \alpha_{k-1}^2) + s^2(\alpha_{2k-1}^2\alpha_{2k-2}^2 \cdots \alpha_k^2 - \alpha_{k-1}^2\alpha_{k-2}^2 \cdots \alpha_0^2) & (j = k) \end{cases}$$

$$\gamma_{k,0} = s\alpha_0\alpha_1 \cdots \alpha_{k-2}\alpha_{k-1}^2$$

$$\gamma_{k,1} = s\alpha_1\alpha_2 \cdots \alpha_{k-1}(\alpha_k^2 - \alpha_0^2).$$

Thus

$$\det D_k = \begin{cases} (q_{k,k}q_{k,1} - \gamma_{k,1}^2)(q_{k,k-1}q_{k,0} - \gamma_{k,0}^2)q_{k,k-2}q_{k,k-3} \cdots q_{k,2} & (k \ge 4) \\ (q_{3,3}q_{3,1} - \gamma_{3,1}^2)(q_{3,2}q_{3,0} - \gamma_{3,0}^2) & (k = 3) \end{cases}$$

If $\alpha_0 = \alpha_1 = 0$ and if we let $t := s^2$ then

$$\lim_{t \to 0+} \frac{\det D_k}{t^k} = (2\alpha_k^2 - \alpha_{k-1}^2\alpha_k^2 - 1) \prod_{j=2}^{k-1} \alpha_j^2(\alpha_j^2 - \alpha_{j-1}^2).$$

Since $\det D_k \ge 0$ it follows that

$$(2 - \alpha_{k-1}^2)\alpha_k^2 - 1 \ge 0,$$

which proves (4.3). If $\lim_{t \to 0+} \alpha_n^2 = \alpha$ then $(2 - \alpha^2)\alpha - 1 \ge 0$, i.e.,

$$(\alpha - 1)^2 \le 0, \quad \text{i.e.,} \quad \alpha = 1.$$

□

Consider the case of cubic hyponormality. Let W_α be a hyponormal weighted shift with $\{\alpha_n\}_{n=0}^\infty$. For $s, t \in \mathbb{C}$, let

$$C_n(s,t) := P_n[(W_\alpha + sW_\alpha^2 + tW_\alpha^3)^* W_\alpha + sW_\alpha^2 + tW_\alpha^3]P_n.$$

Then $C_n(s,t)$ is a pentadiagonal matrix :

$$C_n(s,t) = \begin{bmatrix} q_0 & \gamma_0 & \upsilon_0 & 0 & 0 & \cdots & 0 & \\ \overline{\gamma_0} & q_1 & \gamma_1 & \upsilon_1 & 0 & 0 & \cdots & 0 \\ \overline{\upsilon_0} & \overline{\gamma_1} & q_2 & \gamma_2 & \upsilon_2 & 0 & \cdots & 0 \\ 0 & \overline{\upsilon_1} & \overline{\gamma_2} & q_3 & \gamma_3 & \upsilon_3 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \upsilon_{n-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \gamma_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \overline{\upsilon_{n-2}} & \overline{\gamma_{n-1}} & q_n \end{bmatrix},$$

where

$$\begin{cases} q_n = (\alpha_n^2 - \alpha_{n-1}^2) + (\alpha_n^2\alpha_{n+1}^2|t|^2 + (\alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-3}^2\alpha_{n-2}^2\alpha_{n-1}^2)|t|^2 \\ \gamma_n = \alpha_n(\alpha_{n+1}^2 - \alpha_{n-1}^2)\overline{s} + \alpha_n(\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-1}^2\alpha_{n-2}^2)s\overline{t} \\ \upsilon_n = \alpha_n\alpha_{n+1}(\alpha_{n+2}^2 - \alpha_{n-1}^2)\overline{t} \end{cases}$$

and $\alpha_{-1} = \alpha_{-2} = \alpha_{-3} = 0$. Then

$$W_\alpha \text{ is cubically hyponormal} \iff \det C_n(s,t) \geq 0 \ (s,t \in \mathbb{C},\ n \geq 0).$$

In particular, if $d_k := \det C_k(s,t)$ then

$$\begin{aligned} d_k =& \Big(q_{k-1} - \frac{\gamma_{k-3}\gamma_{k-2}\overline{\upsilon_{k-3}}}{|\gamma_{k-3}|^2}\Big) d_{k-1} - \Big(|\gamma_{k-2}|^2 - \frac{q_{k-2}\gamma_{k-3}\gamma_{k-2}\overline{\upsilon_{k-3}}}{|\gamma_{k-3}|^2}\Big) d_{k-2} \\ &+ \Big(|\upsilon_{k-3}|^2 q_{k-2} - \gamma_{k-3}\gamma_{k-2}\overline{\upsilon_{k-3}}\Big) d_{k-3} + |\upsilon_{k-4}|^2 \Big(|\upsilon_{k-3}|^2 - \frac{q_{k-3}\gamma_{k-3}\gamma_{k-2}\overline{\upsilon_{k-3}}}{|\gamma_{k-3}|^2}\Big) d_{k-4} \\ &+ \frac{|\upsilon_{k-4}|^2 |\upsilon_{k-2}|^2 \gamma_{k-3}\gamma_{k-2}\overline{\upsilon_{k-3}}}{|\gamma_{k-3}|^2} d_{k-5}. \end{aligned}$$

4.4 The Perturbations

Recall the Bram-Halmos criterion for subnormality, which states that an operator T is subnormal if and only if

$$\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$, or equivalently,

$$\begin{bmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{bmatrix} \geq 0 \qquad (\text{all } k \geq 1). \tag{4.4}$$

Condition (4.4) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (4.4) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (4.4) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{4.5}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (4.5) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (4.4); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$.

Recall also that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e.,

$$(M_k(T) \begin{bmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{bmatrix}, \begin{bmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{bmatrix}) \geq 0 \qquad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C}. \tag{4.6}$$

If $k = 2$ then T is said to be *quadratically hyponormal*, and if $k = 3$ then T is said to be *cubically hyponormal.* Similarly, $T \in \mathcal{B}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general.

In the present section we renew our efforts to help describe the gap between subnormality and hyponormality, with particular emphasis on polynomial hyponormality. We focus on the class of unilateral weighted shifts, and initiate a study of how the above mentioned notions behave under finite perturbations of the weight sequence. We first obtain three concrete results:

(i) the subnormality of W_α is never stable under nonzero finite rank perturbations unless the perturbation is confined to the zeroth weight;

(ii) 2-hyponormality implies positive quadratic hyponormality, in the sense that the Maclaurin coefficients of $D_n(s) := \det P_n\,[(W_\alpha + sW_\alpha^2)^*,\, W_\alpha + sW_\alpha^2]\,P_n$ are nonnegative, for every $n \ge 0$, where P_n denotes the orthogonal projection onto the basis vectors $\{e_0, \cdots, e_n\}$; and

(iii) if α is strictly increasing and W_α is 2-hyponormal then for α' a small perturbation of α, the shift $W_{\alpha'}$ remains positively quadratically hyponormal.

Along the way we establish two related results, each of independent interest:

(iv) an integrality criterion for a subnormal weighted shift to have an n-step subnormal extension; and

(v) a proof that the sets of k-hyponormal and weakly k-hyponormal operators are closed in the strong operator topology.

C. Berger's characterization of subnormality for unilateral weighted shifts states that W_α is subnormal if and only if there exists a Borel probability measure μ supported in $[0, ||W_\alpha||^2]$, with $||W_\alpha||^2 \in \operatorname{supp}\mu$, such that

$$\gamma_n = \int t^n d\mu(t) \quad \text{for all } n \ge 0.$$

Given an initial segment of weights $\alpha : \alpha_0, \cdots \alpha_m$, the sequence $\widehat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\widehat{\alpha} : \alpha_i$ $(i = 0, \cdots, m)$ is said to be *recursively generated* by α if there exists $r \ge 1$ and $\varphi_0, \cdots, \varphi_{r-1} \in \mathbb{R}$ such that

$$\gamma_{n+r} = \varphi_0\gamma_n + \cdots + \varphi_{r-1}\gamma_{n+r-1} \text{ (all } n \ge 0),$$

where $\gamma_0 = 1,\ \gamma_n = \alpha_0^2 \cdots \alpha_{n-1}^2$ $(n \ge 1)$. In this case, $W_{\widehat{\alpha}}$ with weights $\widehat{\alpha}$ is said to be recursively generated. If we let

$$g(t) := t^r - (\varphi_{r-1}t^{r-1} + \cdots + \varphi_0)$$

then g has r distinct real roots $0 \le s_0 < \cdots < s_{r-1}$. Then $W_{\widehat{\alpha}}$ is a subnormal shift whose Berger measure μ is given by

$$\mu = \rho_0\delta_{s_0} + \cdots + \rho_{r-1}\delta_{s_r-1},$$

where $(\rho_0, \cdots, \rho_{r-1})$ is the unique solution of the Vandermonde equation

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_0 & s_1 & \cdots & s_{r-1} \\ \vdots & \vdots & & \vdots \\ s_0^{r-1} & s_1^{r-1} & \cdots & s_{r-1}^{r-1} \end{bmatrix} \begin{bmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_{r-1} \end{bmatrix} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{r-1} \end{bmatrix}.$$

For example, given $\alpha_0 < \alpha_1 < \alpha_2$, $W_{\widehat{(\alpha_0,\alpha_1,\alpha_2)}}$ is the recursive weighted shift whose weights are calculated according to the recursive relation

$$\alpha_{n+1}^2 = \varphi_1 + \varphi_0 \frac{1}{\alpha_n^2},$$

where $\varphi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2-\alpha_1^2)}{\alpha_1^2-\alpha_0^2}$ and $\varphi_1 = -\frac{\alpha_1^2(\alpha_2^2-\alpha_0^2)}{\alpha_1^2-\alpha_0^2}$. In this case, $W_{\widehat{(\alpha_0,\alpha_1,\alpha_2)}}$ is subnormal with 2-atomic Berger measure. Write $W_{x\widehat{(\alpha_0,\alpha_1,\alpha_2)}}$ for the weighted shift whose weight sequence consists of the initial weight x followed by the weight sequence of $W_{\widehat{(\alpha_0,\alpha_1,\alpha_2)}}$.

By the Density Theorem ([CuF2, Theorem 4.2 and Corollary 4.3]), we know that if W_α is a subnormal weighted shift with weights $\alpha = \{\alpha_n\}$ and $\epsilon > 0$, then there exists a nonzero compact operator K with $||K|| < \epsilon$ such that $W_\alpha + K$ is a recursively generated subnormal weighted shift; in fact $W_\alpha + K = W_{\widehat{\alpha^{(m)}}}$ for some $m \geq 1$, where $\alpha^{(m)} : \alpha_0, \cdots, \alpha_m$. The following result shows that K cannot generally be taken to be finite rank.

Theorem 4.4.1. (Finite Rank Perturbations of Subnormal Shifts) *If W_α is a subnormal weighted shift then there exists no nonzero finite rank operator $F(\neq cP_{\{e_0\}})$ such that $W_\alpha + F$ is a subnormal weighted shift. Concretely, suppose W_α is a subnormal weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ and assume $\alpha' = \{\alpha'_n\}$ is a nonzero perturbation of α in a finite number of weights except the initial weight; then $W_{\alpha'}$ is not subnormal.*

We next consider the selfcommutator $[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$. Let W_α be a hyponormal weighted shift. For $s \in \mathbb{C}$, we write

$$D(s) := [(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$$

and we let

$$D_n(s) := P_n[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]P_n = \begin{bmatrix} q_0 & \bar{r}_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \dots & r_{n-1} & q_n \end{bmatrix}, \quad (4.7)$$

where P_n is the orthogonal projection onto the subspace generated by $\{e_0, \cdots, e_n\}$,

$$\begin{cases} q_n := u_n + |s|^2 v_n \\ r_n := s\sqrt{w_n} \\ u_n := \alpha_n^2 - \alpha_{n-1}^2 \\ v_n := \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2 \\ w_n := \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2, \end{cases} \quad (4.8)$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \ge 0$ for all $s \in \mathbb{C}$ and all $n \ge 0$. Let $d_n(\cdot) := \det(D_n(\cdot))$. Then d_n satisfies the following 2–step recursive formula:

$$d_0 = q_0, \quad d_1 = q_0 q_1 - |r_0|^2, \quad d_{n+2} = q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n.$$

If we let $t := |s|^2$, we observe that d_n is a polynomial in t of degree $n+1$, and if we write $d_n \equiv \sum_{i=0}^{n+1} c(n,i) t^i$, then the coefficients $c(n,i)$ satisfy a double-indexed recursive formula, namely

$$\begin{aligned} c(n+2,i) &= u_{n+2}\, c(n+1,i) + v_{n+2}\, c(n+1,i-1) - w_{n+1}\, c(n,i-1), \\ c(n,0) &= u_0 \cdots u_n, \quad c(n,n+1) = v_0 \cdots v_n, \quad c(1,1) = u_1 v_0 + v_1 u_0 - w_0 \end{aligned} \tag{4.9}$$

($n \ge 0$, $i \ge 1$). We say that W_α is *positively quadratically hyponormal* if $c(n,i) \ge 0$ for every $n \ge 0$, $0 \le i \le n+1$. Evidently, positively quadratically hyponormal $\Longrightarrow$ quadratically hyponormal. The converse, however, is not true in general.

The following theorem establishes a useful relation between 2-hyponormality and positive quadratic hyponormality.

Theorem 4.4.2. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ be a weight sequence and assume that W_α is 2-hyponormal. Then W_α is positively quadratically hyponormal. More precisely, if W_α is 2-hyponormal then*

$$c(n,i) \;\ge\; v_0 \cdots v_{i-1} u_i \cdots u_n \qquad (n \ge 0,\ 0 \le i \le n+1). \tag{4.10}$$

In particular, if α is strictly increasing and W_α is 2-hyponormal then the Maclaurin coefficients of $d_n(t)$ are positive for all $n \ge 0$.

If W_α is a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, then the *moments* of W_α are usually defined by $\beta_0 := 1$, $\beta_{n+1} := \alpha_n \beta_n$ ($n \ge 0$); however, we prefer to reserve this term for the sequence $\gamma_n := \beta_n^2$ ($n \ge 0$). A criterion for k-hyponormality can be given in terms of these moments ([Cu2, Theorem 4]): if we build a $(k+1) \times (k+1)$ Hankel matrix $A(n;k)$ by

$$A(n;k) := \begin{bmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{bmatrix} \quad (n \ge 0),$$

then

$$W_\alpha \text{ is } k\text{-hyponormal} \iff A(n;k) \ge 0 \quad (n \ge 0). \tag{4.11}$$

In particular, for α strictly increasing, W_α is 2-hyponormal if and only if

$$\det \begin{bmatrix} \gamma_n & \gamma_{n+1} & \gamma_{n+2} \\ \gamma_{n+1} & \gamma_{n+2} & \gamma_{n+3} \\ \gamma_{n+2} & \gamma_{n+3} & \gamma_{n+4} \end{bmatrix} \ge 0 \quad (n \ge 0). \tag{4.12}$$

One might conjecture that if W_α is a k-hyponormal weighted shift whose weight sequence is strictly increasing then W_α remains weakly k-hyponormal under a small perturbation of the weight sequence. We will show below that this is true for $k = 2$ ([**?**]).

In [CuF3, Theorem 4.3], it was shown that the gap between 2-hyponormality and quadratic hyponormality can be detected by unilateral shifts with a weight sequence $\alpha : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$. In particular, there exists a maximum value $H_2 \equiv H_2(a, b, c)$ of x that makes $W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ 2-hyponormal; H_2 is called the modulus of 2-hyponormality (cf. citeCuF3). Any value of $x > H_2$ yields a non-2-hyponormal weighted shift. However, if $x - H_2$ is small enough, $W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ is still quadratically hyponormal. The following theorem shows that, more generally, for finite rank perturbations of weighted shifts with strictly increasing weight sequences, there always exists a gap between 2-hyponormality and quadratic hyponormality.

Theorem 4.4.3. (Finite Rank Perturbations of 2-hyponormal Shifts) *Let $\alpha = \{\alpha_n\}_{n=0}^\infty$ be a strictly increasing weight sequence. If W_α is 2-hyponormal then W_α remains positively quadratically hyponormal under a small nonzero finite rank perturbation of α.*

We are ready for:

Proof of Theorem 4.4.1. It suffices to show that if T is a weighted shift whose restriction to $\bigvee\{e_n, e_{n+1}, \cdots\}$ $(n \geq 2)$ is subnormal then there is at most one α_{n-1} for which T is subnormal.

Let $W := T|_{\bigvee\{e_{n-1},e_n,e_{n+1},\cdots\}}$ and $S := T|_{\bigvee\{e_n,e_{n+1},\cdots\}}$, where $n \geq 2$. Then W and S have weights $\alpha_k(W) := \alpha_{k+n-1}$ and $\alpha_k(S) := \alpha_{k+n}$ $(k \geq 0)$. Thus the corresponding moments are related by the equation

$$\gamma_k(S) = \alpha_n^2 \cdots \alpha_{n+k-1}^2 = \frac{\gamma_{k+1}(W)}{\alpha_{n-1}^2}.$$

We now adapt the proof of [Cu2, Proposition 8]. Suppose S is subnormal with associated Berger measure μ. Then $\gamma_k(S) = \int_0^{||T||^2} t^k \, d\mu$. Thus W is subnormal if and only if there exists a probability measure ν on $[0, ||T||^2]$ such that

$$\frac{1}{\alpha_{n-1}^2} \int_0^{||T||^2} t^{k+1} \, d\nu(t) = \int_0^{||T||^2} t^k \, d\mu(t) \quad \text{for all } k \geq 0,$$

which readily implies that $t \, d\nu = \alpha_{n-1}^2 \, d\mu$. Thus W is subnormal if and only if the formula

$$d\nu := \lambda \cdot \delta_0 + \frac{\alpha_{n-1}^2}{t} d\mu$$

defines a probability measure for some $\lambda \geq 0$, where δ_0 is the point mass at the origin. In particular $\frac{1}{t} \in L^1(\mu)$ and $\mu(\{0\}) = 0$ whenever W is subnormal. If we repeat the above argument for W and $V := T|_{\bigvee\{e_{n-2},e_{n-1},\cdots\}}$, then we should have that $\nu(\{0\}) = 0$ whenever V is subnormal. Therefore we can conclude that if V is subnormal then $\lambda = 0$, and hence

$$d\nu = \frac{\alpha_{n-1}^2}{t} d\mu.$$

Thus we have

$$1 = \int_0^{||T||^2} d\nu(t) = \alpha_{n-1}^2 \int_0^{||T||^2} \frac{1}{t} d\mu(t),$$

so that

$$\alpha_{n-1}^2 = \left(\int_0^{||T||^2} \frac{1}{t} d\mu(t) \right)^{-1},$$

which implies that α_{n-1} is determined uniquely by $\{\alpha_n, \alpha_{n+1}, \cdots\}$ whenever T is subnormal. This completes the proof. $\square$

Theorem 4.4.1 says that a nonzero finite rank perturbation of a subnormal shift is never subnormal unless the perturbation occurs at the initial weight. However, this is not the case for k-hyponormality. To see this we use a close relative of the Bergman shift B_+ (whose weights are given by $\alpha = \{\sqrt{\frac{n+1}{n+2}}\}_{n=0}^\infty$); it is well known that B_+ is subnormal.

Example 4.4.4. *For $x > 0$, let T_x be the weighted shift whose weights are given by*

$$\alpha_0 := \sqrt{\frac{1}{2}}, \quad \alpha_1 := \sqrt{x}, \quad \text{and} \quad \alpha_n := \sqrt{\frac{n+1}{n+2}} \ (n \geq 2).$$

Then we have:

(i)

T_x is subnormal $\Longleftrightarrow x = \frac{2}{3}$;

(ii)

T_x is 2-hyponormal $\Longleftrightarrow \frac{63-\sqrt{129}}{80} \leq x \leq \frac{24}{35}$.

Proof. Assertion (i) follows from Theorem 4.4.1. For assertion (ii) we use (4.12): T_x is 2-hyponormal if and only if

$$\det \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}x \\ \frac{1}{2} & \frac{1}{2}x & \frac{3}{8}x \\ \frac{1}{2}x & \frac{3}{8}x & \frac{3}{10}x \end{bmatrix} \geq 0 \quad \text{and} \quad \det \begin{bmatrix} \frac{1}{2} & \frac{1}{2}x & \frac{3}{8}x \\ \frac{1}{2}x & \frac{3}{8}x & \frac{3}{10}x \\ \frac{3}{8}x & \frac{3}{10}x & \frac{1}{4}x \end{bmatrix} \geq 0,$$

or equivalently, $\frac{63-\sqrt{129}}{80} \leq x \leq \frac{24}{35}$. $\square$

For perturbations of recursive subnormal shifts of the form $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$, subnormality and 2-hyponormality coincide.

Theorem 4.4.5. *Let* $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ *be recursively generated by* $\sqrt{a}, \sqrt{b}, \sqrt{c}$. *If* T_x *is the weighted shift whose weights are given by* $\alpha_x : \alpha_0, \cdots, \alpha_{j-1}, \sqrt{x}, \alpha_{j+1}, \cdots$, *then we have*

$$T_x \text{ is subnormal} \Longleftrightarrow T_x \text{ is 2-hyponormal} \Longleftrightarrow \begin{cases} x = \alpha_j^2 & \text{if } j \geq 1; \\ x \leq a & \text{if } j = 0. \end{cases}$$

Proof. Since α is recursively generated by $\sqrt{a}, \sqrt{b}, \sqrt{c}$, we have that $\alpha_0^2 = a,\ \alpha_1^2 = b,\ \alpha_2^2 = c$,

$$\alpha_3^2 = \frac{b(c^2 - 2ac + ab)}{c(b-a)}, \quad \text{and} \quad \alpha_4^2 = \frac{bc^3 - 4abc^2 + 2ab^2c + a^2bc - a^2b^2 + a^2c^2}{(b-a)(c^2 - 2ac + ab)}. \tag{4.13}$$

Case 1 ($j = 0$): It is evident that T_x is subnormal if and only if $x \leq a$. For 2-hyponormality observe by (4.11) that T_x is 2-hyponormal if and only if

$$\det \begin{bmatrix} 1 & x & bx \\ x & bx & bcx \\ bx & bcx & \alpha_3^2 bcx \end{bmatrix} \geq 0,$$

or equivalently, $x \leq a$.

Case 2 ($j \geq 1$): Without loss of generality we may assume that $j = 1$ and $a = 1$. Thus $\alpha_1 = \sqrt{x}$. Then by Theorem 4.4.1, T_x is subnormal if and only if $x = b$. On the other hand, by (4.12), T_x is 2-hyponormal if and only if

$$\det \begin{bmatrix} 1 & 1 & x \\ 1 & x & cx \\ x & cx & \alpha_3^2 cx \end{bmatrix} \geq 0 \quad \text{and} \quad \det \begin{bmatrix} 1 & x & cx \\ x & cx & \alpha_3^2 cx \\ cx & \alpha_3^2 cx & \alpha_3^2 \alpha_4^2 cx \end{bmatrix} \geq 0.$$

Thus a direct calculation with the specific forms of α_3, α_4 given in (4.13) shows that T_x is 2-hyponormal if and only if $(x - b)\left(x - \frac{b(c^2-2c+b)}{b-1}\right) \leq 0$ and $x \leq b$. Since $b \leq \frac{b(c^2-2c+b)}{b-1}$, it follows that T_x is 2-hyponormal if and only if $x = b$. This completes the proof. □

With the notation in (4.8), we let

$$p_n := u_n\, v_{n+1} - w_n \qquad (n \geq 0).$$

We then have:

Lemma 4.4.6. *If* $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ *is a strictly increasing weight sequence then the following statements are equivalent:*

(i)
W_α *is 2-hyponormal;*

(ii)
$\alpha_{n+1}^2 (u_{n+1} + u_{n+2})^2 \leq u_{n+1} v_{n+2} \qquad (n \geq 0)$*;*

(iii)
$$\frac{\alpha_n^2}{\alpha_{n+2}^2}\frac{u_{n+2}}{u_{n+3}} \ \le \ \frac{u_{n+1}}{u_{n+2}} \quad (n \ge 0);$$
(iv)
$$p_n \ge 0 \quad (n \ge 0).$$

Proof. This follows from a straightforward calculation. □

We are ready for:

Proof of Theorem 4.4.2. If α is not strictly increasing then α is flat, by the argument of [Cu2, Corollary 6], i.e., $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$. Then

$$D_n(s) = \begin{bmatrix} \alpha_0^2 + |s|^2\alpha_0^4 & \bar{s}\alpha_0^3 \\ s\alpha_0^3 & |s|^2\alpha_0^4 \end{bmatrix} \oplus 0_\infty$$

(cf. (4.7)), so that (4.10) is evident. Thus we may assume that α is strictly increasing, so that $u_n > 0,\ v_n > 0$ and $w_n > 0$ for all $n \ge 0$. Recall that if we write $d_n(t) := \sum_{i=0}^{n+1} c(n,i)t^i$ then the $c(n,i)$'s satisfy the following recursive formulas (cf. (4.9)):

$$c(n+2,i) = u_{n+2}\,c(n+1,i) + v_{n+2}\,c(n+1,i-1) - w_{n+1}\,c(n,i-1) \quad (n \ge 0,\ 1 \le i \le n). \tag{4.14}$$

Also, $c(n,n+1) = v_0 \cdots v_n$ (again by (4.9) and $p_n := u_n v_{n+1} - w_n \ge 0$ $(n \ge 0)$, by Lemma 4.4.6. A straightforward calculation shows that

$$\begin{aligned} &d_0(t) = u_0 + v_0\,t; \\ &d_1(t) = u_0u_1 + (v_0u_1 + p_0)\,t + v_0v_1\,t^2; \\ &d_2(t) = u_0u_1u_2 + (v_0u_1u_2 + u_0p_1 + u_2p_0)\,t + (v_0v_1u_2 + v_0p_1 + v_2p_0)\,t^2 + v_0v_1v_2\,t^3. \end{aligned} \tag{4.15}$$

Evidently,
$$c(n,\, i) \ge 0 \qquad (0 \le n \le 2,\ 0 \le i \le n+1). \tag{4.16}$$

Define
$$\beta(n,i) := c(n,i) - v_0 \cdots v_{i-1}u_i \cdots u_n \qquad (n \ge 1,\ 1 \le i \le n).$$

For every $n \ge 1$, we now have

$$c(n,i) = \begin{cases} u_0 \cdots u_n \ge 0 & (i = 0) \\ v_0 \cdots v_{i-1}u_i \cdots u_n \ + \ \beta(n,i) & (1 \le i \le n) \\ v_0 \cdots v_n \ge 0 & (i = n+1). \end{cases} \tag{4.17}$$

For notational convenience we let $\beta(n,0) := 0$ for every $n \ge 0$.

Claim 1. *For $n \ge 1$,*
$$c(n,\, n) \ge u_n\,c(n-1,\, n) \ge 0.$$

Proof of Claim 1. We use mathematical induction. For $n = 1$,

$$c(1,1) = v_0u_1 + p_0 \ \ge \ u_1\,c(0,1) \ge 0,$$

and

$$\begin{aligned} c(n+1,n+1) &= u_{n+1}\,c(n,n+1) + v_{n+1}\,c(n,n) - w_n c(n-1,n) \\ &\geq u_{n+1}\,c(n,n+1) + v_{n+1}\,u_n c(n-1,n) - w_n\,c(n-1,n) \quad \text{(by inductive hypothesis)} \\ &= u_{n+1}\,c(n,n+1) + p_n\,c(n-1,n) \\ &\geq u_{n+1}\,c(n,n+1), \end{aligned}$$

which proves Claim 1.

Claim 2. *For $n \geq 2$,*

$$\beta(n,\,i) \geq u_n\,\beta(n-1,i) \geq 0 \qquad (0 \leq i \leq n-1). \tag{4.18}$$

Proof of Claim 2. We use mathematical induction. If $n = 2$ and $i = 0$, this is trivial. Also,

$$\beta(2,1) = u_0\,p_1 + u_2\,p_0 = u_0\,p_1 + u_2\,\beta(1,1) \geq u_2\,\beta(1,1) \geq 0.$$

Assume that (4.18) holds. We shall prove that

$$\beta(n+1,\,i) \geq u_{n+1}\,\beta(n,i) \geq 0 \qquad (0 \leq i \leq n).$$

For,

$$\begin{aligned} \beta(n+1,i) + v_0 \cdots v_{i-1}u_i \cdots u_{n+1} &= c(n+1,i) \qquad \text{(by (4.14))} \\ &= u_{n+1}c(n,i) + v_{n+1}c(n,i-1) - w_n c(n-1,i-1) \\ &= u_{n+1}\Big(\beta(n,i) + v_0 \cdots v_{i-1}u_i \cdots u_n\Big) \\ &\quad + v_{n+1}\Big(\beta(n,i-1) + v_0 \cdots v_{i-2}u_{i-1} \cdots u_n\Big) \\ &\quad - w_n\Big(\beta(n-1,i-1) + v_0 \cdots v_{i-2}u_{i-1} \cdots u_{n-1}\Big), \end{aligned}$$

so that

$$\begin{aligned} \beta(n+1,i) &= u_{n+1}\beta(n,i) + v_{n+1}\beta(n,i-1) - w_n\beta(n-1,i-1) \\ &\quad + v_0 \cdots v_{i-2}u_{i-1} \cdots u_{n-1}\,(u_n v_{n+1} - w_n) \\ &= u_{n+1}\beta(n,i) + v_{n+1}\beta(n,i-1) - w_n\beta(n-1,i-1) + (v_0 \cdots v_{i-2}u_{i-1} \cdots u_{n-1})\,p_n \\ &\geq u_{n+1}\beta(n,i) + v_{n+1}u_n\beta(n-1,i-1) - w_n\beta(n-1,i-1) \\ &\qquad \text{(by the inductive hypothesis and Lemma 4.4.6;} \\ &\qquad \text{ observe that } i-1 \leq n-1, \text{ so (4.18) applies)} \\ &= u_{n+1}\beta(n,i) + p_n\,\beta(n-1,i-1) \\ &\geq u_{n+1}\,\beta(n,i), \end{aligned}$$

which proves Claim 2.

By Claim 2 and (4.17), we can see that $c(n,i) \geq 0$ for all $n \geq 0$ and $1 \leq i \leq n-1$. Therefore (4.16), (4.17), Claim 1 and Claim 2 imply

$$c(n,i) \;\geq\; v_0 \cdots v_{i-1}u_i \cdots u_n \qquad (n \geq 0,\; 0 \leq i \leq n+1).$$

This completes the proof. $\square$

To prove Theorem 4.4.3 we need:

Lemma 4.4.7. ([CuF3, Lemma 2.3]) *Let* $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ *be a strictly increasing weight sequence. If* W_α *is 2-hyponormal then the sequence of quotients*

$$\Theta_n := \frac{u_{n+1}}{u_{n+2}} \qquad (n \geq 0)$$

is bounded away from 0 *and from* ∞*. More precisely,*

$$1 \leq \Theta_n \leq \frac{u_1}{u_2}\left(\frac{||W_\alpha||^2}{\alpha_0\alpha_1}\right)^2 \qquad \textit{for sufficiently large } n.$$

In particular, $\{u_n\}_{n=0}^{\infty}$ *is eventually decreasing.*

We are ready for:

Proof of Theorem 4.4.3. By Theorem 4.4.2, W_α is *strictly* positively quadratically hyponormal, in the sense that all coefficients of $d_n(t)$ are *positive* for all $n \geq 0$. Note that finite rank perturbations of α affect a finite number of values of $u_n,\ v_n$ and w_n. More concretely, if α' is a perturbation of α in the weights $\{\alpha_0, \cdots, \alpha_N\}$, then $u_n,\ v_n$, w_n and p_n are invariant under α' for $n \geq N+3$. In particular, $p_n \geq 0$ for $n \geq N+3$.

Claim 1. For $n \geq 3,\ 0 \leq i \leq n+1$,

$$\begin{aligned} c(n,i) =& u_n\, c(n-1,i) + p_{n-1}\, c(n-2,i-1) + \sum_{k=4}^{n} p_{k-2}\left(\prod_{j=k}^{n} v_j\right) c(k-3,\ i-n+k-2) \\ &+ v_n \cdots v_3\, \rho_{i-n+1}, \end{aligned} \tag{4.19}$$

where

$$\rho_{i-n+1} = \begin{cases} 0 & (i < n-1) \\ u_0 p_1 & (i = n-1) \\ v_0 p_1 + v_2 p_0 & (i = n) \\ v_0 v_1 v_2 & (i = n+1) \end{cases}$$

(cf. [CuF3, Proof of Theorem 4.3]).

Proof of Claim 1. We use induction. For $n = 3,\ 0 \leq i \leq 4$,

$$\begin{aligned} c(3,i) &= u_3\, c(2,\ i) + v_3\, c(2,\ i-1) - w_2\, c(1,\ i-1) \\ &= u_3\, c(2,i) + v_3\Big(u_2\, c(1,i-1) + v_2\, c(1,\ i-2) - w_1\, c(0,\ i-2)\Big) - w_2\, c(1,i-1) \\ &= u_3\, c(2,i) + p_2\, c(1,i-1) + v_3\Big(v_2\, c(1,i-2) - w_1\, c(0,i-2)\Big) \\ &= u_3\, c(2,i) + p_2\, c(1,i-1) + v_3\, \rho_{i-2}, \end{aligned}$$

where by (4.15),

$$\rho_{i-2} = \begin{cases} 0 & (i < 2) \\ u_0 p_1 & (i = 2) \\ v_0 p_1 + v_2 p_0 & (i = 3) \\ v_0 v_1 v_2 & (i = 4). \end{cases}$$

Now,

$$\begin{aligned} c(n+1,i) &= u_{n+1}c(n,i) + v_{n+1}c(n,i-1) - w_n c(n-1,i-1) \\ &= u_{n+1}c(n,i) + v_{n+1}\Bigg(u_n c(n-1,i-1) + p_{n-1}c(n-2,i-2) \\ &\quad + \sum_{k=4}^{n} p_{k-2}\left(\prod_{j=k}^{n} v_j\right) c(k-3,i-n+k-3) + v_n \cdots v_3 \rho_{i-n}\Bigg) - w_n\, c(n-1,i-1) \\ &= u_{n+1}c(n,i) + p_n c(n-1,i-1) + v_{n+1}p_{n-1}c(n-2,i-2) \\ &\quad + v_{n+1}\sum_{k=4}^{n} p_{k-2}\left(\prod_{j=k}^{n} v_j\right) c(k-3,i-n+k-3) + v_{n+1}\cdots v_3\rho_{i-n} \\ &\qquad \text{(by inductive hypothesis)} \\ &= u_{n+1}c(n,i) + p_n c(n-1,i-1) + \sum_{k=4}^{n+1} p_{k-2}\left(\prod_{j=k}^{n+1} v_j\right) c(k-3,i-n+k-3) \\ &\quad + v_{n+1}\cdots v_3\rho_{i-n}, \end{aligned}$$

which proves Claim 1.

Write $u_n',\ v_n', w_n', p_n', \rho_n'$, and $c'(\cdot,\cdot)$ for the entities corresponding to α'. If $p_n > 0$ for every $n = 0, \cdots, N+2$, then in view of Claim 1, we can choose a small perturbation such that $p_n' > 0$ $(0 \le n \le N+2)$ and therefore $c'(n,i) > 0$ for all $n \ge 0$ and $0 \le i \le n+1$, which implies that $W_{\alpha'}$ is also positively quadratically hyponormal. If instead $p_n = 0$ for some $n = 0, \cdots, N+2$, careful inspection of (4.19) reveals that without loss of generality we may assume $p_0 = \cdots = p_{N+2} = 0$. By Theorem 4.4.2, we have that for a sufficiently small perturbation α' of α,

$$c'(n,i) > 0 \ \ (0 \le n \le N+2,\ 0 \le i \le n+1) \quad \text{and} \quad c'(n,n+1) > 0 \ \ (n \ge 0). \tag{4.20}$$

Write

$$k_n := \frac{v_n}{u_n} \qquad (n = 2, 3, \cdots).$$

Claim 2. $\{k_n\}_{n=2}^{\infty}$ is bounded.

Proof of Claim 2. Observe that

$$\begin{aligned} k_n = \frac{v_n}{u_n} &= \frac{\alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2}{\alpha_n^2 - \alpha_{n-1}^2} \\ &= \alpha_n^2 + \alpha_{n-1}^2 + \alpha_n^2\frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2} + \alpha_{n-1}^2\frac{\alpha_{n-1}^2 - \alpha_{n-2}^2}{\alpha_n^2 - \alpha_{n-1}^2}. \end{aligned}$$

Therefore if W_α is 2-hyponormal then by Lemma 4.4.7, the sequences

$$\left\{\frac{\alpha_{n+1}^2-\alpha_n^2}{\alpha_n^2-\alpha_{n-1}^2}\right\}_{n=2}^{\infty} \quad \text{and} \quad \left\{\frac{\alpha_{n-1}^2-\alpha_{n-2}^2}{\alpha_n^2-\alpha_{n-1}^2}\right\}_{n=2}^{\infty}$$

are both bounded, so that $\{k_n\}_{n=2}^{\infty}$ is bounded. This proves Claim 2.

Write $k := \sup_n k_n$. Without loss of generality we assume $k < 1$ (this is possible from the observation that $c\alpha$ induces $\{c^2 k_n\}$). Choose a sufficiently small perturbation α' of α such that if we let

$$h := \sup_{0\le \ell\le N+2;\, 0\le m\le 1} \left| \sum_{k=4}^{N+4} p'_{k-2} \left(\prod_{j=k}^{N+3} v'_j\right) c'(k-3,\, \ell) + v'_{N+3}\cdots v'_3\, \rho'_m \right| \tag{4.21}$$

then

$$c'(N+3,\, i) - \frac{1}{1-k} h \ > \ 0 \qquad (0 \le i \le N+3) \tag{4.22}$$

(this is always possible because by Theorem 4.4.2, we can choose a sufficiently small $|p'_i|$ such that

$$c'(N+3, i) > v_0\cdots v_{i-1}u_i\cdots u_{N+3} - \epsilon \quad \text{and} \quad |h| < (1-k)\big(v_0\cdots v_{i-1}u_i\cdots u_{N+3} - \epsilon\big)$$

for any small $\epsilon > 0$).

Claim 3. For $j \ge 4$ and $0 \le i \le N+j$,

$$c'(N+j,\, i) \ \ge \ u_{N+j}\cdots u_{N+4} \left(c'(N+3,\, i) - \sum_{n=1}^{j-3} k^n\, h \right). \tag{4.23}$$

Proof of Claim 3. We use induction. If $j = 4$ then by Claim 1 and (4.21),

$$\begin{aligned} c'(N+4,\, i) &= u'_{N+4}c'(N+3, i) + p'_{N+3}c'(N+2, i-1) \\ &\qquad + v'_{N+4}\sum_{k=4}^{N+4} p'_{k-2}\left(\prod_{j=k}^{N+3} v'_j\right) c'(k-3, i-N+k-6) + v'_{N+4}\cdots v'_3\rho'_{i-(N+3)} \\ &\ge u'_{N+4}c'(N+3, i) + p'_{N+3}c'(N+2, i-1) - v'_{N+4}h \\ &\ge u_{N+4}\big(c'(N+3, i) - k_{N+4}h\big) \\ &\ge u_{N+4}\big(c'(N+3, i) - k\, h\big) \end{aligned}$$

because $u'_{N+4} = u_{N+4}$, $v'_{N+4} = v_{N+4}$ and $p'_{N+3} = p_{N+3} \ge 0$. Now suppose (4.23) holds for some $j \geq 4$. By Claim 1, we have that for $j \geq 4$,

$$\begin{aligned} c'(N+j+1,i) &= u'_{N+j+1}c'(N+j,\, i) + p'_{N+j}c(N+j-1,\, i-1) \\ &\quad + \sum_{k=4}^{N+j+1} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3, i-N+k-j-3) + v'_{N+j+1}\cdots v'_3\rho'_{i-(N+j)} \\ &= u'_{N+j+1}c'(N+j,\, i) + p'_{N+j}c(N+j-1,\, i-1) \\ &\quad + \sum_{k=N+5}^{N+j+1} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3, i-N+k-j-3) \\ &\quad + \sum_{k=4}^{N+4} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3, i-N+k-j-3) + v'_{N+j+1}\cdots v'_3\rho'_{i-(N+j)}. \end{aligned}$$

Since $p'_n = p_n > 0$ for $n \geq N+3$ and $c'(n,\ell) > 0$ for $0 \leq n \leq N+j$ by the inductive hypothesis, it follows that

$$p'_{N+j}c(N+j-1,\, i-1) + \sum_{k=N+5}^{N+j+1} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3, i-N+k-j-3) \geq 0. \tag{4.24}$$

By inductive hypothesis and (4.24),

$$\begin{aligned} &c'(N+j+1,i) \\ &\geq u'_{N+j+1}c'(N+j,\, i) + \sum_{k=4}^{N+4} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3, i-N+k-j-3) + v'_{N+j+1}\cdots v'_3\rho'_{i-(N+j)} \\ &\geq u_{N+j+1}u_{N+j}\cdots u_{N+4}\left(c'(N+3,i) - \sum_{n=1}^{j-3} k^n h\right) \\ &\quad + v_{N+j+1}v_{N+j}\cdots v_{N+4}\left(\sum_{k=4}^{N+4} p'_{k-2}\left(\prod_{j=k}^{N+3} v'_j\right) c'(k-3, i-N+k-j-3) + v'_{N+3}\cdots v'_3\rho'_{i-(N+j)}\right) \\ &\geq u_{N+j+1}u_{N+j}\cdots u_{N+4}\left(c'(N+3,i) - \sum_{n=1}^{j-3} k^n h\right) - v_{N+j+1}v_{N+j}\cdots v_{N+4}\, h \\ &= u_{N+j+1}u_{N+j}\cdots u_{N+4}\left(c'(N+3,i) - \sum_{n=1}^{j-3} k^n h - k_{N+j+1}k_{N+j}\cdots k_{N+4}\, h\right) \\ &\geq u_{N+j+1}u_{N+j}\cdots u_{N+4}\left(c'(N+3,i) - \sum_{n=1}^{j-2} k^n h\right), \end{aligned}$$

which proves Claim 3.

Since $\sum_{n=1}^{j} k^n < \frac{1}{1-k}$ for every $j > 1$, it follows from Claim 3 and (4.22) that

$$c'(N+j,\, i) > 0 \quad \text{for } j \geq 4 \text{ and } 0 \leq i \leq N+j. \tag{4.25}$$

It thus follows from (4.20) and (4.25) that $c'(n,i) > 0$ for every $n \geq 0$ and $0 \leq i \leq n+1$. Therefore $W_{\alpha'}$ is also positively quadratically hyponormal. This completes the proof. $\square$

Corollary 4.4.8. *Let W_α be a weighted shift such that $\alpha_{j-1} < \alpha_j$ for some $j \geq 1$, and let T_x be the weighted shift with weight sequence*

$$\alpha_x : \alpha_0, \cdots, \alpha_{j-1}, x, \alpha_{j+1}, \cdots .$$

Then $\{x : T_x \text{ is 2-hyponormal}\}$ is a proper closed subset of $\{x : T_x \text{ is quadratically hyponormal}\}$ whenever the latter set is non-empty.

Proof. Write

$$H_2 := \{x : T_x \text{ is 2-hyponormal}\}.$$

Without loss of generality, we can assume that H_2 is non-empty, and that $j = 1$. Recall that a 2-hyponormal weighted shift with two equal weights is of the form $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$ or $\alpha_0 < \alpha_1 = \alpha_2 = \alpha_3 = \cdots$. Let $x_m := \inf H_2$. By Proposition 4.4.14 below, T_{x_m} is hyponormal. Then $x_m > \alpha_0$. By assumption, $x_m < \alpha_2$. Thus $\alpha_0, x_m, \alpha_2, \alpha_3, \cdots$ is strictly increasing. Now we apply Theorem 4.4.3 to obtain x' such that $\alpha_0 < x' < x_m$ and $T_{x'}$ is quadratically hyponormal. However $T_{x'}$ is not 2-hyponormal by the definition of x_m. The proof is complete. $\square$

The following question arises naturally:

Question. *Let α be a strictly increasing weight sequence and let $k \geq 3$. If W_α is a k-hyponormal weighted shift, does it follow that W_α is weakly k-hyponormal under a small perturbation of the weight sequence?*

Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence, let $x_i > 0$ for $1 \leq i \leq n$, and let $(x_n, \cdots x_1)\alpha : x_n, \cdots, x_1, \alpha_0, \alpha_1, \cdots$ be the augmented weight sequence. We say that $W_{(x_n,\cdots,x_1)\alpha}$ is an *extension* (or *n-step extension*) of W_α. Observe that

$$W_{(x_n,\cdots,x_1)\alpha}|_{\bigvee\{e_n,e_{n+1},\cdots\}} \cong W_\alpha.$$

The hypothesis $F \neq c\,P_{\{e_0\}}$ in Theorem 4.4.1 is essential. Indeed, there exist infinitely many one-step subnormal extension of a subnormal weighted shift whenever one such extension exists. Recall ([Cu2, Proposition 8]) that if W_α is a weighted shift whose restriction to $\bigvee\{e_1, e_2, \cdots\}$ is subnormal with associated measure μ, then W_α is subnormal if and only if

(i) $\frac{1}{t} \in L^1(\mu)$;
(ii) $\alpha_0^2 \leq \left(||\frac{1}{t}||_{L^1(\mu)}\right)^{-1}$.

Also note that there may not exist any one-step subnormal extension of the subnormal weighted shift: for example, if W_α is the Bergman shift then the corresponding Berger measure is $\mu(t) = t$, and hence $\frac{1}{t}$ is not integrable with respect to μ; therefore W_α does not admit any subnormal extension. A similar situation arises when μ has an atom at $\{0\}$.

More generally we have:

Theorem 4.4.9. (Subnormal Extensions) *Let* W_α *be a subnormal weighted shift with weights* $\alpha : \alpha_0, \alpha_1, \cdots$ *and let* μ *be the corresponding Berger measure. Then* $W_{(x_n,\cdots,x_1)\alpha}$ *is subnormal if and only if*

(i) $\frac{1}{t^n} \in L^1(\mu)$;

(ii) $x_j = \left(\frac{||\frac{1}{t^{j-1}}||_{L^1(\mu)}}{||\frac{1}{t^j}||_{L^1(\mu)}} \right)^{\frac{1}{2}}$ *for* $1 \le j \le n-1$;

(iii) $x_n \le \left(\frac{||\frac{1}{t^{n-1}}||_{L^1(\mu)}}{||\frac{1}{t^n}||_{L^1(\mu)}} \right)^{\frac{1}{2}}$.

In particular, if we put

$$S := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : W_{(x_n,\cdots,x_1)\alpha} \ \text{ is subnormal}\}$$

then either $S = \emptyset$ *or* S *is a line segment in* $\mathbb{R}^n$.

Proof. Write $W_j := W_{(x_n,\cdots,x_1)\alpha}|_{\bigvee\{e_{n-j},e_{n-j+1},\cdots\}}$ $(1 \le j \le n)$ and hence $W_n = W_{(x_n,\cdots,x_1)\alpha}$. By the argument used to establish (3.2) we have that W_1 is subnormal with associated measure ν_1 if and only if

(i) $\frac{1}{t} \in L^1(\mu)$;

(ii) $d\nu_1 = \frac{x_1^2}{t} d\mu$, or equivalently, $x_1^2 = \left(\int_0^{||W_\alpha||^2} \frac{1}{t}\, d\mu(t) \right)^{-1}$.

Inductively W_{n-1} is subnormal with associated measure ν_{n-1} if and only if

(i) W_{n-2} is subnormal;

(ii) $\frac{1}{t^{n-1}} \in L^1(\mu)$;

(iii) $d\nu_{n-1} = \frac{x_{n-1}^2}{t} d\nu_{n-2} = \cdots = \frac{x_{n-1}^2 \cdots x_1^2}{t^{n-1}} d\mu$, or equivalently, $x_{n-1}^2 = \frac{\int_0^{||W_\alpha||^2} \frac{1}{t^{n-2}}\, d\mu(t)}{\int_0^{||W_\alpha||^2} \frac{1}{t^{n-1}}\, d\mu(t)}$.

Therefore W_n is subnormal if and only if

(i) W_{n-1} is subnormal;

(ii) $\frac{1}{t^n} \in L^1(\mu)$;

(iii) $x_n^2 \le \left(\int_0^{||W_\alpha||^2} \frac{1}{t} d\nu_{n-1} \right)^{-1} = \left(\int_0^{||W_\alpha||^2} \frac{x_{n-1}^2 \cdots x_1^2}{t^n} d\mu(t) \right)^{-1} = \frac{\int_0^{||W_\alpha||^2} \frac{1}{t^{n-1}}\, d\mu(t)}{\int_0^{||W_\alpha||^2} \frac{1}{t^n}\, d\mu(t)}$. □

Corollary 4.4.10. *If* W_α *is a subnormal weighted shift with associated measure* μ, *there exists an* n*-step subnormal extension of* W_α *if and only if* $\frac{1}{t^n} \in L^1(\mu)$.

Corollary 4.4.11. *A recursively generated subnormal shift with* $\varphi_0 \ne 0$ *admits an* n*-step subnormal extension for every* $n \ge 1$.

Proof. The assumption about φ_0 implies that the zeros of $g(t)$ are positive, so that $s_0 > 0$. Thus for every $n \ge 1$, $\frac{1}{t^n}$ is integrable with respect to the corresponding Berger measure $\mu = \rho_0 \delta_{s_0} + \cdots + \rho_{r-1} \delta_{s_{r-1}}$. By Corollary 4.4.10, there exists an n-step subnormal extension. □

We need not expect that for arbitrary recursively generated shifts, 2-hyponormality and subnormality coincide as in Theorem 4.4.5. For example, if $\alpha : \sqrt{\frac{1}{2}}, \sqrt{x}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$ then by (4.12) and Theorem 4.4.9,

(i) T_x is 2-hyponormal $\Longleftrightarrow 4-\sqrt{6} \le x \le 2$;
(ii) T_x is subnormal $\Longleftrightarrow x = 2$.

A straightforward calculation shows, however, that T_x is 3-*hyponormal* if and only if $x = 2$; for,

$$A(0;3) := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}x & \frac{3}{2}x \\ \frac{1}{2} & \frac{1}{2}x & \frac{3}{2}x & 5x \\ \frac{1}{2}x & \frac{3}{2}x & 5x & 17x \\ \frac{3}{2}x & 5x & 17x & 58x \end{bmatrix} \ge 0 \iff x = 2.$$

This behavior is typical of general recursively generated weighted shifts: we show in [CJL] that subnormality is equivalent to k-hyponormality for some $k \ge 2$.

Next, we will show that canonical rank-one perturbations of k-hyponormal weighted shifts which preserve k-hyponormality form a convex set. To see this we need an auxiliary result.

Lemma 4.4.12. *Let $I = \{1, \cdots, n\} \times \{1, \cdots, n\}$ and let J be a symmetric subset of I. Let $A = (a_{ij}) \in M_n(\mathbb{C})$ and let $C = (c_{ij}) \in M_n(\mathbb{C})$ be given by*

$$c_{ij} = \begin{cases} c\, a_{ij} & \text{if } (i,j) \in J \\ a_{ij} & \text{if } (i,j) \in I \setminus J \end{cases} \qquad (c > 0).$$

If A and C are positive semidefinite then $B = (b_{ij}) \in M_n(\mathbb{C})$ defined by

$$b_{ij} = \begin{cases} b\, a_{ij} & \text{if } (i,j) \in J \\ a_{ij} & \text{if } (i,j) \in I \setminus J \end{cases} \qquad (b \in [1,c] \text{ or } [c,1])$$

is also positive semidefinite.

Proof. Without loss of generality we may assume $c > 1$. If $b = 1$ or $b = c$ the assertion is trivial. Thus we assume $1 < b < c$. The result is now a consequence of the following observation. If $[D]_{(i,j)}$ denotes the (i,j)-entry of the matrix D then

$$\begin{aligned} \left[\frac{c-b}{c-1}\left(A + \frac{b-1}{c-b}C\right)\right]_{(i,j)} &= \begin{cases} \frac{c-b}{c-1}\left(1 + \frac{b-1}{c-b}c\right) a_{ij} & \text{if } (i,j) \in J \\ \frac{c-b}{c-1}\left(1 + \frac{b-1}{c-b}\right) a_{ij} & \text{if } (i,j) \in I \setminus J \end{cases} \\ &= \begin{cases} b\, a_{ij} & \text{if } (i,j) \in J \\ a_{ij} & \text{if } (i,j) \in I \setminus J \end{cases} \\ &= [B]_{(i,j)}, \end{aligned}$$

which is positive semidefinite because positive semidefinite matrices in $M_n(\mathbb{C})$ form a cone. □

An immediate consequence of Lemma 4.4.12 is that positivity of a matrix forms a convex set with respect to a fixed diagonal location; i.e., if

$$A_x = \begin{bmatrix} * & * & * \\ * & x & * \\ * & * & * \end{bmatrix}$$

then $\{x : A_x \text{ is positive semidefinite}\}$ is convex.

We now have:

Theorem 4.4.13. *Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence, let $k \geq 1$, and let $j \geq 0$. Define $\alpha^{(j)}(x) : \alpha_0, \cdots, \alpha_{j-1}, x, \alpha_{j+1}, \cdots$. Assume W_α is k-hyponormal and define*

$$\Omega_\alpha^{k,j} := \{x : \ W_{\alpha^{(j)}(x)} \text{ is } k\text{-hyponormal}\}.$$

Then $\Omega_\alpha^{k,j}$ is a closed interval.

Proof. Suppose $x_1, x_2 \in \Omega_\alpha^{k,j}$ with $x_1 < x_2$. Then by ([**?**]), the $(k+1) \times (k+1)$ Hankel matrix

$$A_{x_i}(n;k) := \begin{bmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{bmatrix} \qquad (n \geq 0;\ i = 1, 2)$$

is positive, where A_{x_i} corresponds to $\alpha^{(j)}(x_i)$. We must show that $tx_1 + (1-t)x_2 \in \Omega_\alpha^{k,j}$ $(0 < t < 1)$, i.e.,

$$A_{tx_1+(1-t)x_2}(n;k) \ \geq \ 0 \quad (n \geq 0,\ 0 < t < 1).$$

Observe that it suffices to establish the positivity of the $2k$ Hankel matrices corresponding to $\alpha^{(j)}(tx_1 + (1-t)x_2)$ such that $tx_1 + (1-t)x_2$ appears as a factor in at least one entry but not in every entry. A moment's thought reveals that without loss of generality we may assume $j = 2k$. Observe that

$$A_{z_1}(n;k) - A_{z_2}(n;k) = (z_1^2 - z_2^2)\, H(n;k)$$

for some Hankel matrix $H(n;k)$. For notational convenience, we abbreviate $A_z(n;k)$ as A_z. Then

$$A_{tx_1+(1-t)x_2} = \begin{cases} t^2 A_{x_1} + (1-t)^2 A_{x_2} + 2t(1-t) A_{\sqrt{x_1 x_2}} & \text{for } 0 \leq n \leq 2k \\ \left(t + (1-t)\frac{x_2}{x_1}\right)^2 A_{x_1} & \text{for } n \geq 2k+1. \end{cases}$$

Since $A_{x_1} \geq 0$, $A_{x_2} \geq 0$ and $A_{\sqrt{x_1 x_2}}$ have the form described by Lemma 4.4.12 and since $x_1 < \sqrt{x_1 x_2} < x_2$ it follows from Lemma 4.4.12 that $A_{\sqrt{x_1 x_2}} \geq 0$. Thus evidently, $A_{tx_1+(1-t)x_2} \geq 0$, and therefore $tx_1 + (1-t)x_2 \in \Omega_\alpha^{k,j}$. This shows that $\Omega_\alpha^{k,j}$ is an interval. The closedness of the interval follows from Proposition 4.4.14 below. ☐

In [CP1] and [CP2], it was shown that there exists a non-subnormal polynomially hyponormal operator. Also in [McCP], it was shown that there exists a non-subnormal polynomially hyponormal operator if and only if there exists one which is also a weighted shift. However, no concrete weighted shift has yet been found. As a strategy for finding such a shift, we would like to suggest the following:

Question *Does it follow that the polynomial hyponormality of the weighted shift is stable under small perturbations of the weight sequence?*

If the answer to the above question were affirmative then we would easily find a polynomially hyponormal non-subnormal (even non-2-hyponormal) weighted shift; for example, if

$$\alpha : 1, \sqrt{x}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^{\wedge}$$

and T_x is the weighted shift associated with α, then by Theorem 4.4.5, T_x is subnormal $\Leftrightarrow x = 2$, whereas T_x is polynomially hyponormal $\Leftrightarrow 2 - \delta_1 < x < 2 + \delta_2$ for some $\delta_1, \delta_2 > 0$ provided the answer to the above question is yes; therefore for sufficiently small $\epsilon > 0$,

$$\alpha_\epsilon : 1, \sqrt{2+\epsilon}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^{\wedge}$$

would induce a non-2-hyponormal polynomially hyponormal weighted shift.

The answer to the above question for weak k-hyponormality is negative. In fact we have:

Proposition 4.4.14. (i) *The set of k-hyponormal operators is sot-closed.*
(ii) *The set of weakly k-hyponormal operators is sot-closed.*

Proof. Suppose $T_\eta \in \mathcal{L}(\mathcal{H})$ and $T_\eta \to T$ in *sot*. Then, by the Uniform Boundedness Principle, $\{||T_\eta||\}_\eta$ is bounded. Thus $T_\eta^{*i} T_\eta^j \to T^{*i} T^j$ in *sot* for every i, j, so that $M_k(T_\eta) \to M_k(T)$ in *sot* (where $M_k(T)$ is as in (4.5). (i) In this case $M_k(T_\eta) \geq 0$ for all η, so $M_k(T) \geq 0$, i.e., T is k-hyponormal.

(ii) Here, $M_k(T_\eta)$ is weakly positive for all η. By (4.6), $M_k(T)$ is also weakly positive, i.e., T is weakly k-hyponormal. □

4.5 The Extensions

In [Sta3], J. Stampfli showed that given $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}$ with $0 < a < b < c$, there always exists a subnormal completion of α, but that for $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}$ $(a < b < c < d)$ such a subnormal completion may not exist.

There are instances where k-hyponormality implies subnormality for weighted shifts. For example, in [CuF3], it was shown that if $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ $(a < b < c)$ then $W_{\alpha(x)}$ is 2-hyponormal if and only if it is subnormal: more concretely, $W_{\alpha(x)}$ is 2-hyponormal if and only if

$$\sqrt{x} \le H_2(\sqrt{a}, \sqrt{b}, \sqrt{c}) := \sqrt{\frac{ab(c-b)}{(b-a)^2 + b(c-b)}},$$

in which case $W_{\alpha(x)}$ is subnormal. In this section we extend the above result to weight sequences of the form $\alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$ with $0 < \alpha_0 < \cdots < \alpha_k$. We here show:

Extensions of Recursively Generated Weighted Shifts.
If $\alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$ then

$$W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } ([\frac{k+1}{2}]+1)\text{-hyponormal} & (n=1) \\ W_\alpha \text{ is } ([\frac{k+1}{2}]+2)\text{-hyponormal} & (n>1). \end{cases}$$

In particular, the above theorem shows that the subnormality of an extension of the recursive shift is independent of its length if the length is bigger than 1.

Given an initial segment of weights

$$\alpha : \ \alpha_0, \cdots, \alpha_{2k} \quad (k \ge 0),$$

suppose $\hat{\alpha} \equiv (\alpha_0, \cdots, \alpha_{2k})^\wedge$, i.e., $\hat{\alpha}$ is recursively generated by α. Write

$$\mathbf{v}_n := \begin{bmatrix} \gamma_n \\ \vdots \\ \gamma_{n+k} \end{bmatrix} \qquad (0 \le n \le k+1).$$

Then $\{\mathbf{v}_0, \cdots, \mathbf{v}_{k+1}\}$ is linearly dependent in $\mathbb{R}^{k+1}$. Now the *rank* of α is defined by the smallest integer i $(1 \le i \le k+1)$ such that $\mathbf{v}_i$ is a linear combination of $\mathbf{v}_0, \cdots, \mathbf{v}_{i-1}$. Since $\{\mathbf{v}_0, \cdots, \mathbf{v}_{i-1}\}$ is linearly independent, there exists a unique i-tuple $\varphi \equiv (\varphi_0, \cdots, \varphi_{i-1}) \in \mathbb{R}^i$ such that $\mathbf{v}_i = \varphi_0 \mathbf{v}_0 + \cdots + \varphi_{i-1}\mathbf{v}_{i-1}$, or equivalently,

$$\gamma_j = \varphi_{i-1}\gamma_{j-1} + \cdots + \varphi_0 \gamma_{j-i} \quad (i \le j \le k+i),$$

which says that $(\alpha_0, \cdots, \alpha_{k+i})$ is recursively generated by $(\alpha_0, \cdots, \alpha_i)$. In this case, W_α is said to be *i-recursive* (cf. [CuF3, Definition 5.14]).

We begin with:

Lemma 4.5.1. ([CuF2, Propositions 2.3, 2.6, and 2.7]) *Let* $A, B \in M_n(\mathbb{C})$, $\tilde{A}, \tilde{B} \in M_{n+1}(\mathbb{C})$ $(n \geq 1)$ *be such that*

$$\tilde{A} = \begin{bmatrix} A & * \\ & * \end{bmatrix} \quad \text{and} \quad \tilde{B} = \begin{bmatrix} * & * \\ * & B \end{bmatrix}.$$

Then we have:

(i) *If* $A \geq 0$ *and if* $\tilde{A}$ *is a flat extension of* A *(i.e., rank*$(\tilde{A})$ = *rank*(A)*) then* $\tilde{A} \geq 0$*;*
(ii) *If* $A \geq 0$ *and* $\tilde{A} \geq 0$ *then det*$(A) = 0$ *implies det*$(\tilde{A}) = 0$*;*
(iii) *If* $B \geq 0$ *and* $\tilde{B} \geq 0$ *then det*$(B) = 0$ *implies det*$(\tilde{B}) = 0$.

Lemma 4.5.2. *If* $\alpha \equiv (\alpha_0, \cdots, \alpha_k)^\wedge$ *then*

$$W_\alpha \text{ is subnormal} \iff W_\alpha \text{ is } ([\frac{k}{2}] + 1)\text{-hyponormal.} \tag{4.26}$$

In the cases where W_α *is subnormal and* $i :=$ *rank*(α)*, then we have that* $\alpha = (\alpha_0, \cdots, \alpha_{2i-2})^\wedge$.

Proof. We only need to establish the sufficiency condition in (4.26). Let $i := \operatorname{rank}(\alpha)$. Since W_α is i-recursive, [CuF3, Proposition 5.15] implies the subnormality of W_α follows after we verify that $A(0, i-1) \geq 0$ and $A(1, i-1) \geq 0$. Now observe that $i - 1 \leq [\frac{k}{2}] + 1$ and

$$A(j, [\frac{k}{2}] + 1) = \begin{bmatrix} A(j, i-1) & * \\ * & * \end{bmatrix} \quad (j = 0, 1),$$

so the positivity of $A(0, i-1)$ and $A(1, i-1)$ is a consequence of the positivity of the $([\frac{k}{2}] + 1)$-hyponormality of W_α. For the second assertion, observe that if $i := \operatorname{rank}(\alpha)$ then $\det A(n, i) = 0$ for all $n \geq 0$. By assumption $A(n, i+1) \geq 0$, so by Lemma 4.5.1 (ii) we have $\det A(n, i+1) = 0$, which says that $(\alpha_0, \cdots, \alpha_{2i-1}) \subset (\alpha_0, \cdots, \alpha_{2i-2})^\wedge$. By iteration we obtain $(\alpha_0, \cdots, \alpha_k) \subset (\alpha_0, \cdots, \alpha_{2i-2})^\wedge$, and therefore $(\alpha_0, \cdots, \alpha_k)^\wedge = (\alpha_0, \cdots, \alpha_{2i-2})^\wedge$. This proves the lemma. □

In what follows, and for notational convenience, we shall set $x_{-j} := \alpha_j$ $(0 \leq j \leq k)$.

Theorem 4.5.3. (Subnormality Criterion) *If* $\alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$ *then*

$$W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } ([\frac{k+1}{2}] + 1)\text{-hyponormal} & (n = 1) \\ W_\alpha \text{ is } ([\frac{k+1}{2}] + 2)\text{-hyponormal} & (n > 1). \end{cases} \tag{4.27}$$

Furthermore, in the cases where the above equivalence holds, if $\operatorname{rank}(\alpha_0, \cdots, \alpha_k) = i$ *then*

$$W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } i\text{-hyponormal} & (n = 1) \\ W_\alpha \text{ is } (i+1)\text{-hyponormal} & (n > 1). \end{cases} \tag{4.28}$$

In fact,

$$\begin{cases} x_1 = H_i(\alpha_0, \cdots, \alpha_{2i-2}) \\ \quad\cdots\cdots\cdots \\ x_{n-1} = H_i(x_{n-2}, \cdots, \cdots, \alpha_{2i-n}) \\ x_n \le H_i(x_{n-1}, \cdots, \alpha_{2i-n-1}), \end{cases}$$

where H_i is the modulus of i-hyponormality (cf. [CuF3, Proposition 3.4 and (3.4)]*), i.e.,*

$$H_i(\alpha) := \sup\{x > 0 : \ W_{x\alpha} \text{ is } i\text{-hyponormal}\}.$$

Therefore, $W_\alpha = W_{x_n(x_{n-1},\cdots,\alpha_{2i-n-1})^\wedge}$.

Proof. Consider the $(k+1) \times (l+1)$ "Hankel" matrix $A(n; k, l)$ by

$$A(n; k, l) := \begin{bmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+l} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+1+l} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+k+l} \end{bmatrix} \quad (n \ge 0).$$

Case 1 ($\alpha : x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$): Let $\hat{A}(n; k, l)$ and $A(n; k, l)$ denote the Hankel matrices corresponding to the weight sequences $(\alpha_0, \cdots, \alpha_k)^\wedge$ and α, respectively. Suppose W_α is $([\frac{k+1}{2}] + 1)$-hyponormal. Then by Lemma 4.5.2, $W_{(\alpha_0,\cdots,\alpha_k)^\wedge}$ is subnormal. Observe that

$$A(n+1; m, m) = x_1^2 \, \hat{A}(n; m, m) \quad \text{for all } n \ge 0 \text{ and all } m \ge 0.$$

Thus it suffices to show that $A(0; m, m) \ge 0$ for all $m \ge [\frac{k+1}{2}] + 2$. Also note that if $\tilde{B}$ denotes the $(k-1) \times k$ matrix obtained by eliminating the first row of a $k \times k$ matrix B then

$$\tilde{A}(0; m, m) = x_1^2 \, \hat{A}(0; m-1, m) \quad \text{for all } m \ge [\frac{k+1}{2}] + 2.$$

Therefore for every $m \ge [\frac{k+1}{2}] + 2$, $A(0; m, m)$ is a flat extension of $A(0; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1)$. This implies $A(0; m, m) \ge 0$ for all $m \ge [\frac{k+1}{2}] + 2$ and therefore W_α is subnormal.

Case 2 ($\alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge)$): As in Case 1, let $\hat{A}(n; k, l)$ and $A(n; k, l)$ denote the Hankel matrices corresponding to the weight sequences $(\alpha_0, \cdots, \alpha_k)^\wedge$ and α, respectively. Observe that $\det \hat{A}(n; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = 0$ for all $n \ge 0$. Suppose W_α is $([\frac{k+1}{2}] + 2)$-hyponormal. Observe that

$$A(n+1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = x_1^2 \cdots x_n^2 \hat{A}(1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1),$$

so that

$$\det A(n+1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = 0. \tag{4.29}$$

Also observe

$$A(n-1;[\frac{k+1}{2}]+2,[\frac{k+1}{2}]+2)=\begin{bmatrix} x_2^2\cdots x_n^2 & * \\ * & A(n+1;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1)\end{bmatrix}.$$

Since W_α is $([\frac{k+1}{2}]+1)$-hyponormal it follows from Lemma 4.5.1 (iii) and (4.29) that $\det A(n-1;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1)=0$. Note that

$$A(n-1;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1)=x_1^2\cdots x_n^2\begin{bmatrix} \frac{1}{x_1^2} & \hat\gamma_0 & \cdots & \hat\gamma_{[\frac{k+1}{2}]+1} \\ \hat\gamma_0 & \hat\gamma_1 & \cdots & \hat\gamma_{[\frac{k+1}{2}]+2} \\ \vdots & \vdots & & \vdots \\ \hat\gamma_{[\frac{k+1}{2}]+1} & \hat\gamma_{[\frac{k+1}{2}]+2} & \cdots & \hat\gamma_{2[\frac{k+1}{2}]+2}\end{bmatrix},$$

where $\hat\gamma_j$ denotes the moments corresponding to the weight sequence $(\alpha_0,\cdots,\alpha_k)^\wedge$. Therefore x_1 is determined uniquely by $\{\alpha_0,\cdots,\alpha_k\}$ such that $(x_1,\alpha_0,\cdots,\alpha_{k-1})^\wedge = x_1,(\alpha_0,\cdots,\alpha_k)^\wedge$: more precisely, if $i:=\text{rank}\,(\alpha)$ and $\varphi_0,\cdots,\varphi_{i-1}$ denote the coefficients of recursion in $(\alpha_0,\cdots,\alpha_k)^\wedge$ then

$$x_1=H_i[(\alpha_0,\cdots,\alpha_k)^\wedge]=\left[\frac{\varphi_0}{\hat\gamma_{i-1}-\varphi_{i-1}\hat\gamma_{i-2}-\cdots-\varphi_1\hat\gamma_0}\right]^{\frac{1}{2}}$$

(cf. [CuF3, (3.4)]). Continuing this process we can see that $x_1,\cdots,x_{n-1}$ are determined uniquely by a telescoping method such that

$$(x_{n-1},\cdots,x_{n-1-k})^\wedge = x_{n-1},\cdots,x_1,(\alpha_0,\cdots,\alpha_k)^\wedge$$

and $W_{(x_{n-1},\cdots,x_{n-1-k})^\wedge}$ is subnormal. Therefore, after $(n-1)$ steps, Case 2 reduces to Case 1. This proves the first assertion. For the second assertion, note that if $\text{rank}(\alpha_0,\cdots,\alpha_k)=i$ then

$$\det\hat A(n;i,i)=0.$$

Now applying the above argument with i in place of $[\frac{k+1}{2}]+1$ gives that $x_1,\cdots,x_{n-1}$ are determined uniquely by $\alpha_0,\cdots,\alpha_{2i-2}$ such that $W_{(x_{n-1},\cdots,x_{n-2i-1})^\wedge}$ is subnormal. Thus the second assertion immediately follows. Finally, observe that the preceding argument also establish the remaining assertions. $\square$

Remark 4.5.4. (a) From Theorem 4.5.3 we note that the subnormality of an extension of a recursive shift is independent of its length if the length is bigger than 1.

(b) In Theorem 4.5.3, "$[\frac{k+1}{2}]$" can not be relaxed to "$[\frac{k}{2}]$". For example consider the following weight sequences:

(i) $\alpha:\sqrt{\frac{1}{2}},(\sqrt{\frac{3}{2}},\sqrt{3},\sqrt{\frac{10}{3}},\sqrt{\frac{17}{5}})^\wedge$ with $\varphi_0=0$;

(ii) $\alpha':\sqrt{\frac{1}{2}},\sqrt{\frac{3}{2}},(\sqrt{3},\sqrt{\frac{10}{3}},\sqrt{\frac{17}{5}})^\wedge$.

Observe that α equals α'. Then a straightforward calculation shows that W_α (and hence $W_{\alpha'}$) is 2-hyponormal but not 3-hyponormal (and hence, not subnormal). Note that $k = 3$ and $n = 1$ in (i) and $k = 2$ and $n = 2$ in (ii).

(c) The second assertion of Theorem 4.5.3 does *not* imply that if $\operatorname{rank}(\alpha_0, \cdots, \alpha_k) = i$ then (4.28) holds in general. Theorem 4.5.3 says only that when W_α is $([\frac{k+1}{2}]+1)$-hyponormal ($n = 1$), i-hyponormality and subnormality coincide, and that when W_α is $([\frac{k+1}{2}]+2)$-hyponormal ($n > 1$), $(i+1)$-hyponormality and subnormality coincide. For example consider the weight sequence

$$\hat{\alpha} \equiv (\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}, 2)^\wedge \quad \text{with } \varphi_0 = 0 \text{ (here } \varphi_1 = 0 \text{ also).}$$

Since $(\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}) \subset (\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}})^\wedge$, we can see that $\operatorname{rank}(\alpha) = 2$. Put

$$\beta \equiv 1, (\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}, 2)^\wedge.$$

If (4.28) held true without assuming (4.27), then 2-hyponormality would imply subnormality for W_β. However, a straightforward calculation shows that W_β is 2-hyponormal but not 3-hyponormal (and hence not subnormal): in fact, $\det A(n, 2) = 0$ for all $n \geq 0$ except for $n = 2$ and $\det A(2, 2) = 160 > 0$, while since

$$\varphi_3 = -\frac{\alpha_3^2\alpha_4^2(\alpha_5^2 - \alpha_4^2)}{\alpha_4^2 - \alpha_3^2} = -102 \quad \text{and} \quad \varphi_4 = \frac{\alpha_4^2(\alpha_5^2 - \alpha_3^2)}{\alpha_4^2 - \alpha_3^2} = 34$$

(so that $\alpha_6 = \sqrt{\varphi_4 - \frac{\varphi_3}{\alpha_5^2}} = \sqrt{\frac{17}{2}}$), we have that

$$\det A(1,3) = \det \begin{bmatrix} 1 & 2 & 6 & 20 \\ 2 & 6 & 20 & 68 \\ 6 & 20 & 68 & 272 \\ 20 & 68 & 272 & 2312 \end{bmatrix} = -3200 < 0.$$

(d) On the other hand, Theorem 4.5.3 does show that if $\alpha \equiv (\alpha_0, \cdots, \alpha_k)$ is such that $\operatorname{rank}(\alpha) = i$ and $W_{\hat{\alpha}}$ is subnormal with associated Berger measure μ, then $W_{\hat{\alpha}}$ has an n-step $(i+1)$-hyponormal extension $W_{x_n, \cdots, x_1, \hat{\alpha}}$ ($n \geq 2$) if and only if $\frac{1}{t^n} \in L^1(\mu)$,

$$x_{j+1} = \left[\frac{\varphi_0}{\gamma_{i-1}^{(j)} - \varphi_{i-1}\gamma_{i-2}^{(j)} - \cdots - \varphi_1\gamma_0^{(j)}}\right]^{\frac{1}{2}} \quad (0 \leq j \leq n-2),$$

and

$$x_n \leq \left[\frac{\varphi_0}{\gamma_{i-1}^{(n-1)} - \varphi_{i-1}\gamma_{i-2}^{(n-1)} - \cdots - \varphi_1\gamma_0^{(n-1)}}\right]^{\frac{1}{2}},$$

where $\varphi_0, \cdots, \varphi_{i-1}$ denote the coefficients of recursion in $(\alpha_0, \cdots, \alpha_{2i-2})^\wedge$ and $\gamma_m^{(j)}$ $(0 \leq m \leq i-1)$ are the moments corresponding to the weight sequence

$$(x_j, \cdots, x_1, \alpha_0, \cdots, \alpha_{k-j})^\wedge \quad \text{with } \gamma_m^{(0)} = \gamma_m.$$

We now observe that the determination of k-hyponormality and subnormality for canonical perturbations of recursive shifts falls within the scope of the theory of extensions.

Corollary 4.5.5. *Let* $\alpha \equiv \{\alpha_n\}_{n=0}^\infty = (\alpha_0, \cdots, \alpha_k)^\wedge$. *If* $W_{\alpha'}$ *is a perturbation of* W_α *at the* j*-th weight then*

$$W_{\alpha'} \textit{ is subnormal } \Longleftrightarrow \begin{cases} W_{\alpha'} \textit{ is } ([\frac{k+1}{2}]+1)\textit{-hyponormal} & (j=0) \\ W_{\alpha'} \textit{ is } ([\frac{k+1}{2}]+2)\textit{-hyponormal} & (j \geq 1) \end{cases}.$$

Proof. Observe that if $j=0$ then $\alpha' = x, (\alpha_1, \cdots, \alpha_{k+1})^\wedge$ and if instead $j \geq 1$ then $\alpha' = \alpha_0, \cdots, \alpha_{j-1}, x, (\alpha_{j+1}, \cdots, \alpha_{j+k+1})^\wedge$. Thus the result immediately follows from Theorem 4.5.3. $\square$

In Corollary 4.5.5, we showed that if $\alpha(x)$ is a canonical rank-one perturbation of a recursive weight sequence then subnormality and k-hyponormality for the corresponding shift coincide. We now consider a converse - an "extremality" problem: *Let* $\alpha(x)$ *be a canonical rank-one perturbation of a weight sequence* α. *If there exists* $k \geq 1$ *such that* $(k+1)$*-hyponormality and* k*-hyponormality for the corresponding shift* $W_{\alpha(x)}$ *coincide, does it follow that* $\alpha(x)$ *is recursively generated?*

In [CuF3], the following extremality criterion was established.

Lemma 4.5.6. (Extremality Criterion) [CuF3, Theorem 5.12, Proposition 5.13] *Let* α *be a weight sequence and let* $k \geq 1$.

(i) *If* W_α *is* k*-extremal (i.e.,* $\det A(j,k) = 0$ *for all* $j \geq 0$*) then* W_α *is recursive subnormal.*

(ii) *If* W_α *is* k*-hyponormal and if* $\det A(i_0, j_0) = 0$ *for some* $i_0 \geq 0$ *and some* $j_0 < k$ *then* W_α *is recursive subnormal.*

In particular, Lemma 4.5.6 (ii) shows that if W_α is subnormal and if $\det A(i_0, j_0) = 0$ for some $i \geq 0$ and some $j \geq 0$ then W_α is recursive subnormal.

We now answer the above question affirmatively.

Theorem 4.5.7. *Let* $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ *be a weight sequence and let* $\alpha_j(x)$ *be a canonical perturbation of* α *in the* j*-th weight. Write*

$$\mathfrak{H}_k := \{x \in \mathbb{R}^+ : \ W_{\alpha_j(x)} \textit{ is } k\textit{-hyponormal}\}.$$

If $\mathfrak{H}_k = \mathfrak{H}_{k+1}$ *for some* $k \geq 1$ *and* $x \in \mathfrak{H}_k$, *then* $\alpha_j(x)$ *is recursively generated, i.e.,* $W_{\alpha_j(x)}$ *is recursive subnormal.*

Proof. Suppose $\mathfrak{H}_k = \mathfrak{H}_{k+1}$ and let $H_k := \sup_x \mathfrak{H}_k$. To avoid triviality we assume $\alpha_{j-1} < x < \alpha_{j+1}$.

Case 1 ($j=0$): In this case, clearly H_k^2 is the nonzero root of the equation $\det A(0,k) = 0$ and for $x \in (0, H_k]$, $W_{\alpha_0(x)}$ is k-hyponormal. By assumption $H_k =$

H_{k+1}, so $W_{\alpha_0(H_{k+1})}$ is $(k+1)$-hyponormal. The result now follows from Lemma 4.5.6 (ii).

Case 2 ($j \geq 1$): Let $A_x(n,k)$ denote the Hankel matrix corresponding to $\alpha_j(x)$. Since $W_{\alpha_j(x)}$ is $(k+1)$-hyponormal for $x \in \mathfrak{H}_k$, we have that $A_x(n,k+1) \geq 0$ for all $n \geq 0$ and all $x \in \mathfrak{H}_k$. Observe that if $n \geq j+1$ then

$$A_x(n,k) = \alpha_0^2 \cdots \alpha_{j-1}^2\, x^2 \begin{bmatrix} \tilde{\gamma}_{n-j-1} & \cdots & \tilde{\gamma}_{n-j-1+k} \\ \vdots & & \vdots \\ \tilde{\gamma}_{n-j-1+k} & \cdots & \tilde{\gamma}_{n-j-1+2k} \end{bmatrix},$$

where $\tilde{\gamma}_*$ denotes the moments corresponding to the subsequence $\alpha_{j+1}, \alpha_{j+2}, \cdots$. Therefore for $n \geq j+1$, the positivity of $A_x(n,k)$ is independent of the values of $x > 0$. This gives

$$W_{\alpha_j(x)} \text{ is } k\text{-hyponormal} \Longleftrightarrow A_x(n,k) \geq 0 \text{ for all } n \leq j.$$

Write

$$\mathfrak{H}_k^{(i)} := \Big\{ x : \ \det A_x(i,k) \geq 0 \ \text{ and } \ \alpha_{j-1} < x < \alpha_{j+1} \Big\} \quad (0 \leq i \leq j)$$

and

$$H_k^{(i)} = \sup_x \mathfrak{H}_k^{(i)} \quad (0 \leq i \leq j).$$

Since $\det A_x(i,k)$ is a polynomial in x we have $\det A_{H_k^{(i)}}(i,k) = 0$. Observe that

$$\cap_{i=0}^{j} \mathfrak{H}_k^{(i)} = \mathfrak{H}_k \quad \text{and} \quad \sup_{0 \leq i \leq j} H_k^{(i)} = H_k.$$

Since by [CuL2, Theorem 2.11], $\mathfrak{H}_k$ is a closed interval, it follows that $H_k \in \mathfrak{H}_k$. Say, $H_k = H_k^{(p)}$ for some $0 \leq p \leq j$. Then $\det A_{H_k^{(p)}}(p,k) = 0$ and $W_{\alpha(H_k^{(p)})}$ is $(k+1)$-hyponormal. Therefore it follows from Lemma 4.1 (ii) that W_α is recursive subnormal. This completes the proof. □

We conclude this section with two corollaries of independent interest.

Corollary 4.5.8. *With the notations in Theorem 4.5.7, if $j \geq 1$ and $\mathfrak{H}_k = \mathfrak{H}_{k+1}$ for some k, then $\mathfrak{H}_k$ is a singleton set.*

Proof. By [CuL2, Theorem 2.2],

$$\mathfrak{H}_\infty := \{ x \in \mathbb{R}^+ : W_{\alpha_j(x)} \text{ is subnormal} \}$$

is a singleton set. By Theorem 4.5.7, we have that $\mathfrak{H}_k = \mathfrak{H}_\infty$. □

Corollary 4.5.9. *If W_α is a nonrecursive shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ and if $\alpha(x)$ is a canonical rank-one perturbation of α, then for every $k \geq 1$ there always exists a gap between k-hyponormality and $(k+1)$-hyponormality for $W_{\alpha(x)}$. More concretely, if we let*

$$\mathfrak{H}_k := \{x : W_{\alpha(x)} \text{ is } k\text{-hyponormal}\}$$

then $\{\mathfrak{H}_k\}_{k=1}^\infty$ is a strictly decreasing nested sequence of closed intervals in $(0,\infty)$ except when the perturbation occurs in the first weight. In that case, the intervals are of the form $(0, H_k]$.

Proof. Straightforward from Theorem 4.5.7. □

We now illustrate our result with some examples. Consider

$$\alpha(y,x) : \sqrt{y}, \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge \quad (a < b < c).$$

Without loss of generality, we assume $a = 1$. Observe that

$$H_2(1, \sqrt{b}, \sqrt{c}) = \sqrt{\frac{bc - b^2}{1 + bc - 2b}} \quad \text{and} \quad \left(H_2(\sqrt{x}, 1, \sqrt{b})\right)^2 = \frac{x(b-1)}{(x-1)^2 + (b-1)} := f(x).$$

According to Theorem **??**, $W_{\alpha(y,x)}$ is 2-hyponormal if and only if $0 < x \leq \sqrt{\frac{bc-b^2}{1+bc-2b}}$ and $0 < y \leq f(x)$. To completely describe the region

$$\mathcal{R} := \{(x,y) : \ W_{\alpha(y,x)} \text{ is 2-hyponormal}\},$$

we study the graph of f. Observe that

$$f'(x) = \frac{(b-1)(b-x^2)}{(b-2x+x^2)^2} > 0 \quad \text{and} \quad f''(x) = \frac{2(b-1)(2b-3bx+x^3)}{(b-2x+x^2)^3}.$$

Note that $b - 2x + x^2 = (b-1) + (1-x)^2 > 0$ and $f'(\sqrt{b}) = 0$. To consider the sign of f'', we let $g(x) := 2b - 3bx + x^3$. Then $g'(\sqrt{b}) = 0$, $g(1) = -b + 1 < 0$, and $g''(x) > 0$ $(x > 0)$. Hence there exists $x_0 \in (0,1)$ such that $f''(x_0) = 0$, $f''(x) > 0$ on $0 < x < x_0$, and $f''(x) < 0$ on $x_0 < x \leq 1$. We investigate which of the two values x_0 or $\widetilde{H} := H_2(1, \sqrt{b}, \sqrt{c})^2$ is bigger. By a simple calculation, we have

$$g(\widetilde{H}) = \frac{(-1+b)b \cdot g_1(b,c)}{(1-2b+bc)^3},$$

where

$$g_1(b,c) = -(2 - 10b + 17b^2 - 11b^3 + b^4 + 3bc - 9b^2c + 9b^3c - 3b^3c^2 + b^2c^3).$$

For notational convenience we let $b := 1 + h$, $c := 1 + h + k$. Then

$$g_1(b,c) = 2h^5 + (3h^3 + 3h^4)k + (-1 - 2h - h^2)k^3.$$

If h is sufficiently small (i.e., b is sufficiently close to 1), then $g_1 < 0$, i.e., $\widetilde{H} > x_0$. If k is sufficiently small (i.e., b is sufficiently close to c), then $g_1 > 0$, i.e., $\widetilde{H} < x_0$. Thus, if $\widetilde{H} \leq x_0$, then f is convex downward on $x \leq \widetilde{H}$. If $\widetilde{H} > x_0$, then $(x_0, f(x_0))$ is an inflection point. Thus, f is convex downward on $0 < x < x_0$ and convex upward on $x_0 < x \leq \widetilde{H}$. Moreover, $W_{\alpha(y,x)}$ is 2-hyponormal if and only if $(x, y) \in \{(x, y) | 0 \leq y \leq f(x),\ 0 < x \leq \widetilde{H}\}$, and $W_{\alpha(y,x)}$ is k-hyponormal ($k \geq 3$) if and only if $(x, y) \in \{(\widetilde{H}, y) | 0 \leq y \leq f(x)\}$.

Example 4.5.10. ($b = 2$, $c = 3$)

$$f(x) = \frac{x}{1 + (1 - x)^2}.$$

Notice that f is convex in this case.

Example 4.5.11. ($b = \frac{11}{10}$, $c = 10$)

$$f(x) = \frac{x}{11 - 20x + 10x^2}.$$

In this case, f has an inflection point at $x \approx 0.633892$.

4.6 The Completion Problem

We begin with:

Definition 4.6.1. Let $\alpha : \alpha_0, \cdots \alpha_m,\ (m \geq 0)$ be an initial segment of positive weights and let $\varpi = \{\varpi_n\}_{n=0}^{\infty}$ be a bounded sequence. We say that W_ϖ is a *completion of* α if

$$\varpi_n = \alpha_n \ (0 \leq n \leq m)$$

and we write $\alpha \subseteq \varpi$.

The *completion problem* for a property (P) entails finding necessary and sufficient conditions on α to ensure the existence of a weight sequence $\varpi \supset \alpha$ such that

$$W_\alpha \text{ satisfies } (P).$$

In 1966, Stampfli [Sta3] showed that for arbitrary $\alpha_0 < \alpha_1 < \alpha_2$, there exists a subnormal shift W_α whose first three weights are $\alpha_0, \alpha_1, \alpha_2$; he also proved that given four or more weights it may not be possible to find a subnormal completion.

Theorem 4.6.2. [CuF3]
(a)(Minimality of Norm)

$$||W_{\widehat{\alpha}}|| = \inf\left\{ \|W_\omega\| : \ \alpha \subseteq \omega \ \textit{and} \ W_\omega \ \textit{is subnormal} \right\}.$$

(b) (Minimality of Moments) *If* $\alpha \subseteq \omega$ *and* W_ω *is subnormal then*

$$\int t^n d\mu_{\widehat{\alpha}}(t) \leq \int t^n d\mu_\omega(t) \qquad (n \geq 0).$$

Proof. See [CuF3]. □

Theorem 4.6.3. (Subnormal Completion Problem) [CuF3] *If* $\alpha : \alpha_0, \alpha_1, \cdots, \alpha_m$ $(m \geq 0)$ *is an initial segment then the followings are equivalent:*

(i) *α has a subnormal completion.*

(ii) *α has a recursively generated subnormal completion.*

(iii) *The Hankel matrices*

$$A(k) := \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_1 & & & \gamma_{k+1} \\ \vdots & & & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k} \end{bmatrix} \quad \textit{and} \quad B(l-1) := \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_2 \\ \gamma_1 & & & \gamma_{l+1} \\ \vdots & & & \vdots \\ \gamma_l & \gamma_{l+1} & \cdots & \gamma_{2l} \end{bmatrix}$$

are both positive $\left(k = \left[\frac{m+1}{2}\right],\ l = \left[\frac{m}{2}\right] + 1\right)$ *and the vector*

$$\begin{bmatrix}\gamma_{k+1}\\ \vdots \\ \gamma_{2k+1}\end{bmatrix}\ \left(\textit{resp.}\ \begin{bmatrix}\gamma_{k+1}\\ \vdots \\ \gamma_{2k}\end{bmatrix}\right)$$

is in the range of $A(k)$ *(resp.* $B(l-1)$*) when* m *is even (resp. odd).*

Theorem 4.6.4. (k-Hyponormal Completion Problem) [CuF3] *If* $\alpha : \alpha_0, \alpha_1 \cdots, \alpha_{2m}$ $(m \geq 1)$ *is an initial segment then for* $1 \leq k \leq m$*, the followings are equivalent:*

(i) α *has* k*-hyponormal completion.*

(ii) *The Hankel matrix*

$$A(j,k) := \begin{bmatrix}\gamma_j & \cdots & \gamma_{j+k}\\ \vdots & & \vdots \\ \gamma_{j+k} & \cdots & \gamma_{j+2k}\end{bmatrix}$$

is positive for all j*,* $0 \leq j \leq 2m - 2k + 1$ *and the vector*

$$\begin{bmatrix}\gamma_{2m-k+2}\\ \vdots \\ \gamma_{2m+1}\end{bmatrix}$$

is in the range of $A(2m - 2k + 2, k - 1)$*.*

Theorem 4.6.5. (Quadraically Hyponormal Completion Problem) *Let* $m \geq 2$ *and let* $\alpha : \alpha_0 < \alpha_1, \cdots < \alpha_m$ *be an initial segment. Then the followings are equivalent:*

(i) α *has a quadratically hyponormal completion.*

(ii) $D_{m-1}(t) > 0$ *for all* $t \geq 0$*.*

Moreover, a quadratically hyponormal completion ω *of* $\mathcal{L}$ *can be obtained by*

$$\omega : \alpha_0, \alpha_1, \cdots, \alpha_{m-2}\ (\alpha_{m-1}, \alpha_m, \alpha_{m+1})^\wedge,$$

where α_{m+1} *is chosen sufficiently large.*

Proof. First of all, note that $D_{m-1}(t) > 0$ for all $t \geq 0$ if and only if $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \cdots, m-1$; this follows from the Nested Determinants Test (see [12, Remark 2.4]) or Choleski's Algorithm (see [CuF2, Proposition 2.3]). A straightforward calculation gives

$$\begin{aligned}
d_0(t) &= \alpha_0^2 + \alpha_0^2\alpha_1^2\, t\\
d_1(t) &= \alpha_0^2(\alpha_1^2 - \alpha_0^2) + \alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_0^2)\, t + \alpha_0^2\alpha_1^4\alpha_2^2\, t^2\\
d_2(t) &= \alpha_0^2(\alpha_1^2 - \alpha_0^2)(\alpha_2^2 - \alpha_1^2) + \alpha_0^2\alpha_2^2(\alpha_1^2 - \alpha_0^2)(\alpha_3^2 - \alpha_1^2)\, t\\
&\quad + \alpha_0^2\alpha_1^2\alpha_2^2\Big\{\alpha_3^2(\alpha_2^2 - \alpha_0^2) - \alpha_1^2(\alpha_1^2 - \alpha_0^2)\Big\}\, t^2 + \alpha_0^2\alpha_1^4\alpha_2^2(\alpha_2^2\alpha_3^2 - \alpha_1^2\alpha_0^2)\, t^3,
\end{aligned}$$

which shows that all coefficients of d_i $(i = 0, 1, 2)$ are positive, so that $d_i(t) > 0$ for all $t \geq 0$ and $i = 0, 1, 2$.

Now suppose α has a quadratically hyponormal completion. Then evidently, $d_n(t) \geq 0$ for all $t \geq 0$ and all $n \geq 0$. In view of the propagation property, $\{\alpha_n\}_{n=m}^{\infty}$ is strictly increasing. Thus $d_n(0) = u_0 \cdots u_n = \prod_{i=0}^{n}(\alpha_i^2 - \alpha_{i-1}^2) > 0$ for all $n \geq 0$. If $d_{n_0}(t_0) = 0$ for some $t_0 > 0$ and the first such $n_0 > 0$ $(3 \leq n_0 \leq m-1)$, then (1.6) implies that $0 \leq d_{n_0+1}(t_0) = -|r_{n_0}(t_0)|^2 d_{n_0-1}(t_0) \leq 0$, which forces $r_{n_0}(t_0) = 0$, so that $\alpha_{n_0+1} = \alpha_{n_0-1}$, a contradiction. Therefore $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \cdots, m-1$. This proves the implication (i) ⇒ (ii).

For the reverse implication, we must find a bounded sequence $\{\alpha_n\}_{n=m+1}^{\infty}$ such that $d_n(t) \geq 0$ for all $t \geq 0$ and all $n \geq 0$. Suppose $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \cdots, m-1$. We now claim that there exists a constant $M_k > 0$ for which

$$\frac{d_{k-1}(t)}{d_k(t)} \leq M_k \qquad \text{for all } t \geq 0 \text{ and for } k = 1, \cdots, m-1.$$

Indeed, since $\frac{d_{k-1}(t)}{d_k(t)}$ is a continuous function of t on $[0, \infty)$, and $\deg(d_{k-1}) < \deg(d_k)$, it follows that

$$\max_{t\in[0,\,\infty)} \frac{d_{k-1}(t)}{d_k(t)} \;\leq\; \max\left\{1,\ \max_{t\in[0,\,\xi]} \frac{d_{k-1}(t)}{d_k(t)}\right\} =: M_k,$$

where ξ is the largest root of the equation $d_{k-1}(t) = d_k(t)$. Now a straightforward calculation shows that

$$\begin{aligned} d_m(t) &= q_m(t) d_{m-1}(t) - |r_{m-1}(t)|^2 d_{m-2}(t) \\ &= \left[u_m + \left(v_m - w_{m-1}\frac{d_{m-2}(t)}{d_{m-1}(t)}\right) t\right] d_{m-1}(t)\,. \end{aligned}$$

So if we write $e_m(t) := v_m - w_{m-1}\frac{d_{m-2}(t)}{d_{m-1}(t)}$, then by (3.1), $e_m(t) \geq v_m - w_{m-1}M_{m-1}$. Now choose α_{m+1} so that $v_m - w_{m-1}M_{m-1} > 0$, i.e.,

$$\alpha_{m+1}^2 > \max\left\{\alpha_m^2,\ \frac{\alpha_{m-1}^2}{\alpha_m^2}\left[M\left(\alpha_m^2 - \alpha_{m-2}^2\right)^2 + \alpha_{m-2}^2\right]\right\},$$

where $M := \max_{t\in[0,\infty)} \frac{d_{m-2}(t)}{d_{m-1}(t)}$. Then $e_m(t) \geq 0$ for all $t \geq 0$, so that

$$d_m(t) = (u_m + e_m(t)\,t) d_{m-1}(t) \geq u_m d_{m-1}(t) > 0.$$

Therefore, $d_{m-1}(t) \leq \frac{d_m(t)}{u_m}$. With α_{m+2} to be chosen later, we now consider d_{m+1}. We have

$$\begin{aligned} d_{m+1}(t) &= q_{m+1}(t) d_m(t) - |r_m(t)|^2 d_{m-1}(t) \\ &\geq \frac{1}{u_m}\left[u_m q_{m+1}(t) - |r_m(t)|^2\right] d_m(t) \\ &= \frac{1}{u_m}\left[u_m u_{m+1} + (u_m v_{m+1} - w_m)t\right] d_m(t) \\ &= u_{m+1} d_m(t) + \frac{t}{u_m}(u_m v_{m+1} - w_m)\, d_m(t). \end{aligned}$$

Write $f_{m+1} := u_m v_{m+1} - w_m$. If we choose α_{m+2} such that $f_{m+1} \geq 0$, then $d_{m+1}(t) \geq 0$ for all $t > 0$. In particular we can choose α_{m+2} so that $f_{m+1} = 0$. i.e., $u_m v_{m+1} = w_m$, or

$$\alpha_{m+2}^2 := \frac{\alpha_m^2(\alpha_{m+1}^2 - \alpha_{m-1}^2)^2 + \alpha_{m-1}^2\alpha_m^2(\alpha_m^2 - \alpha_{m-1}^2)}{\alpha_{m+1}^2(\alpha_m^2 - \alpha_{m-1}^2)},$$

or equivalently,

$$\alpha_{m+2}^2 := \alpha_{m+1}^2 \ + \ \frac{\alpha_{m-1}^2(\alpha_{m+1}^2 - \alpha_m^2)^2}{\alpha_{m+1}^2(\alpha_m^2 - \alpha_{m-1}^2)}.$$

In this case, $d_{m+1}(t) \geq u_{m+1}\, d_m(t) \geq 0$. Repeating the argument (with α_{m+3} to be chosen later), we obtain

$$\begin{aligned}
d_{m+2}(t) &= q_{m+2}(t) d_{m+1}(t) - |r_{m+1}(t)|^2 d_m(t) \\
&\geq \frac{1}{u_{m+1}}\Big[u_{m+1} q_{m+2}(t) - |r_{m+1}(t)|^2\Big] d_{m+1}(t) \\
&= \frac{1}{u_{m+1}}\Big[u_{m+1}u_{m+2} + (u_{m+1}v_{m+2} - w_{m+1})\, t\Big] d_{m+1}(t) \\
&= u_{m+2}\, d_{m+1}(t) + \frac{t}{u_{m+1}}(u_{m+1}v_{m+2} - w_{m+1})\, d_{m+1}(t).
\end{aligned}$$

Write $f_{m+2} := u_{m+1}v_{m+2} - w_{m+1}$. If we choose α_{m+3} such that $f_{m+2} = 0$, i.e.,

$$\alpha_{m+3}^2 := \alpha_{m+2}^2 \ + \ \frac{\alpha_m^2(\alpha_{m+2}^2 - \alpha_{m+1}^2)^2}{\alpha_{m+2}^2(\alpha_{m+1}^2 - \alpha_m^2)},$$

then $d_{m+2}(t) \geq u_{m+2}\, d_{m+1}(t) \geq 0$. Continuing this process with the sequence $\{\alpha_n\}_{n=m+2}^{\infty}$ defined recursively by

$$\varphi_1 := \frac{\alpha_m^2(\alpha_{m+1}^2 - \alpha_{m-1}^2)}{\alpha_m^2 - \alpha_{m-1}^2}, \quad \varphi_0 := -\frac{\alpha_{m-1}^2\alpha_m^2(\alpha_{m+1}^2 - \alpha_m^2)}{\alpha_m^2 - \alpha_{m-1}^2}$$

and

$$\alpha_{n+1}^2 := \varphi_1 + \frac{\varphi_0}{\alpha_n^2} \quad (n \geq m+1),$$

we obtain that $d_n(t) \geq 0$ for all $t > 0$ and all $n \geq m+2$. On the other hand, by an argument of [Sta3, Theorem 5], the sequence $\{\alpha_n\}_{n=m+2}^{\infty}$ is bounded. Therefore, a quadratically hyponormal completion $\{\alpha_n\}_{n=0}^{\infty}$ is obtained. The above recursive relation shows that the sequence $\{\alpha_n\}_{n=m+2}^{\infty}$ is obtained recursively from α_{m-1}, α_m and α_{m+1}, that is, $\{\alpha_n\}_{n=m-1}^{\infty} = (\alpha_{m-1}, \alpha_m, \alpha_{m+1})^\wedge$. This completes the proof. $\square$

Given *four* weights $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$, it may not be possible to find a 2-hyponormal completion. In fact, by the preceding criterion for subnormal and k-hyponormal completions, the following statements are equivalent:

(i) α has a subnormal completion;
(ii) α has a 2-hyponormal completion;

(iii) $\det \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} \geq 0.$

By contrast, a quadratically hyponormal completion *always* exists for *four* weights.

Corollary 4.6.6. *For arbitrary $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$, there always exists a quadratically hyponormal completion ω of α.*

Proof. In the proof of Theorem 4.6.5, we showed that $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, 1, 2$. Thus the result immediately follows from Theorem 4.6.5. $\square$

Remark. To discuss the hypothesis $\alpha_0 < \alpha_1 < \cdots < \alpha_m$ in Theorem 4.6.5, we consider the case where $\alpha : \alpha_0, \alpha_1, \cdots, \alpha_m$ admits equal weights:

(i) If $\alpha_0 < \alpha_1 = \cdots = \alpha_m$ then there exists a trivial quadratically hyponormal completion (in fact, a subnormal completion) $\omega : \alpha_0 < \alpha_1 = \cdots = \alpha_n = \alpha_{n+1} = \cdots$.

(ii) If $\{\alpha_n\}_{n=0}^m$ is such that $\alpha_j = \alpha_{j+1}$ for some $j = 1, 2, \cdots, m-1$, and $\alpha_j \neq \alpha_k$ for some $1 \leq j, k \leq m$, then by the propagation property there does not exist any quadratically hyponormal completion of α.

(iii) If $\alpha_0 = \alpha_1$, the conclusion of 4.6.5 may fail: for example, if $\alpha : 1, 1, 2, 3$ then $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, 1, 2$, whereas α admits no quadratically hyponormal completion because we must have $\alpha_2^2 < 2$.

Problem. Given $\alpha : \alpha_0 = \alpha_1 < \alpha_2 < \cdots < \alpha_m$, find necessary and sufficient conditions for the existence of a quadratically hyponormal completion ω of α.

In [CuJ], related to the above problem, weighted shifts of the form $1, (1, \sqrt{b}, \sqrt{c})^\wedge$ have been studied and their quadratic hyponormality completely characterized in terms of b and c.

Remark. In Theorem 4.6.5, the recursively quadratically hyponormal completion requires a sufficiently large α_{m+1}. One might conjecture that if the quadratically hyponormal completion of $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m$ exists, then

$$\omega : \alpha_0, \cdots, \alpha_{m-3}, (\alpha_{m-2}, \alpha_{m-1}, \alpha_m)^\wedge$$

is such a completion. However, if $\alpha : \sqrt{\frac{9}{10}}, \sqrt{1}, \sqrt{2}, \sqrt{3}$ then $\omega : \sqrt{\frac{9}{10}}, (\sqrt{1}, \sqrt{2}, \sqrt{3})^\wedge$ is not quadratically hyponormal (by [CuF3, Theorem 4.3]), even though by Corollary 4.6.6 a quadratically hyponormal completion does exist.

We conclude this section by establishing that for *five* or more weights, the gap between 2-hyponormal and quadratically hyponormal completions can be extremal.

Proposition 4.6.7. *For $a < b < c$, let $\eta : (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ be a recursively generated weight sequence, and consider $\alpha(x) : \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{x}, \eta_4$ (five weights). Then*

(i) *α has a subnormal completion $\Longleftrightarrow x = \eta_3$;*

(ii) *α has a 2-hyponormal completion $\Longleftrightarrow x = \eta_3$;*

(iii) *α has a quadratically hyponormal completion $\Longleftrightarrow c < x < \eta_4^2$.*

Proof. Assertions (i) and (ii) follow from the argument used in the proof of **??**. For assertion (iii), observe that by Theorem 3.1, α has a quadratically hyponormal completion if and only if $d_3(t) > 0$ for all $t \geq 0$. Without loss of generality, we write $a = 1$, $b = 1 + r$, $c = 1 + r + s$, and $x = 1 + r + s + u$ ($r > 0$, $s > 0$, $u > 0$). A straightforward calculation using *Mathematica* shows that the Maclaurin coefficients $c(3, i)$ of $d_3(t)$ are given by

$$\begin{aligned}
c(3,0) &= rsu; \\
c(3,1) &= s^3(r+s)(1+r+s+u)(r+r^2+2rs+s^2)^{-1}; \\
c(3,2) &= (1+r+s)(s^4+rsu+4r^2su+5r^3su+2r^4su+2rs^2u+7r^2s^2u+5r^3s^2u \\
&\quad +2s^3u+4rs^3u+4r^2s^3u+s^4u+rs^4u+r^2u^2+2r^3u^2+r^4u^2+3r^2su^2 \\
&\quad +3r^3su^2+2rs^2u^2+3r^2s^2u^2+s^3u^2+rs^3u^2)(r+r^2+2rs+s^2)^{-1}; \\
c(3,3) &= (1+r)(r+s)(1+r+s)(1+r+s+u)(r^2s^2+r^3s^2+s^3+2rs^3 \\
&\quad +2r^2s^3+s^4+rs^4+r^2u+2r^3u+r^4u+3r^2su+3r^3su+2rs^2u+3r^2s^2u \\
&\quad +s^3u+rs^3u)(r^2+r^3+2r^2s+rs^2)^{-1};\ \text{and} \\
c(3,4) &= (1+r)^2(1+r+s)(r+r^2+2s+2rs+s^2+u+ru+su)(r^2s+2r^3s+r^4s \\
&\quad +rs^2+4r^2s^2+3r^3s^2+s^3+3rs^3+3r^2s^3+s^4+rs^4+r^2u+2r^3u+r^4u \\
&\quad +3r^2su+3r^3su+2rs^2u+3r^2s^2u+s^3u+rs^3u)(r^2+r^3+2r^2s+rs^2)^{-1}.
\end{aligned}$$

This readily shows that for $c < x < \alpha_4^2$, all Maclaurin coefficients of $d_3(t)$ are positive, so that $d_3(t) > 0$ for all $t \geq 0$. Moreover if $x = c$ or α_4^2 then Theorem 1.2 shows that no quadratically hyponormal completion exists. This proves assertion (iii). $\square$

4.7 Comments and Problems

Problem 4.1. *Let T_x be a weighted shift with weights $\alpha \equiv \{\alpha_n\}$ given by*

$$\alpha : x, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots .$$

Describe the set $\{x : T_x$ is cubically hyponormal$\}$. More generally, describe $\{x : T_x$ is weakly k-hyponormal$\}$.

Problem 4.2. *Let T be the weighted shift with weights $\alpha \equiv \{\alpha_n\}$ given by*

$$\alpha_0 = \alpha_1 \le \alpha_2 \le \alpha_3 \le \cdots .$$

If T is cubically hyponormal, is α flat?

Problem 4.3. (Minimality of Weights Problem) *If $\alpha : \alpha_0, \alpha_1, \cdots, \alpha_{2k}$ admits a subnormal completion and if $\alpha \subseteq \omega$ with W_ω subnormal, does it follow that*

$$\alpha_n \le \omega_n \quad \text{for all } n \ge 0\,?$$

A combination of Theorem 4.6.2 (a) and (b) show that $\alpha_n \le \omega_n$ for $0 \le n \le 2k+1$ and also for large n.

Problem 4.4. *Given $\alpha : \alpha_0 = \alpha_1 < \alpha_2 < \cdots < \alpha_m$, find necessary and sufficient conditions for the existence of a quadratically hyponormal completion ω of α.*

In [CuF2] it was shown that

$$\exists\ 1 < b < c \text{ such that } W_{1(1,\sqrt{b},\sqrt{c})^\wedge}$$

is quadratically hyponormal. In fact, it was shown that if we write

$$\mathfrak{H}_2 := \{(b,c) : W_{1(1,\sqrt{b},\sqrt{c})^\wedge} \text{ is quadratically hyponormal}\}$$

then

$$\mathfrak{H}_2 := \{(b,c) : b(bc-1) + b(b-1)(c-1)K - (b-1)^2K^2 \ge 0\},$$

where

$$K = \frac{b(c-1)^2\Big(b(c-1) + \sqrt{b^2(c-1)^2 - 4b(b-1)(c-b)}\Big)}{2(b-1)^2(c-b)}.$$

Problem 4.5. *Does there exists $1 < b < c$ such that $W_{1,(1,\sqrt{b},\sqrt{c})^\wedge}$ is cubically hyponormal ? More generally, describe the set*

$$\{(b,c) : W_{1,(1,\sqrt{b},\sqrt{c})^\wedge} \textit{ is cubically hyponormal}\}.$$

We remember the following question (Due to P. Halmos):

Whether every polynomially hyponormal operator is subnormal ?

In 1993, R. Curto and M. Putinar [CP2] have answered it negatively:

There exits a polynomially hyponormal operator which is not 2-hyponormal.

In 1989, S. M. McCullough and V. Paulsen [McCP] proved the following: *Every polynomially hyponormal operator is subnormal if and only if every polynomially hyponormal weighted shift is subnormal.*

However we did not find a concrete example of such a weighted shift:

Problem 4.6. *Find a weighted shift which is polynomially hyponormal but not subnormal.*

Problem 4.7. *Does there exists a polynomially hyponormal weighted shift which is not* 2*-hyponormal?*

Let B_1 be the weighted shift whose weight are given by

$$\sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{5}{4}}, \sqrt{\frac{4}{5}}, \cdots$$

Let B_2 be the weighted shift whose weight are given by

$$\sqrt{\frac{1}{2}}, \sqrt{x}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$$

A straightforward calculation shows that

$$\begin{aligned}
&B_1 \text{ subnormal } \Longleftrightarrow 0 < x \le \frac{1}{2};\\
&B_1 \text{ 2-hyponormal } \Longleftrightarrow 0 < x \le \frac{9}{16};\\
&B_1 \text{ quadratically hyponormal } \Longleftrightarrow 0 < x \le \frac{2}{3};\\
&B_2 \text{ subnormal } \Longleftrightarrow x = \frac{2}{3};\\
&B_2 \text{ 2-hyponormal } \Longleftrightarrow x \in \left[\frac{63-\sqrt{129}}{80}, \frac{24}{35}\right].
\end{aligned}$$

We conjecture that

$$\begin{aligned}
&\frac{9}{16} < \sup\{x : B_1 \text{ is polynomially hyponormal}\}\\
&\frac{24}{35} < \sup\{x : B_2 \text{ is polynomially hyponormal}\}
\end{aligned}$$

Problem 4.8. *Is the above converse true ?*

We here suggest related problems:

Problem 4.9.

(a) *Does there exists a Toeplitz operator which is polynomially hyponormal but not subnormal ?*

(b) *Classify the polynomially hyponormal operators with finite rank self commutators.*

(c) *Is there an analogue of Berger's theorem for polynomially hyponormal weighted shift ?*

An operator $T \in B(H)$ is called *M-hyponormal* if

$$\exists\, M > 0 \text{ such that } ||(T-\lambda)^* x|| \le M\, ||(T-\lambda)x|| \quad \text{for any } \lambda \in \mathbb{C} \text{ and for any } x \in H.$$

If $M \le 1$ then M-hyponormality $\Rightarrow$ hyponormality. It was shown [HLL] that it $T \equiv W_\alpha$ is a weighted shift with weight sequence α then

$$\alpha \text{ is eventually increasing } \Longrightarrow T \text{ is hyponormal.}$$

We wonder if the converse is also true.

Problem 4.10. (M-hyponormality of weighted shifts) *Does it follow that*

$$W_\alpha \textit{ is } M\textit{-hyponormal } \Longrightarrow \alpha \textit{ is eventually increasing ?}$$

Problem 4.11 (Perturbations of weighted shifts) *Let α be a strictly increasing weighted sequence.*

(a) *If W_α is k-hyponormal, dose it follow that W_α is weakly k-hyponormal under small perturbations of the weighted shifts ?*

(b) *Does it follow that the polynomiality of the weighted shifts is stable under small perturbations of the weighted sequence ?*

It was shown [CuL5] that the answer to Problem 4.10 (a) is affirmative if $k = 2$.

Chapter 5

Toeplitz Theory

5.1 Preliminaries

5.1.1 Fourier Transform and Beurling's Theorem

A trigonometric polynomial is a function $p \in C(\mathbb{T})$ of the form $\sum_{k=-n}^{n} a_k z^k$. It was well-known that the set of trigonometric polynomials is uniformly dense in $C(\mathbb{T})$ and hence is dense in $\mathbf{L}^2(\mathbb{T})$. In fact, if $e_n := z^n$, $(n \in \mathbb{Z})$ then $\{e_n : n \in \mathbb{Z}\}$ forms an orthonormal basis for $\mathbf{L}^2(\mathbb{T})$. The Hardy space $\mathbf{H}^2(\mathbb{T})$ is spanned by $\{e_n : n = 0, 1, 2, \cdots\}$. Write $\mathbf{H}^\infty(\mathbb{T}) := \mathbf{L}^\infty(\mathbb{T}) \cap \mathbf{H}^2(\mathbb{T})$. Then $\mathbf{H}^\infty$ is a subalgebra of $\mathbf{L}^\infty$.

Let m :=the normalized Lebesgue measure on $\mathbb{T}$ and write $\mathbf{L}^2 := \mathbf{L}^2(\mathbb{T})$. If $f \in \mathbf{L}^2$ then the Fourier transform of f, $\widehat{f} : \mathbb{Z} \to \mathbb{C}$, is defined by

$$\widehat{f}(n) \equiv \langle f, e_n \rangle = \int_{\mathbb{T}} f \overline{z}^n dm = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt,$$

which is called the n-th *Fourier coefficient* of f. By Parseval's identity,

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) z^n,$$

which converges in the norm of $\mathbf{L}^2$. This series is called the *Fourier series* of f.

Proposition 5.1.1. *We have:*

(i) $f \in \mathbf{L}^2 \Rightarrow \widehat{f} \in \ell^2(\mathbb{Z})$;

(ii) *If* $V : \mathbf{L}^2 \to \ell^2(\mathbb{Z})$ *is defined by* $Vf = \widehat{f}$ *then* V *is an isomorphism.*

(iii) *If* $W = N_m$ *on* $\mathbf{L}^2$ *then* VWV^{-1} *is the bilateral shift on* $\ell^2(\mathbb{Z})$.

Proof. (i) Since by Parseval's identity, $\sum |\widehat{f}(n)|^2 = ||f||^2 < \infty$, it follows $\widehat{f} \in \ell^2(\mathbb{Z})$.

(ii) We claim that $||Vf|| = ||f||$: indeed, $||Vf||^2 = ||\widehat{f}||^2 = \sum |\widehat{f}(n)|^2 = ||f||^2$. If $f = z^n$ then

$$\widehat{f}(k) = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n, \end{cases}$$

so that $\widehat{f}$ is the n-th basis vector in $\ell^2(\mathbb{Z})$. Thus $\operatorname{ran} V$ is dense and hence V is an isomorphism.

(iii) If $\{e_n\}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$ then by (ii), $Vz^n = e_n$. Thus $VWz^n = V(z^{n+1}) = e_{n+1} = UVz^n$. □

If $T \in B(H)$, write $\operatorname{Lat} T$ for the set of all invariant subspaces for T, i.e.,

$$\operatorname{Lat} T := \{\mathcal{M} \subset H : T\mathcal{M} \subset \mathcal{M}\}.$$

Theorem 5.1.2. *If μ is a compactly supported measure on $\mathbb{T}$ and $\mathcal{M} \in \operatorname{Lat} N_\mu$ then*

$$\mathcal{M} = \phi \mathbf{H}^2 \oplus \mathbf{L}^2(\mu|\Delta),$$

where $\phi \in \mathbf{L}^\infty(\mu)$ and Δ is a Borel set of $\mathbb{T}$ such that $\phi|\Delta = 0$ a.e. and $|\phi|^2\mu = m$(:=the normalized Lebesgue measure).

Proof. See [Con3, p.121]. □

Now consider the case where $\mu = m$ (in this case, N_μ is the bilateral shift). Observe

$$\phi \in \mathbf{L}^2,\ |\phi|^2 m = m \implies |\phi| = 1 \text{ a.e.},$$

so that there is no Borel set Δ such that $\phi|\Delta = 0$ and $m(\Delta) \neq 0$. Therefore every invariant subspace for the bilateral shift must have one form or the other. We thus have:

Corollary 5.1.3. *If W is the bilateral shift on $\mathbf{L}^2$ and $\mathcal{M} \in \operatorname{Lat} W$ then*

$$\textit{either } \mathcal{M} = \mathbf{L}^2(m|\Delta) \textit{ or } \mathcal{M} = \phi\mathbf{H}^2$$

for a Borel set Δ and a function $\phi \in \mathbf{L}^\infty$ such that $|\phi| = 1$ a.e.

Definition 5.1.4. A function $\phi \in \mathbf{L}^\infty$ [$\phi \in \mathbf{H}^\infty$] is called a *unimodular* [*inner*] function if $|\phi| = 1$ a.e.

The following theorem has had an enormous influence on the development in operator theory and function theory.

Theorem 5.1.5 (Beurling's Theorem). *If U is the unilateral shift on $\mathbf{H}^2$ then*

$$\operatorname{Lat} U = \{\phi\mathbf{H}^2 : \ \phi \textit{ is an inner function}\}.$$

Proof. Let W be the bilateral shift on $\mathbf{L}^2$. If $\mathcal{M} \in \operatorname{Lat} U$ then $\mathcal{M} \in \operatorname{Lat} W$. By Corollary 5.1.3, $\mathcal{M} = \mathbf{L}^2(m|\Delta)$ or $\mathcal{M} = \phi\mathbf{H}^2$, where ϕ is a unimodular function. Since U is a shift,

$$\bigcap U^n\mathcal{M} \subset \bigcap U^n\mathbf{H}^2 = \{0\},$$

so the first alternative is impossible. Hence $\phi\mathbf{H}^2 = \mathcal{M} \subset \mathbf{H}^2$. Since $\phi = \phi \cdot 1 \in \mathcal{M}$, it follows $\phi \in \mathbf{L}^\infty \cap \mathbf{H}^2 = \mathbf{H}^\infty$. $\square$

5.1.2 Hardy Spaces

If $f \in \mathbf{H}^2$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is its Fourier series expansion, this series converges uniformly on compact subsets of $\mathbb{D}$. Indeed, if $|z| \le r < 1$, then

$$\sum_{n=m}^{\infty} |a_n z^n| \le \left(\sum_{n=m}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=m}^{\infty} |z|^{2n}\right)^{\frac{1}{2}} \le ||f||_2 \left(\sum_{n=m}^{\infty} r^{2n}\right)^{\frac{1}{2}}.$$

Therefore it is possible to identify $\mathbf{H}^2$ with the space of analytic functions on the unit disk whose Taylor coefficients are square summable.

Proposition 5.1.6. *If f is a real-valued function in $\mathbf{H}^1$ then f is constant.*

Proof. Let $\alpha = \int f dm$. By hypothesis, we have $\alpha \in \mathbb{R}$. Since $f \in \mathbf{H}^1$, we have $\int f z^n dm = 0$ for $n \ge 1$. So $\int (f - \alpha) z^n dm = 0$ for $n \ge 0$. Also,

$$0 = \overline{\int (f-\alpha) z^n dm} = \int (f - \alpha) z^{-n} dm \quad (n \ge 0),$$

so that $\int (f - \alpha) z^n dm = 0$ for all integers n. Thus $f - \alpha$ annihilates all the trigonometric polynomials. Therefore, $f - \alpha = 0$ in $\mathbf{L}^1$. □

Corollary 5.1.7. *If ϕ is inner such that $\overline{\phi} = \frac{1}{\phi} \in \mathbf{H}^2$ then ϕ is constant.*

Proof. By hypothesis, $\phi + \overline{\phi}$ and $\frac{\phi - \overline{\phi}}{i}$ are real-valued functions in $\mathbf{H}^2$. By Proposition 5.1.6, they are constant, so is ϕ. □

The proof of the following important theorem uses Beurling's theorem.

Theorem 5.1.8 (The F. and M. Riesz Theorem). *If f is a nonzero function in $\mathbf{H}^2$, then $m\Big(\{z \in \partial\mathbb{D} : f(z) = 0\}\Big) = 0$. Hence, in particular, if $f, g \in \mathbf{H}^2$ and if $fg = 0$ a.e. then $f = 0$ a.e. or $g = 0$ a.e.*

Proof. Let $\triangle$ be a Borel set of $\partial\mathbb{D}$ and put

$$\mathcal{M} := \{h \in \mathbf{H}^2 : h(z) = 0 \text{ a.e. on } \triangle\}.$$

Then $\mathcal{M}$ is an invariant subspace for the unilateral shift. By Beurling's theorem, if $\mathcal{M} \ne \{0\}$, then there exists an inner function ϕ such that $\mathcal{M} = \phi\mathbf{H}^2$. Since $\phi = \phi \cdot 1 \in \mathcal{M}$, it follows $\phi = 0$ on $\triangle$. But $|\phi| = 1$ a.e., and hence $\mathcal{M} = \{0\}$. □

A function f in $\mathbf{H}^2$ is called an *outer function* if

$$\mathbf{H}^2 = \bigvee \{z^n f : n \ge 0\}.$$

So f is outer if and only if it is a cyclic vector for the unilateral shift.

Theorem 5.1.9 (Inner-Outer Factorization). *If f is a nonzero function in $\mathbf{H}^2$, then*

$$\exists \text{ an inner function } \phi \text{ and an outer function } g \text{ in } \mathbf{H}^2 \ \text{ such that } \ f = \phi g.$$

In particular, if $f \in \mathbf{H}^\infty$, then $g \in \mathbf{H}^\infty$.

Proof. Observe $\mathcal{M} \equiv \bigvee\{z^n f : n \geq 0\} \in \operatorname{Lat} U$. By Beurling's theorem,

$$\exists \text{ an inner function } \phi \ \text{ s.t. } \ \mathcal{M} = \phi \mathbf{H}^2.$$

Let $g \in \mathbf{H}^2$ be such that $f = \phi g$. We want to show that g is outer. Put $\mathcal{N} \equiv \bigvee\{z^n g : n \geq 0\}$. Again there exists an inner function ψ such that $\mathcal{N} = \psi \mathbf{H}^2$. Note that

$$\phi \mathbf{H}^2 := \bigvee\{z^n f : n \geq 0\} = \bigvee\{z^n \phi g : n \geq 0\} = \phi\psi \mathbf{H}^2.$$

Therefore there exists a function $h \in \mathbf{H}^2$ such that $\phi = \phi\psi h$ so that $\overline{\psi} = h \in \mathbf{H}^2$. Hence ψ is a constant by Corollary 5.1.7. So $\mathcal{N} = \mathbf{H}^2$ and g is outer. Assume $f \in \mathbf{H}^\infty$ with $f = \phi g$. Thus $|g| = |f|$ a.e. on $\partial\mathbb{D}$, so that g must be bounded, i.e., $g \in \mathbf{H}^\infty$. $\square$

5.1.3 Toeplitz Operators

Let P be the orthogonal projection of $\mathbf{L}^2(\mathbb{T})$ onto $\mathbf{H}^2(\mathbb{T})$. For $\varphi \in \mathbf{L}^\infty(\mathbb{T})$, the Toeplitz operator T_φ with symbol φ is defined by

$$T_\varphi f = P(\varphi f) \quad \text{for } f \in \mathbf{H}^2.$$

Remember that $\{z^n : n = 0, 1, 2, \cdots\}$ is an orthonormal basis for $\mathbf{H}^2$. Thus if $\varphi \in \mathbf{L}^\infty$ has the Fourier coefficients

$$\widehat{\varphi}(n) = \frac{1}{2\pi}\int_0^{2\pi} \varphi \overline{z}^n dt,$$

then the matrix (a_{ij}) for T_φ with respect to the basis $\{z^n : n = 0, 1, 2, \cdots\}$ is given by:

$$a_{ij} = (T_\varphi z^j, z^i) = \frac{1}{2\pi}\int_0^{2\pi} \varphi \overline{z}^{i-j} dt = \widehat{\varphi}(i-j).$$

Thus the matrix for T_φ is constant on diagonals:

$$(a_{ij}) = \begin{bmatrix} c_0 & c_{-1} & c_{-2} & c_{-3} & \cdots \\ c_1 & c_0 & c_{-1} & c_{-2} & \cdots \\ c_2 & c_1 & c_0 & c_{-1} & \cdots \\ c_3 & c_2 & c_1 & c_0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \text{where } c_j = \widehat{\varphi}(j):$$

Such a matrix is called a *Toeplitz matrix.*

Lemma 5.1.10. *Let $A \in B(\mathbf{H}^2)$. The matrix A relative to the orthonormal basis $\{z^n : n = 0, 1, 2, \cdots\}$ is a Toeplitz matrix if and only if*

$$U^*AU = A, \text{ where } U \text{ is the unilateral shift.}$$

Proof. The hypothesis on the matrix entries $a_{ij} = \langle Az^j, z^i\rangle$ of A if and only if

$$a_{i+1,j+1} = a_{ij} \ (i, j = 0, 1, 2, \cdots). \tag{5.1}$$

Noting $Uz^n = z^{n+1}$ for $n \geq 0$, we get

$$\begin{aligned} (5.1) &\iff \langle U^*AUz^j, z^i\rangle = \langle AUz^j, Uz^i\rangle = \langle Az^{j+1}, z^{i+1}\rangle = \langle Az^j, z^i\rangle, \quad \forall\, i, j \\ &\iff U^*AU = A. \end{aligned}$$

□

Remark. $AU = UA \Leftrightarrow A$ is an analytic Toeplitz operator (i.e., $A = T_\varphi$ with $\varphi \in \mathbf{H}^\infty$).

Consider the mapping $\xi : \mathbf{L}^\infty \to B(\mathbf{H}^2)$ defined by $\xi(\varphi) = T_\varphi$. We have:

Proposition 5.1.11. *ξ is a contractive $*$-linear mapping from $\mathbf{L}^\infty$ to $B(\mathbf{H}^2)$.*

Proof. It is obvious that ξ is contractive and linear. To show that $\xi(\varphi)^* = \xi(\overline{\varphi})$, let $f, g \in \mathbf{H}^2$. Then

$$\langle T_{\overline{\varphi}} f, g\rangle = \langle P(\overline{\varphi} f), g\rangle = \langle \overline{\varphi} f, g\rangle = \langle f, \varphi g\rangle = \langle f, P(\varphi g)\rangle = \langle f, T_\varphi g\rangle = \langle T_\varphi^* f, g\rangle,$$

so that $\xi(\varphi)^* = T_\varphi^* = T_{\overline{\varphi}} = \xi(\overline{\varphi})$. □

Remark. ξ is not multiplicative. For example, $T_z T_{\overline{z}} \neq I = T_1 = T_{|z|^2} = T_{z\overline{z}}$. Thus ξ is not a homomorphism.

In special cases, ξ is multiplicative.

Proposition 5.1.12. *$T_\varphi T_\psi = T_{\varphi\psi} \iff$ either ψ or $\overline{\varphi}$ is analytic.*

Proof. ($\Leftarrow$) Recall that if $f \in \mathbf{H}^2$ and $\psi \in \mathbf{H}^\infty$ then $\psi f \in \mathbf{H}^2$. Thus, $T_\psi f = P(\psi f) = \psi f$. So

$$T_\varphi T_\psi f = T_\varphi(\psi f) = P(\varphi\psi f) = T_{\varphi\psi} f, \ \text{ i.e., } \ T_\varphi T_\psi = T_{\varphi\psi}.$$

Taking adjoints reduces the second part to the first part.

($\Rightarrow$) From a straightforward calculation. □

Write M_φ for the multiplication operator on $\mathbf{L}^2$ with symbol $\varphi \in \mathbf{L}^\infty$. The *essential range* of $\varphi \in \mathbf{L}^\infty \equiv \mathfrak{R}(\varphi)$:=the set of all λ for which $\mu\Big(\{x : |f(x) - \lambda| < \epsilon\}\Big) > 0$ for any $\epsilon > 0$.

Lemma 5.1.13. *If $\varphi \in \mathbf{L}^\infty(\mu)$ then $\sigma(M_\varphi) = \mathfrak{R}(\varphi)$.*

Proof. If $\lambda \notin \mathfrak{R}(\varphi)$ then

$$\exists\, \varepsilon > 0 \text{ such that } \mu\Big(\{x : |\varphi(x) - \lambda| < \varepsilon\}\Big) = 0, \ \text{ i.e., } |\varphi(x) - \lambda| \geq \ \epsilon \text{ a.e. } [\mu].$$

So

$$g(x) := \frac{1}{\varphi(x) - \lambda} \ \in \ \mathbf{L}^\infty(X, \mu).$$

Hence M_g is the inverse of $M_\varphi - \lambda$, i.e., $\lambda \notin \sigma(M_\varphi)$. For the converse, suppose $\lambda \in \mathfrak{R}(\varphi)$. We will show that

$$\exists \text{ a sequence } \{g_n\} \text{ of unit vectors } \in \mathbf{L}^2 \text{ with the property } ||M_\varphi g_n - \lambda g_n|| \to 0,$$

showing that $M_\varphi - \lambda$ is not bounded below, and hence $\lambda \in \sigma(M_\varphi)$. By assumption, $\{x \in \mathbb{T} : |\varphi(x) - \lambda| \leq \frac{1}{n}\}$ has a positive measure. So we can find a subset

$$E_n \subseteq \left\{ x \in \mathbb{T} : |\varphi(x) - \lambda| \leq \frac{1}{n} \right\}$$

satisfying $0 < \mu(E_n) < \infty$. Letting $g_n := \frac{\chi_{E_n}}{\sqrt{\mu(E_n)}}$, we have that

$$\big|(\varphi(x)-\lambda)g_n(x)\big| \le \frac{1}{n}|g_n(x)|,$$

and hence $||(\varphi-\lambda)g_n||_{\mathbf{L}^2} \le \frac{1}{n} \longrightarrow 0$. □

Proposition 5.1.14. *If $\varphi \in \mathbf{L}^\infty$ is such that T_φ is invertible, then φ is invertible in $\mathbf{L}^\infty$.*

Proof. In view of Lemma 5.1.13, it suffices to show that

$$T_\varphi \text{ is invertible } \Longrightarrow\ M_\varphi \text{ is invertible.}$$

If T_φ is invertible then

$$\exists\, \varepsilon > 0 \ \text{ such that } \ ||T_\varphi f|| \ge \varepsilon ||f||, \ \ \forall f \in \mathbf{H}^2.$$

So for $n \in \mathbb{Z}$ and $f \in \mathbf{H}^2$,

$$||M_\varphi(z^n f)|| = ||\varphi z^n f|| = ||\varphi f|| \ge ||P(\varphi f)|| = ||T_\varphi f|| \ge \varepsilon ||f|| = \varepsilon ||z^n f||.$$

Since $\{z^n f : f \in \mathbf{H}^2, n \in \mathbb{Z}\}$ is dense in $\mathbf{L}^2$, it follows $||M_\varphi g|| \ge \varepsilon ||g||$ for $g \in \mathbf{L}^2$. Similarly, $||M_{\overline{\varphi}} f|| \ge \varepsilon ||f||$ since $T_\varphi^* = T_{\overline{\varphi}}$ is also invertible. Therefore M_φ is invertible. □

Theorem 5.1.15 (Hartman-Wintner). *If $\varphi \in \mathbf{L}^\infty$ then*

(i) $\mathfrak{R}(\varphi) = \sigma(M_\varphi) \subset \sigma(T_\varphi)$

(ii) $||T_\varphi|| = ||\varphi||_\infty$ *(i.e., ξ is an isometry).*

Proof. (i) From Lemma 5.1.13 and Proposition 5.1.14.

(ii) $||\varphi||_\infty = \sup_{\lambda \in \mathfrak{R}(\varphi)} |\lambda| \le \sup_{\lambda \in \sigma(T_\varphi)} |\lambda| = r(T_\varphi) \le ||T_\varphi|| \le ||\varphi||_\infty$.

□

From Theorem 5.1.15 we can see that

(i) If T_φ is quasinilpotent then $T_\varphi = 0$ because $\mathfrak{R}(\varphi) \subseteq \sigma(T_\varphi) = \{0\} \ \Rightarrow\ \varphi = 0$.

(ii) If T_φ is self-adjoint then φ is real-valued because $\mathfrak{R}(\varphi) \subseteq \sigma(T_\varphi) \subseteq \mathbb{R}$.

If $\mathfrak{S} \subseteq \mathbf{L}^\infty$, write $\mathcal{T}(\mathfrak{S}) :=$ the smallest closed subalgebra of $\mathcal{L}(\mathbf{H}^2)$ containing $\{T_\varphi : \varphi \in \mathfrak{S}\}$.

If $\mathcal{A}$ is a C^*-algebra then its *commutator ideal* $\mathcal{C}$ is the closed ideal generated by the commutators $[a,b] := ab - ba$ $(a, b \in \mathcal{A})$. In particular, $\mathcal{C}$ is the smallest closed ideal in $\mathcal{A}$ such that $\mathcal{A}/\mathcal{C}$ is abelian.

Theorem 5.1.16. *If $\mathcal{C}$ is the commutator ideal in $\mathcal{T}(\mathbf{L}^\infty)$, then the mapping ξ_c induced from $\mathbf{L}^\infty$ to $\mathcal{T}(\mathbf{L}^\infty)/\mathcal{C}$ by ξ is a $*$-isometrical isomorphism. Thus there is a short exact sequence*

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{T}(\mathbf{L}^\infty) \longrightarrow \mathbf{L}^\infty \longrightarrow 0.$$

Proof. See [Do1]. □

The commutator ideal $\mathcal{C}$ contains compact operators.

Proposition 5.1.17. *The commutator ideal in $\mathcal{T}(C(\mathbb{T})) = K(\mathbf{H}^2)$. Hence the commutator ideal of $\mathcal{T}(\mathbf{L}^\infty)$ contains $K(\mathbf{H}^2)$.*

Proof. Since T_z is the unilateral shift, we can see that the commutator ideal of $\mathcal{T}(C(\mathbb{T}))$ contains the rank one operator $T_z^*T_z - T_zT_z^*$. Moreover, $\mathcal{T}(C(\mathbb{T}))$ is irreducible since T_z has no proper reducing subspaces by Beurling's theorem. Therefore $\mathcal{T}(C(\mathbb{T}))$ contains $K(\mathbf{H}^2)$. Since T_z is normal modulo a compact operator and generates the algebra $\mathcal{T}(C(\mathbb{T}))$, it follows that $\mathcal{T}(C(\mathbb{T}))/K(\mathbf{H}^2)$ is commutative. Hence $K(\mathbf{H}^2)$ contains the commutator ideal of $\mathcal{T}(C(\mathbb{T}))$. But since $K(\mathbf{H}^2)$ is simple (i.e., it has no nontrivial closed ideal), we can conclude that $K(\mathbf{H}^2)$ is the commutator ideal of $\mathcal{T}(C(\mathbb{T}))$. □

Corollary 5.1.18. *There exists a $*$-homomorphism $\zeta : \mathcal{T}(\mathbf{L}^\infty)/K(\mathbf{H}^2) \longrightarrow \mathbf{L}^\infty$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{T}(\mathbf{L}^\infty) \arrow[rr, "\pi"] \arrow[dr, "\rho"'] & & \mathcal{T}(\mathbf{L}^\infty)/K(\mathbf{H}^2) \arrow[dl, "\zeta"] \\
& \mathbf{L}^\infty(\mathbb{T}) &
\end{tikzcd}$$

Corollary 5.1.19. *Let $\varphi \in \mathbf{L}^\infty$. If T_φ is Fredholm then φ is invertible in $\mathbf{L}^\infty$.*

Proof. If T_φ is Fredholm then $\pi(T_\varphi)$ is invertible in $\mathcal{T}(\mathbf{L}^\infty)/\mathcal{K}(\mathbf{H}^2)$, so $\varphi = \rho(T_\varphi) = (\zeta \circ \pi)(T_\varphi)$ is invertible in $\mathbf{L}^\infty$. □

From Corollary 5.1.18, we have:

(i) $||T_\varphi|| \leq ||T_\varphi + K||$ for every compact operator K because $||T_\varphi|| = ||\varphi||_\infty = ||\zeta(T_\varphi + K)|| \leq ||T_\varphi + K||$.

(ii) The only compact Toeplitz operator is 0 because $||K|| \leq ||K + K|| \Rightarrow K = 0$.

Proposition 5.1.20. *If φ is invertible in $\mathbf{L}^\infty$ such that $\Re(\varphi) \subseteq$ the open right half-plane, then T_φ is invertible.*

Proof. If $\Delta \equiv \{z \in \mathbb{C} : |z-1| < 1\}$ then there exists $\epsilon > 0$ such that $\epsilon\,\mathfrak{R}(\varphi) \subseteq \Delta$. Hence $||\epsilon\varphi - 1|| < 1$, which implies $||I - T_{\epsilon\varphi}|| < 1$. Therefore $T_{\epsilon\varphi} = \epsilon T_\varphi$ is invertible. □

Corollary 5.1.21 (Brown-Halmos). *If* $\varphi \in \mathbf{L}^\infty$*, then* $\sigma(T_\varphi) \subseteq \operatorname{conv}\mathfrak{R}(\varphi)$*.*

Proof. It is sufficient to show that every open half-plane containing $\mathfrak{R}(\varphi)$ contains $\sigma(T_\varphi)$. This follow at once from Proposition 5.1.20 after a translation and rotation of the open half-plane to coincide with the open right half-plane. □

Proposition 5.1.22. *If* $\varphi \in C(\mathbb{T})$ *and* $\psi \in \mathbf{L}^\infty$ *then*

$$T_\varphi T_\psi - T_{\varphi\psi} \quad \textit{and} \quad T_\psi T_\varphi - T_{\psi\varphi} \quad \textit{are compact.}$$

Proof. If $\psi \in \mathbf{L}^\infty,\ f \in \mathbf{H}^2$ then

$$\begin{aligned} T_\psi T_{\overline{z}} f &= T_\psi P(\overline{z} f) = T_\psi(\overline{z} f - \widehat{f}(0)\overline{z}) \\ &= P M_\psi\Big(\overline{z} f - \widehat{f}(0)\overline{z}\Big) \\ &= P(\psi\overline{z} f) - \widehat{f}(0) P(\psi\overline{z}) \\ &= T_{\psi\overline{z}} f - \widehat{f}(0) P(\psi\overline{z}), \end{aligned}$$

which implies that $T_\psi T_{\overline{z}} - T_{\psi\overline{z}}$ is at most a rank one operator. Suppose $T_\psi T_{\overline{z}^n} - T_{\psi\overline{z}^n}$ is compact for every $\psi \in \mathbf{L}^\infty$ and $n = 1, \cdots, N$. Then

$$T_\psi T_{\overline{z}^{N+1}} - T_{\psi\overline{z}^{N+1}} = (T_\psi T_{\overline{z}^N} - T_{\psi\overline{z}^N})\, T_{\overline{z}} + (T_{\psi\overline{z}^N} T_{\overline{z}} - T_{(\psi\overline{z}^N)\overline{z}}),$$

which is compact. Also, since $T_\psi T_{z^n} = T_{\psi z^n}$ $(n \geq 0)$, it follows that $T_\psi T_p - T_{\psi p}$ is compact for every trigonometric polynomial p. But since the set of trigonometric polynomials is dense in $C(\mathbb{T})$ and ξ is isometric, we can conclude that $T_\psi T_\varphi - T_{\psi\varphi}$ is compact for $\psi \in \mathbf{L}^\infty$ and $\varphi \in C(\mathbb{T})$. □

Theorem 5.1.23. $\mathcal{T}(C(\mathbb{T}))$ *contains* $K(\mathbf{H}^2)$ *as its commutator and the sequence*

$$0 \longrightarrow K(\mathbf{H}^2) \longrightarrow \mathcal{T}(C(\mathbb{T})) \longrightarrow C(\mathbb{T}) \longrightarrow 0$$

is a short exact sequence, i.e., $\mathcal{T}(C(\mathbb{T}))/K(\mathbf{H}^2)$ *is* $*$*-isometrically isomorphic to* $C(\mathbb{T})$*.*

Proof. By Proposition 5.1.22 and Corollary 5.1.18. □

Proposition 5.1.24. [Co] *If* $\varphi \neq 0$ *a.e. in* $\mathbf{L}^\infty$*, then*

$$\textit{either } \ker T_\varphi = \{0\} \textit{ or } \ker T_\varphi^* = \{0\}.$$

Proof. If $f \in \ker T_\varphi$ and $g \in \ker T_\varphi^*$, i.e., $P(\varphi f) = 0$ and $P(\overline{\varphi} g)$=0, then

$$\overline{\varphi}\overline{f} \in z\mathbf{H}^2 \quad \text{and} \quad \varphi\overline{g} \in z\mathbf{H}^2.$$

Thus $\overline{\varphi}\overline{f}g,\ \varphi\overline{g}f \in z\mathbf{H}^1$ and therefore $\varphi f\overline{g} = 0$. If neither f nor g is 0, then by F. and M. Riesz theorem, $\varphi = 0$ a.e. on $\mathbb{T}$, a contradiction. □

Corollary 5.1.25. *If $\varphi \in C(\mathbb{T})$ then T_φ is Fredholm if and only if φ vanishes nowhere.*

Proof. By Theorem 5.1.23,

$$\begin{aligned} T_\varphi \text{ is Fredholm } &\iff \pi(T_\varphi) \text{ is invertible in } \mathcal{T}(C(\mathbb{T}))/\mathcal{K}(\mathbf{H}^2) \\ &\iff \varphi \text{ is invertible in } C(\mathbb{T}). \end{aligned}$$

□

Corollary 5.1.26. *If $\varphi \in C(\mathbb{T})$, then $\sigma_e(T_\varphi) = \varphi(\mathbb{T})$.*

Proof. $\sigma_e(T_\varphi) = \sigma(T_\varphi + \mathcal{K}(\mathbf{H}^2)) = \sigma(\varphi) = \varphi(\mathbb{T})$. □

Theorem 5.1.27. *If $\varphi \in C(\mathbb{T})$ is such that T_φ is Fredholm, then*

$$\text{index}\,(T_\varphi) = -\text{wind}\,(\varphi).$$

Proof. We claim that if φ and ψ determine homotopic curves in $\mathbb{C} \setminus \{0\}$, then

$$\text{index}\,(T_\varphi) = \text{index}\,(T_\psi).$$

To see this, let Φ be a constant map from $[0,1] \times \mathbb{T}$ to $\mathbb{C} \setminus \{0\}$ such that

$$\Phi(0, e^{it}) = \varphi(e^{it}) \quad \text{and} \quad \Phi(1, e^{it}) = \psi(e^{it}).$$

If we set $\Phi_\lambda(e^{it}) = \Phi(\lambda, e^{it})$, then the mapping $\lambda \mapsto T_{\Phi_\lambda}$ is norm continuous and each T_{Φ_λ} is a Fredholm operator. Since the map index is continuous, index(T_φ) = index(T_ψ). Now if n = wind(φ) then φ is homotopic in $\mathbb{C} \setminus \{0\}$ to z^n. Since index$\,(T_{z^n}) = -n$, we have that index$\,(T_\varphi) = -n$. □

Theorem 5.1.28. *If U is the unilateral shift on $\mathbf{H}^2$ then* $\text{comm}(U) = \{T_\varphi : \varphi \in \mathbf{H}^\infty\}$.

Proof. It is straightforward that $UT_\varphi = T_\varphi U$ for $\varphi \in \mathbf{H}^\infty$, i.e., $\{T_\varphi : \varphi \in \mathbf{H}^\infty\} \subset \text{comm}(U)$. For the reverse we suppose $T \in \text{comm}(U)$, i.e., $TU = UT$. Put $\varphi := T(1)$. So $\varphi \in \mathbf{H}^2$ and $T(p) = \varphi p$ for every polynomial p. If $f \in \mathbf{H}^2$, let $\{p_n\}$ be a sequence of polynomials such that $p_n \to f$ in $\mathbf{H}^2$. By passing to a subsequence, we can assume $p_n(z) \to f(z)$ a.e. $[m]$. Thus $\varphi p_n = T(p_n) \to T(f)$ in $\mathbf{H}^2$ and $\varphi p_n \to \varphi f$ a.e. $[m]$.

Therefore $Tf = \varphi f$ for all $f \in \mathbf{H}^2$. We want to show that $\varphi \in \mathbf{L}^\infty$ and hence $\varphi \in \mathbf{H}^\infty$. We may assume, without loss of generality, that $||T|| = 1$. Observe

$$T^k f = \varphi^k f \quad \text{for } f \in \mathbf{H}^2,\ k \geq 1.$$

Hence $||\varphi^k f||_2 \leq ||f||_2$ for all $k \geq 1$. Taking $f = 1$ shows that $\int |\varphi|^{2k} dm \leq 1$ for all $k \geq 1$. If $\Delta := \{z \in \partial\mathbb{D} : |\varphi(z)| > 1\}$ then $\int_\Delta |\varphi|^{2k} dm \leq 1$ for all $k \geq 1$. If $m(\Delta) \neq 0$ then $\int_\Delta |\varphi|^{2k} dm \to \infty$ as $k \to \infty$, a contradiction. Therefore $m(\Delta) = 0$ and hence φ is bounded. Therefore $T = T_\varphi$ for $\varphi \in \mathbf{H}^\infty$. □

D. Sarason [Sa] gave a generalization of Theorem 5.1.28.

Theorem 5.1.29 (Sarason's Interpolation Theorem). *Let*

(i) $U =$*the unilateral shift on* $\mathbf{H}^2$*;*
(ii) $\mathcal{K} := \mathbf{H}^2 \ominus \psi\mathbf{H}^2$ *(*ψ *is an inner function);*
(iii) $S := PU|_{\mathcal{K}}$*, where* P *is the projection of* $\mathbf{H}^2$ *onto* $\mathcal{K}$*.*

If $T \in \text{comm}(S)$ *then there exists a function* $\varphi \in \mathbf{H}^\infty$ *such that* $T = T_\varphi|_{\mathcal{K}}$ *with* $||\varphi||_\infty = ||T||$.

Proof. See [Sa]. □

5.2 Hyponormality of Toeplitz operators

An elegant and useful theorem of C. Cowen [Cow3] characterizes the hyponormality of a Toeplitz operator T_φ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} \subset \mathbb{C}$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. This result makes it possible to answer an algebraic question coming from operator theory – namely, is T_φ hyponormal? - by studying the function φ itself. Normal Toeplitz operators were characterized by a property of their symbol in the early 1960's by A. Brown and P.R. Halmos [BH], and so it is somewhat of a surprise that 25 years passed before the exact nature of the relationship between the symbol $\varphi \in L^\infty$ and the positivity of the selfcommutator $[T_\varphi^*, T_\varphi]$ was understood (via Cowen's theorem). As Cowen notes in his survey paper [Cow2], the intensive study of subnormal Toeplitz operators in the 1970's and early 80's is one explanation for the relatively late appearance of the sequel to the Brown-Halmos work. The characterization of hyponormality via Cowen's theorem requires one to solve a certain functional equation in the unit ball of H^∞. However the case of arbitrary trigonometric polynomials φ, though solved in principle by Cowen's theorem, is in practice very complicated. Indeed it may not even be possible to find tractable necessary and sufficient conditions for the hyponormality of T_φ in terms of the Fourier coefficients of φ unless certain assumptions are made about φ. In this chapter we present some recent development in this research.

5.2.1 Cowen's Theorem

In this section we present Cowen's theorem. Cowen's method is to recast the operator-theoretic problem of hyponormality of Toeplitz operators into the problem of finding a solution of a certain functional equation involving its symbol. This approach has been put to use in the works [CLL, CuL1, CuL2, CuL3, FL1, FL2, Gu1, HKL1, HKL2, HwL3, KL, NaT, Zh] to study Toeplitz operators.

We begin with:

Lemma 5.2.1. *A necessary and sufficient condition that two Toeplitz operators commute is that either both be analytic or both be co-analytic or one be a linear function of the other.*

Proof. Let $\varphi = \sum_i \alpha_i z^i$ and $\psi = \sum_j \beta_j z^j$. Then a straightforward calculation shows that

$$T_\varphi T_\psi = T_\psi T_\varphi \iff \alpha_{i+1}\beta_{-j-1} = \beta_{i+1}\alpha_{-j-1} \quad (i, j \geq 0).$$

Thus either $\alpha_{-j-1} = \beta_{-j-1} = 0$ for $j \geq 0$, i.e., φ and ψ are both analytic, or $\alpha_{i+1} = \beta_{i+1} = 0$ for $i \geq 0$, i.e., φ and ψ are both co-analytic, or there exist i_0, j_0 such that $\alpha_{i_0+1} \neq 0$ and $\alpha_{-j_0-1} \neq 0$. So for the last case, if the common value of $\beta_{-j_0-1}/\alpha_{-j_0-1}$ and $\beta_{i_0+1}/\alpha_{i_0+1}$ is denoted by λ, then

$$\beta_{i+1} = \lambda\, \alpha_{i+1} \;\; (i \geq 0) \quad \text{and} \quad \beta_{-j-1} = \lambda\, \alpha_{-j-1} \;\; (j \geq 0).$$

Therefore, $\beta_k = \lambda\, \alpha_k$ $(k \neq 0)$. □

Theorem 5.2.2 (Brown-Halmos). *Normal Toeplitz operators are translations and rotations of hermitian Toeplitz operators i.e.,*

$$T_\varphi \text{ normal} \iff \exists\ \alpha, \beta \in \mathbb{C},\ \text{a real valued } \psi \in \mathbf{L}^\infty \text{ such that } T_\varphi = \alpha T_\psi + \beta 1.$$

Proof. If $\varphi = \sum_i \alpha_i z^i$, then

$$\overline{\varphi} = \sum_i \overline{\alpha_i}\, \overline{z}^i = \sum_i \overline{\alpha_{-i}} z^i.$$

So if φ is real, then $\alpha_i = \overline{\alpha_{-i}}$. Thus no real φ can be analytic or co-analytic unless φ is a constant. Write $T_\varphi = T_{\varphi_1 + i\varphi_2}$, where φ_1, φ_2 are real-valued. Then by Lemma 5.2.1, $T_\varphi T_{\overline{\varphi}} = T_{\overline{\varphi}} T_\varphi$ iff $T_{\varphi_1} T_{\varphi_2} = T_{\varphi_2} T_{\varphi_1}$ iff either φ_1 and φ_2 are both analytic or φ_1 and φ_2 are both co-analytic or $\varphi_1 = \alpha\varphi_2 + \beta$ $(\alpha, \beta \in \mathbb{C})$. So if $\varphi \neq$ a constant, then $\varphi = \alpha\varphi_2 + \beta + i\varphi_2 = (\alpha + i)\varphi_2 + \beta$. □

For $\psi \in \mathbf{L}^\infty$, the Hankel operator H_ψ is the operator on $\mathbf{H}^2$ defined by

$$H_\psi f = J(I - P)(\psi f) \quad (f \in \mathbf{H}^2),$$

where J is the unitary operator from $(\mathbf{H}^2)^\perp$ onto $\mathbf{H}^2$:

$$J(z^{-n}) = z^{n-1}\ (n \geq 1).$$

Denoting $v^*(z) := \overline{v(\overline{z})}$, another way to put this is that H_ψ is the operator on $\mathbf{H}^2$ defined by

$$< zuv, \overline{\psi} > = < H_\psi u, v^* > \ \text{ for all } v \in \mathbf{H}^\infty.$$

If ψ has the Fourier series expansion $\psi := \sum_{n=-\infty}^{\infty} a_n z^n$, then the matrix of H_ψ is given by

$$H_\psi \equiv \begin{bmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & & \\ a_{-3} & & \ddots & \\ \vdots & & & \ddots \end{bmatrix}.$$

The following are basic properties of Hankel operators.

1. $H_\psi^* = H_{\psi^*}$;
2. $H_\psi U = U^* H_\psi$ (U is the unilateral shift);
3. Ker$H_\psi = \{0\}$ or $\theta\mathbf{H}^2$ for some inner function θ (by Beurling's theorem);
4. $T_{\varphi\psi} - T_\varphi T_\psi = H_{\overline{\varphi}}^* H_\psi$;
5. $H_\varphi T_h = H_{\varphi h} = T_{h^*}^* H_\varphi$ $(h \in \mathbf{H}^\infty)$.

We are ready for:

Theorem 5.2.3 (Cowen's Theorem). *If $\varphi \in \mathbf{L}^\infty$ is such that $\varphi = \overline{g} + f$ $(f, g \in \mathbf{H}^2)$, then*

$$T_\varphi \text{ is hyponormal} \iff g = c + T_{\overline{h}} f$$

for some constant c and some $h \in \mathbf{H}^\infty(\mathbb{D})$ with $||h||_\infty \leq 1$.

Proof. Let $\varphi = f + \overline{g}$ ($f, g \in \mathbf{H}^2$). For every polynomial $p \in \mathbf{H}^2$,

$$\begin{aligned}
\langle (T_\varphi^* T_\varphi - T_\varphi T_\varphi^*)p,\, p\rangle &= \langle T_\varphi p, T_\varphi p\rangle - \langle T_\varphi^* p, T_\varphi^* p\rangle \\
&= \langle fp + P\overline{g}p, fp + P\overline{g}p\rangle - \langle P\overline{f}p + gp, P\overline{f}p + gp\rangle \\
&= \langle \overline{f}p, \overline{f}p\rangle - \langle P\overline{f}p, P\overline{f}p\rangle - \langle \overline{g}p, \overline{g}p\rangle + \langle P\overline{g}p, P\overline{g}p\rangle \\
&= \langle \overline{f}p, (I-P)\overline{f}p\rangle - \langle \overline{g}p, (I-p)\overline{g}p\rangle \\
&= \langle (I-P)\overline{f}p, (I-P)\overline{f}p\rangle - \langle (I-P)\overline{g}p, (I-P)\overline{g}p\rangle \\
&= ||H_{\overline{f}}p||^2 - ||H_{\overline{g}}p||^2.
\end{aligned}$$

Since polynomials are dense in $\mathbf{H}^2$,

$$T_\varphi \text{ hyponormal} \iff ||H_{\overline{g}}u|| \le ||H_{\overline{f}}u||, \quad \forall u \in \mathbf{H}^2 \tag{5.2}$$

Write $\mathcal{K} := \operatorname{cl}\operatorname{ran}(H_{\overline{f}})$ and let S be the compression of the unilateral shift U to $\mathcal{K}$. Since $\mathcal{K}$ is invariant for U^* (why: $H_{\overline{f}}U = U^* H_{\overline{f}}$), we have $S^* = U^*|_{\mathcal{K}}$. Suppose T_φ is hyponormal. Define A on $\operatorname{ran}(H_{\overline{f}})$ by

$$A(H_{\overline{f}}u) = H_{\overline{g}}u. \tag{5.3}$$

Then A is well defined because by (5.3)

$$H_{\overline{f}}u_1 = H_{\overline{f}}u_2 \implies H_{\overline{f}}(u_1 - u_2) = 0 \implies H_{\overline{g}}(u_1 - u_2) = 0.$$

By (5.2), $||A|| \le 1$, so A has an extension to $\mathcal{K}$, which will also be denoted A. Observe that

$$H_{\overline{g}}U = AH_{\overline{f}}U = AU^*H_{\overline{f}} = AS^*H_{\overline{f}} \quad \text{and} \quad H_{\overline{g}}U = U^*H_{\overline{g}} = U^*AH_{\overline{f}} = S^*AH_{\overline{f}}.$$

Thus $AS^* = S^*A$ on $\mathcal{K}$ since $\operatorname{ran}H_{\overline{f}}$ is dense in $\mathcal{K}$, and hence $SA^* = A^*S$. By Sarason's interpolation theorem,

$$\exists\, k \in \mathbf{H}^\infty(\mathbb{D}) \text{ with } ||k||_\infty = ||A^*|| = ||A|| \text{ s.t. } A^* = \text{the compression of } T_k \text{ to } \mathcal{K}.$$

Since $T_k^* H_{\overline{f}} = H_{\overline{f}} T_{k^*}$, we have that $\mathcal{K}$ is invariant for $T_k^* = T_{\overline{k}}$, which means that A is the compression of $T_{\overline{k}}$ to $\mathcal{K}$ and

$$H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}} \qquad (\text{by } (5.3)). \tag{5.4}$$

Conversely, if (5.4) holds for some $k \in \mathbf{H}^\infty(\mathbb{D})$ with $||k||_\infty \le 1$, then (5.2) holds for all u, and hence T_φ is hyponormal. Consequently,

$$T_\varphi \text{ hyponormal} \iff H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}}.$$

But $H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}}$ if and only if $\forall\, u, v \in \mathbf{H}^\infty$,

$$\begin{aligned}
\langle zuv, g\rangle = \langle H_{\overline{g}}u, v^*\rangle &= \langle T_{\overline{k}} H_{\overline{f}}u, v^*\rangle = \langle H_{\overline{f}}u, kv^*\rangle \\
&= \langle zuk^*v, f\rangle = \langle zuv, \overline{k^*}f\rangle = \langle zuv, T_{\overline{k^*}}f\rangle.
\end{aligned}$$

Since $\bigvee\{zuv : u, v \in \mathbf{H}^\infty\} = z\mathbf{H}^2$, it follows that

$$H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}} \iff g = c + T_{\overline{h}} f \text{ for } h = k^*.$$

$\square$

Theorem 5.2.4 (Nakazi-Takahashi Variation of Cowen's Theorem). *For* $\varphi \in \mathbf{L}^\infty$*, put*

$$\mathcal{E}(\varphi) := \{k \in \mathbf{H}^\infty : ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in \mathbf{H}^\infty\}.$$

Then T_φ *is hyponormal if and only if* $\mathcal{E}(\varphi) \neq \varnothing$*.*

Proof. Let $\varphi = f + \overline{g} \in \mathbf{L}^\infty$ $(f, g \in \mathbf{H}^2)$. By Cowen's theorem,

$$T_\varphi \text{ is hyponormal } \iff g = c + T_{\overline{k}} f$$

for some constant c and some $k \in \mathbf{H}^\infty$ with $||k||_\infty \le 1$. If $\varphi = k\overline{\varphi} + h$ $(h \in H^\infty)$ then $\varphi - k\overline{\varphi} = \overline{g} - k\overline{f} + f - kg \in H^\infty$. Thus $\overline{g} - k\overline{f} \in \mathbf{H}^2$, so that $P(g - \overline{k}f) = c$ ($c =$ a constant), and hence $g = c + T_{\overline{k}} f$ for some constant c. Thus T_φ is hyponormal. The argument is reversible. $\square$

5.2.2 The Case of Trigonometric Polynomial Symbols

In this section we consider the hyponormality of Toeplitz operators with trigonometric polynomial symbols. To do this we first review the dilation theory.

If $B = \begin{bmatrix} A & * \\ * & * \end{bmatrix}$, then B is called a *dilation* of A and A is called a *compression* of B. It was well-known that every contraction has a unitary dilation: indeed if $||A|| \le 1$, then

$$B \equiv \begin{bmatrix} A & (I - AA^*)^{\frac{1}{2}} \\ (I - A^*A)^{\frac{1}{2}} & -A^* \end{bmatrix}$$

is unitary.

On the other hand, an operator B is called a *power* (or *strong*) *dilation* of A if B^n is a dilation of A^n for all $n = 1, 2, 3, \cdots$. So if B is a (power) dilation of A then B should be of the form $B = \begin{bmatrix} A & 0 \\ * & * \end{bmatrix}$. Sometimes, B is called a lifting of A and A is said to be lifted to B. It was also well-known that every contraction has a isometric (power) dilation. In fact, the minimal isometric dilation of a contraction A is given by

$$B \equiv \begin{bmatrix} A & 0 & 0 & 0 & \cdots \\ (I - A^*A)^{\frac{1}{2}} & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix}.$$

We then have:

Theorem 5.2.5 (Commutant Lifting Theorem). *Let A be a contraction and T be a minimal isometric dilation of A. If $BA = AB$ then there exists a dilation S of B such that*

$$S = \begin{bmatrix} B & 0 \\ * & * \end{bmatrix}, \quad ST = TS, \quad \text{and} \quad ||S|| = ||B||.$$

Proof. See [GGK, p.658]. □

We next consider the following interpolation problem, called the Carathéodory-Schur Interpolation Problem (CSIP).

Given $c_0, \cdots, c_{N-1}$ in $\mathbb{C}$, find an analytic function φ on $\mathbb{D}$ such that

(i) $\widehat{\varphi}(j) = c_j$ $(j = 0, \cdots, N-1)$;
(ii) $||\varphi||_\infty \le 1$.

The following is a solution of CSIP.

Theorem 5.2.6.

$$\text{CSIP is solvable} \iff C \equiv \begin{bmatrix} c_0 & & & & \\ c_1 & c_0 & & \text{O} & \\ c_2 & c_1 & c_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{bmatrix} \text{ is a contraction.}$$

Moreover, φ is a solution if and only if T_φ is a contractive lifting of C which commutes with the unilateral shift.

Proof. ($\Rightarrow$) Assume that we have a solution φ. Then the condition (ii) implies

$$T_\varphi = \begin{bmatrix} \varphi_0 & & & \\ \varphi_1 & \varphi_0 & \text{O} & \\ \varphi_2 & \varphi_1 & \varphi_0 & \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \qquad (\varphi_j := \widehat{\varphi}(j))$$

is a contraction because $||T_\varphi|| = ||\varphi||_\infty \leq 1$. So the compression of T_φ is also contractive. In particular,

$$\begin{bmatrix} \varphi_0 & & & \\ \varphi_1 & \varphi_0 & \text{O} & \\ \vdots & \vdots & \ddots & \\ \varphi_{n-1} & \varphi_{n-2} & \cdots & \varphi_0 \end{bmatrix}$$

must have norm less than or equal to 1 for all n. Therefore if CSIP is solvable, then $||C|| \leq 1$.

($\Leftarrow$) Let

$$C \equiv \begin{bmatrix} c_0 & & & & \\ c_1 & c_0 & & \text{O} & \\ c_2 & c_1 & c_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{bmatrix} \text{ with } ||C|| \leq 1$$

and let

$$A := \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} : \mathbb{C}^N \rightarrow \mathbb{C}^N.$$

Then A and C are contractions and $AC = CA$. Observe that the unilateral shift U is the minimal isometric dilation of A (please check it!). By the Commutant Lifting Theorem, C can be lifted to a contraction S such that $SU = US$. But then S is an analytic Toeplitz operator, i.e., $S = T_\varphi$ with $\varphi \in \mathbf{H}^\infty$. Since S is a lifting of C we must have

$$\widehat{\varphi}(j) = c_j \ (j = 0, 1, \cdots, N-1).$$

Since S is a contraction, it follows that $||\varphi||_\infty = ||T_\varphi|| \le 1$. $\square$

Now suppose φ is a trigonometric polynomial of the form

$$\varphi(z) = \sum_{n=-N}^{N} a_n z^n \ (a_N \ne 0).$$

If a function $k \in \mathbf{H}^\infty(\mathbb{T})$ satisfies $\varphi - k\overline{\varphi} \in \mathbf{H}^\infty$ then k necessarily satisfies

$$k \sum_{n=1}^{N} \overline{a_n} z^{-n} - \sum_{n=1}^{N} a_{-n} z^{-n} \in \mathbf{H}^\infty. \tag{5.5}$$

From (5.5) one compute the Fourier coefficients $\widehat{k}(0), \cdots, \widehat{k}(N-1)$ to be $\widehat{k}(n) = c_n$ $(n = 0, 1, \cdots, N-1)$, where $c_0, c_1, \cdots, c_{N-1}$ are determined uniquely from the coefficients of φ by the following relation

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} \overline{a_1} & \overline{a_2} & \overline{a_3} & \cdots & \overline{a_N} \\ \overline{a_2} & \overline{a_3} & \cdots & \cdot & \\ \overline{a_3} & \cdots & \cdots & & \\ \vdots & \cdots & & \text{O} & \\ \overline{a_N} & & & & \end{bmatrix}^{-1} \begin{bmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-N} \end{bmatrix}. \tag{5.6}$$

Thus if $k(z) = \sum_{j=0}^{\infty} c_j z^j$ is a function in $\mathbf{H}^\infty$ then

$$\varphi - k\overline{\varphi} \in \mathbf{H}^\infty \iff c_0, c_1, \cdots, c_{N-1} \text{ are given by (5.6).}$$

Thus by Cowen's theorem, if $c_0, c_1, \cdots, c_{N-1}$ are given by (5.6) then the hyponormality of T_φ is equivalent to the existence of a function $k \in \mathbf{H}^\infty$ such that

$$\begin{cases} \widehat{k}(j) = c_j \ (j = 0, \cdots, N-1) \\ ||k||_\infty \le 1, \end{cases}$$

which is precisely the formulation of CSIP. Therefore we have:

Theorem 5.2.7. *If $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_N \ne 0$ and if $c_0, c_1, \cdots, c_{N-1}$ are given by (5.6) then*

$$T_\varphi \text{ is hyponormal} \iff C \equiv \begin{bmatrix} c_0 & & & & \\ c_1 & c_0 & & \text{O} & \\ c_2 & c_1 & c_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{bmatrix} \text{ is a contraction.}$$

5.2.3 The Case of Rational Symbols

A function $\varphi \in \mathbf{L}^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are functions ψ_1, ψ_2 in $\mathbf{H}^\infty(\mathbb{D})$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$$

for almost all z in $\mathbb{T}$. Evidently, rational functions in $\mathbf{L}^\infty$ are of bounded type.

If θ is an inner function, the degree of θ, denoted by $\deg(\theta)$, is defined by the number of zeros of θ lying in the open unit disk $\mathbb{D}$ if θ is a finite Blaschke product of the form

$$\theta(z) = e^{i\xi} \prod_{j=1}^{n} \frac{z-\beta_j}{1-\overline{\beta}_j z} \qquad (|\beta_j| < 1 \text{ for } j = 1, \cdots, n),$$

otherwise the degree of θ is infinite. For an inner function θ, write

$$\mathcal{H}(\theta) := \mathbf{H}^2 \ominus \theta \mathbf{H}^2.$$

Note that for $f \in \mathbf{H}^2$,

$$\begin{aligned}\langle [T_\varphi^*, T_\varphi] f, f\rangle = ||T_\varphi f||^2 - ||T_{\overline{\varphi}} f||^2 &= ||\varphi f||^2 - ||H_\varphi f||^2 - (||\overline{\varphi} f||^2 - ||H_{\overline{\varphi}} f||^2) \\ &= ||H_{\overline{\varphi}} f||^2 - ||H_\varphi f||^2.\end{aligned}$$

Thus we have

$$T_\varphi \text{ hyponormal} \iff ||H_{\overline{\varphi}} f|| \geq ||H_\varphi f|| \quad (f \in H^2).$$

Now let $\varphi = \overline{g} + f \in \mathbf{L}^\infty$, where f and g are in $\mathbf{H}^2$. Since $H_\varphi U = U^* H_\varphi$ (U =the unilateral shift), it follows from the Beurling's theorem that

$$\ker H_{\overline{f}} = \theta_0 \mathbf{H}^2 \quad \text{and} \quad \ker H_{\overline{g}} = \theta_1 \mathbf{H}^2 \quad \text{for some inner functions } \ \theta_0, \theta_1.$$

Thus if T_φ is hyponormal then since $||H_{\overline{f}} h|| \geq ||H_{\overline{g}} h||$ ($h \in \mathbf{H}^2$), we have

$$\theta_0 \mathbf{H}^2 = \ker H_{\overline{f}} \subset \ker H_{\overline{g}} = \theta_1 \mathbf{H}^2, \tag{5.7}$$

which implies that θ_1 divides θ_0, so that $\theta_0 = \theta_1\theta_2$ for some inner function θ_2.

On the other hand, note that if $f \in \mathbf{H}^2$ and $\overline{f}$ is of bounded type, i.e., $\overline{f} = \psi_2/\psi_1$ ($\psi_i \in \mathbf{H}^\infty$), then dividing the outer part of ψ_1 into ψ_2 one obtain $\overline{f} = \psi/\theta$ with θ inner and $\psi \in \mathbf{H}^\infty$, and hence $f = \theta\overline{\psi}$. But since $f \in \mathbf{H}^2$ we must have $\psi \in \mathcal{H}(\theta)$. Thus if $f \in \mathbf{H}^2$ and $\overline{f}$ is of bounded type then we can write

$$f = \theta\overline{\psi} \quad (\theta \text{ inner}, \ \psi \in \mathcal{H}(\theta)). \tag{5.8}$$

Therefore if $\varphi = \overline{g} + f$ is of bounded type and T_φ is hyponormal then by (5.7) and (5.8), we can write

$$f = \theta_1\theta_2\overline{a} \quad \text{and} \quad g = \theta_1\overline{b},$$

where $a \in \mathcal{H}(\theta_1\theta_2)$ and $b \in \mathcal{H}(\theta_1)$.

We now have:

Lemma 5.2.8. *Let* $\varphi = \overline{g} + f \in \mathbf{L}^\infty$*, where* f *and* g *are in* $\mathbf{H}^2$*. Assume that*

$$f = \theta_1\theta_2\overline{a} \quad \text{and} \quad g = \theta_1\overline{b} \tag{5.9}$$

for $a \in \mathcal{H}(\theta_1\theta_2)$ *and* $b \in \mathcal{H}(\theta_1)$*. Let* $\psi := \theta_1\overline{P_{\mathcal{H}(\theta_1)}(a)} + \overline{g}$*. Then* T_φ *is hyponormal if and only if* T_ψ *is.*

Proof. This assertion follows at once from [Gu2, Corollary 3.5]. $\square$

In view of Lemma 5.2.8, when we study the hyponormality of Toeplitz operators with bounded type symbols φ, we may assume that the symbol $\varphi = \overline{g} + f \in \mathbf{L}^\infty$ is of the form

$$f = \theta\overline{a} \quad \text{and} \quad g = \theta\overline{b}, \tag{5.10}$$

where θ is an inner function and $a, b \in \mathcal{H}(\theta)$ such that the inner parts of a, b and θ are coprime.

On the other hand, let $f \in \mathbf{H}^\infty$ be a rational function. Then we may write

$$f = p_m(z) + \sum_{i=1}^{n}\sum_{j=0}^{l_i-1} \frac{a_{ij}}{(1-\overline{\alpha_i}z)^{l_i-j}} \qquad (0 < |\alpha_i| < 1),$$

where $p_m(z)$ denotes a polynomial of degree m. Let θ be a finite Blaschke product of the form

$$\theta = z^m \prod_{i=1}^{n}\left(\frac{z-\alpha_i}{1-\overline{\alpha_i}z}\right)^{l_i}.$$

Observe that

$$\frac{a_{ij}}{1-\overline{\alpha_i}z} = \frac{\overline{\alpha_i}a_{ij}}{1-|\alpha_i|^2}\Big(\frac{z-\alpha_i}{1-\overline{\alpha_i}z} + \frac{1}{\overline{\alpha_i}}\Big).$$

Thus $f \in \mathcal{H}(z\theta)$. Letting $a := \theta\overline{f}$, we can see that $a \in \mathcal{H}(z\theta)$ and $f = \theta\overline{a}$. Thus if $\varphi = \overline{g} + f \in \mathbf{L}^\infty$, where f and g are rational functions and if T_φ is hyponormal, then we can write

$$f = \theta\overline{a} \quad \text{and} \quad g = \theta\overline{b}$$

for a finite Blaschke product θ with $\theta(0) = 0$ and $a, b \in \mathcal{H}(\theta)$.

Now let θ be a finite Blaschke product of degree d. We can write

$$\theta = e^{i\xi}\prod_{i=1}^{n} B_i^{n_i}, \tag{5.11}$$

where $B_i(z) := \frac{z-\alpha_i}{1-\overline{\alpha_i}z}$, $(|\alpha_i| < 1)$, $n_i \geq 1$ and $\sum_{i=1}^{n} n_i = d$. Let $\theta = e^{i\xi}\prod_{j=1}^{d} B_j$ and each zero of θ be repeated according to its multiplicity. Note that this Blaschke product is precisely the same Blaschke product in (5.11). Let

$$\phi_j := \frac{d_j}{1-\overline{\alpha_j}z}\, B_{j-1}B_{j-2}\cdots B_1 \quad (1 \leq j \leq d),$$

where $\phi_1 := d_1(1-\overline{\alpha_1}z)^{-1}$ and $d_j := (1-|\alpha_j|^2)^{\frac{1}{2}}$. It is well known that $\{\phi_j\}_1^d$ is an orthonormal basis for $\mathcal{H}(\theta)$ (cf. [FF, Theorem X.1.5]). Let $\varphi = \overline{g} + f \in \mathbf{L}^\infty$, where $g = \theta\overline{b}$ and $f = \theta\overline{a}$ for $a, b \in \mathcal{H}(\theta)$ and write

$$\mathcal{C}(\varphi) := \{k \in \mathbf{H}^\infty : \varphi - k\overline{\varphi} \in \mathbf{H}^\infty\}.$$

Then k is in $\mathcal{C}(\varphi)$ if and only if $\overline{\theta}b - k\overline{\theta}a \in \mathbf{H}^2$, or equivalently,

$$b - ka \in \theta\mathbf{H}^2. \tag{5.12}$$

Note that $\theta^{(n)}(\alpha_i) = 0$ for all $0 \le n < n_i$. Thus the condition (5.12) is equivalent to the following equation: for all $1 \le i \le n$,

$$\begin{bmatrix} k_{i,0} \\ k_{i,1} \\ k_{i,2} \\ \vdots \\ k_{i,n_i-2} \\ k_{i,n_i-1} \end{bmatrix} = \begin{bmatrix} a_{i,0} & 0 & 0 & 0 & \cdots & 0 \\ a_{i,1} & a_{i,0} & 0 & 0 & \cdots & 0 \\ a_{i,2} & a_{i,1} & a_{i,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{i,n_i-2} & a_{i,n_i-3} & \ddots & \ddots & a_{i,0} & 0 \\ a_{i,n_i-1} & a_{i,n_i-2} & \cdots & a_{i,2} & a_{i,1} & a_{i,0} \end{bmatrix}^{-1} \begin{bmatrix} b_{i,0} \\ b_{i,1} \\ b_{i,2} \\ \vdots \\ b_{i,n_i-2} \\ b_{i,n_i-1} \end{bmatrix}, \tag{5.13}$$

where

$$k_{i,j} := \frac{k^{(j)}(\alpha_i)}{j!}, \quad a_{i,j} := \frac{a^{(j)}(\alpha_i)}{j!} \quad \text{and} \quad b_{i,j} := \frac{b^{(j)}(\alpha_i)}{j!}.$$

Conversely, if $k \in \mathbf{H}^\infty$ satisfies the equality (5.13) then k must be in $\mathcal{C}(\varphi)$. Thus k belongs to $\mathcal{C}(\varphi)$ if and only if k is a function in $\mathbf{H}^\infty$ for which

$$\frac{k^{(j)}(\alpha_i)}{j!} = k_{i,j} \qquad (1 \le i \le n,\ 0 \le j < n_i), \tag{5.14}$$

where the $k_{i,j}$ are determined by the equation (5.13). If in addition $||k||_\infty \le 1$ is required then this is exactly the classical Hermite-Fejér Interpolation Problem (HFIP). Therefore we have:

Theorem 5.2.9. *Let $\varphi = \overline{g} + f \in \mathbf{L}^\infty$, where f and g are rational functions. Then T_φ is hyponormal if and only if the corresponding HFIP (5.14) is solvable.*

Now we can summarize that tractable criteria for the hyponormality of Toeplitz operators T_φ are accomplished for the cases where the symbol φ is a trigonometric polynomial or a rational function via solutions of some interpolation problems.

We conclude this section with:

Problem 5.1. *Let $\varphi \in \mathbf{L}^\infty$ be arbitrary. Find necessary and sufficient conditions, in terms of the coefficients of φ, for T_φ to be hyponormal. In particular, for the cases where φ is of bounded type.*

5.3 Subnormality of Toeplitz operators

The present chapter concerns the question: *Which Toeplitz operators are subnormal?* Recall that a Toeplitz operator T_φ is called analytic if φ is in $\mathbf{H}^\infty$, that is, φ is a bounded analytic function on $\mathbb{D}$. These are easily seen to be subnormal: $T_\varphi h = P(\varphi h) = \varphi h = M_\varphi h$ for $h \in \mathbf{H}^2$, where M_φ is the normal operator of multiplication by φ on $\mathbf{L}^2$. P.R. Halmos raised the following problem, so-called the *Halmos's Problem 5* in his 1970 lectures "Ten Problems in Hilbert Space" [Ha1], [Ha2]:

> Is every subnormal Toeplitz operator either normal or analytic ?

The question is natural because the two classes, the normal and analytic Toeplitz operators, are fairly well understood and are obviously subnormal.

5.3.1 Halmos's Problem 5

We begin with a brief survey of research related to P.R. Halmos's Problem 5.

In 1976, M. Abrahamse [Ab] gave a general sufficient condition for the answer to the Halmos's Problem 5 to be affirmative.

Theorem 5.3.1 (Abrahamse's Theorem). *If*

(i) T_φ *is hyponormal;*
(ii) φ *or* $\overline{\varphi}$ *is of bounded type;*
(iii) $\ker[T_\varphi^*, T_\varphi]$ *is invariant for* T_φ,

then T_φ *is normal or analytic.*

Proof. See [Ab]. □

On the other hand, observe that if S is a subnormal operator on $\mathcal{H}$ and if $N := \text{mne}\,(S)$ then

$$\ker[S^*, S] = \{f : < f, [S^*, S]f >= 0\} = \{f : \; ||S^* f|| = ||Sf||\} = \{f : \; N^* f \in \mathcal{H}\}.$$

Therefore, $S(\ker[S^*, S]) \subseteq \ker[S^*, S]$.

By Theorem 5.3.1 and the preceding remark we get:

Corollary 5.3.2. *If* T_φ *is subnormal and if* φ *or* $\overline{\varphi}$ *is of bounded type, then* T_φ *is normal or analytic.*

Lemma 5.3.3. *A function* φ *is of bounded type if and only if* $\ker H_\varphi \neq \{0\}$.

Proof. If $\ker H_\varphi \neq \{0\}$ then since $H_\varphi f = 0 \Rightarrow (1-P)\varphi f = 0 \Rightarrow \varphi f = P\varphi f := g$, we have

$$\exists\ f, g \in \mathbf{H}^2 \text{ s.t. } \varphi f = g.$$

Hence $\varphi = \frac{g}{f}$. Remembering that if $\frac{1}{\varphi} \in \mathbf{L}^\infty$ then φ is outer if and only if $\frac{1}{\varphi} \in \mathbf{H}^\infty$ and dividing the outer part of f into g gives

$$\varphi = \frac{\psi}{\theta} \quad (\psi \in \mathbf{H}^\infty, \theta \text{ inner}).$$

Conversely, if $\varphi = \frac{\psi}{\theta}$ ($\psi \in \mathbf{H}^\infty$, θ inner), then $\theta \in \ker H_\varphi$ because $\varphi\theta = \psi \in \mathbf{H}^\infty \Rightarrow (1-P)\varphi\theta = 0 \Rightarrow H_\varphi\theta = 0$. □

From Theorem 5.3.1 we can see that

$$\varphi = \frac{\psi}{\theta}\ (\theta, \psi \text{ inner}),\ T_\varphi \text{ subnormal } \Rightarrow\ T_\varphi \text{ normal or analytic} \tag{5.15}$$

The following proposition strengthen the conclusion of (5.15), whereas weakens the hypothesis of (5.15).

Proposition 5.3.4. *If* $\varphi = \frac{\psi}{\theta}$ (θ, ψ inner) *and if* T_φ *is hyponormal, then* T_φ *is analytic.*

Proof. Observe that

$$\begin{aligned} 1 = ||\theta|| = ||P(\theta)|| = ||P(\overline{\varphi}\theta\varphi)|| = ||P(\overline{\varphi}\psi)|| \\ = ||T_{\overline{\varphi}}(\psi)|| \leq ||T_\varphi(\psi)|| = ||P(\frac{\psi^2}{\theta})|| \leq ||\frac{\psi^2}{\theta}|| = 1, \end{aligned}$$

which implies that $\frac{\psi^2}{\theta} \in \mathbf{H}^2$, so θ divides ψ^2. Thus if one choose ψ and θ to be relatively prime (i.e., if $\varphi = \frac{\psi}{\theta}$ is in lowest terms), then θ is constant. Therefore T_φ is analytic. □

Proposition 5.3.5. *If A is a weighted shift with weights $a_0, a_1, a_2, \cdots$ such that*

$$0 \leq a_0 \leq a_1 \leq \cdots < a_N = a_{N+1} = \cdots = 1,$$

then A is not unitarily equivalent to any Toeplitz operator.

Proof. Note that A is hyponormal, $||A|| = 1$ and A attains its norm. If A is unitarily equivalent to T_φ then by a result of Brown and Douglas [BD], T_φ is hyponormal and $\varphi = \frac{\psi}{\theta}$ (θ, ψ inner). By Proposition 5.3.4, $T_\varphi \equiv T_\psi$ is an isometry, so $a_0 = 1$, a contradiction. □

Recall that the Bergman shift (whose weights are given by $\sqrt{\frac{n+1}{n+2}}$) is subnormal. The following question arises naturally:

$$\textit{Is the Bergman shift unitarily equivalent to a Toeplitz operator?} \tag{5.16}$$

An affirmative answer to the question (5.16) gives a negative answer to Halmos's Problem 5. To see this, assume that the Bergman shift S is unitarily equivalent to T_φ, then

$$\mathfrak{R}(\varphi) \subseteq \sigma_e(T_\varphi) = \sigma_e(S) = \text{the unit circle } \mathbb{T}.$$

Thus φ is unimodular. Since S is not an isometry it follows that φ is not inner. Therefore T_φ is not an analytic Toeplitz operator.

To answer the question (5.16) we need an auxiliary lemma:

Lemma 5.3.6. *If a Toeplitz operator T_φ is a weighted shift with weights $\{a_n\}_{n=0}^\infty$ with respect to the orthonormal basis $\{e_n\}_{n=0}^\infty$, i.e.,*

$$T_\varphi e_n = a_n e_{n+1} \ (n \geq 0) \tag{5.17}$$

then $e_0(z)$ is an outer function.

Proof. By Coburn's theorem, $\ker T_\varphi = \{0\}$ or $\ker T_\varphi^* = \{0\}$. The expression (5.17) gives $e_0 \in \ker T_\varphi^*$, and hence $\ker T_\varphi = \{0\}$. Thus $a_n > 0$ $(n \geq 0)$. Write

$$e_0 := gF, \text{ where } g \text{ is inner and } F \text{ is outer.}$$

Because $T_\varphi^* e_0 = 0$, we get

$$T_\varphi^* F = T_{\overline{\varphi}}(\overline{g} e_0) = T_{\overline{g}} T_{\overline{\varphi}} e_0 = T_{\overline{g}} T_\varphi^* e_0 = 0.$$

Note that $\dim \ker T_\varphi^* = 1$. So we have $F = ce_0$ (c =a constant), so that g is a constant, and hence e_0 is an outer function. $\square$

Theorem 5.3.7 (Sun's Theorem). *Let T be a weighted shift with a strictly increasing weight sequence $\{a_n\}_{n=0}^\infty$. If $T \cong T_\varphi$ then*

$$a_n = \sqrt{1 - \alpha^{2n+2}}\, ||T_\varphi|| \quad (0 < \alpha < 1).$$

Proof. Assume $T \cong T_\varphi$. We assume, without loss of generality, that $||T|| = 1$ (so $a_n < 1$). Since T is a weighted shift, $\sigma_e(T) = \{z : |z| = 1\}$. Since $\mathfrak{R}(\varphi) \subset \sigma_e(T_\varphi)$, it follows that $|\varphi| = 1$, i.e., φ is unimodular. By Lemma 5.3.6,

$$\exists \text{ an orthonormal basis } \{e_n\}_{n=0}^\infty \text{ such that (5.17) holds.}$$

Expression (5.17) can be written as follows:

$$\begin{cases} \varphi e_n = a_n e_{n+1} + \sqrt{1 - a_n^2}\, \eta_n \\ \overline{\varphi} e_{n+1} = a_n e_n + \sqrt{1 - a_n^2}\, \xi_n \end{cases} \tag{5.18}$$

where $\eta_n, \xi_n \in (\mathbf{H}^2)^\perp$ and $||\eta_n|| = ||\xi_n|| = 1$. Since $\{\varphi e_n\}_{n=0}^\infty$ is an orthonomal system and $a_n < 1$, we have

$$< \eta_\ell, \eta_k > = < \xi_\ell, \xi_k > = \begin{cases} 0, & \ell \neq k \\ 1, & \ell = k \end{cases} \tag{5.19}$$

From (5.18) we have

$$e_n = \overline{\varphi}\left(a_n e_{n+1} + \sqrt{1-a_n^2}\;\eta_n\right) = a_n^2 e_n + a_n\sqrt{1-a_n^2}\;\xi_n + \sqrt{1-a_n^2}\;\overline{\varphi}\,\eta_n. \tag{5.20}$$

Then (5.20) is equivalent to

$$\varphi\overline{\eta_n} = -a_n\xi_n + \sqrt{1-a_n^2}\;\overline{e_n}. \tag{5.21}$$

Set $d_n := \frac{\overline{\eta_n}}{t}$ and $\rho_n := \frac{\overline{\xi_n}}{t}$ ($|t|=1$). Then () is equivalent to

$$\varphi d_n = -a_n\rho_n + \sqrt{1-a_n^2}\;\frac{\overline{e_n}}{t}. \tag{5.22}$$

Since $\frac{\overline{e_n}}{t} \in (\mathbf{H}^2)^\perp$ and $\{d_n\}_{n=0}^\infty$ is an orthonormal basis for $\mathbf{H}^2$, we can see that

$$\begin{cases} ||T_\varphi d_0|| = a_0 = \inf_{||x||=1}||T_\varphi x|| = ||T_\varphi e_0|| \\ ||T_\varphi d_\ell|| = a_\ell = ||T_\varphi e_\ell||\ . \end{cases} \tag{5.23}$$

Then (5.17) and (5.23) imply

$$d_n = r_n e_n \quad (|r_n| = 1). \tag{5.24}$$

Substituting (5.24) into (5.23) and comparing it with (5.18) gives

$$a_n e_{n+1} + \sqrt{1-a_n^2}\,\eta_n = \varphi e_n = -\frac{a_n}{r_n}\,\rho_n + \frac{\sqrt{1-a_n^2}}{r_n}\;\frac{\overline{e_n}}{t},$$

which implies

$$\begin{cases} -\overline{r_n}\rho_n = e_{n+1} \\ \overline{r_n}\frac{\overline{e_n}}{t} = \eta_n\ . \end{cases} \tag{5.25}$$

Therefore (5.18) is reduced to:

$$\begin{cases} \varphi e_n = a_n e_{n+1} + \sqrt{1-a_n^2}\,\overline{r_n}\,\frac{\overline{e_n}}{t} \\ \overline{\varphi} e_{n+1} = a_n e_n - \sqrt{1-a_n^2}\,\overline{r_n}\,\frac{\overline{e_{n+1}}}{t} \end{cases} \tag{5.26}$$

Put $e_{-(n+1)} := \frac{\overline{e_n}}{t} \in (\mathbf{H}^2)^\perp$ ($n \geq 0$). We now claim that

$$\overline{\varphi} e_0 = r e_{-1}\ (|r| = 1) : \tag{5.27}$$

indeed, $T_{\overline{\varphi}}\left(\frac{\varphi\overline{e_0}}{t}\right) = P(\frac{\overline{e_0}}{t}) = 0$, so $e_0 = r\frac{\varphi\overline{e_0}}{t}$ for $|r| = 1$, and hence $\overline{\varphi} e_0 = r e_{-1}$. From (5.26) we have

$$\varphi e_0 = a_0 e_1 + \overline{r_0}\,\sqrt{1-a_0^2}\,e_{-1} = a_0 e_1 + \overline{r_0}\,\overline{r}\,\sqrt{1-a_0^2}\,\overline{\varphi} e_0, \tag{5.28}$$

or, equivalently,

$$\left(\varphi - \overline{r_0}\,\overline{r}\sqrt{1-a_0^2}\,\overline{\varphi}\right) e_0 = a_0 e_1. \tag{5.29}$$

Write

$$\psi \equiv \varphi - \overline{r_0}\,\overline{r}\sqrt{1-a_0^2}\,\overline{\varphi}. \tag{5.30}$$

Evidently,

$$V := \{x \in \mathbf{H}^2 : \psi x \in \mathbf{H}^2\}$$

is not empty. Moreover, since V is invariant for U, it follows from Beurling's theorem that

$$V = \chi\mathbf{H}^2 \text{ for an inner function } \chi.$$

Since $e_0 \in V$ and e_0 is an outer function, we must have $\chi = 1$. This means that $\psi = \psi \cdot 1 \in \mathbf{H}^2$. Therefore $\psi e_1 = T_\psi e_1 \in \mathbf{H}^2$. On the other hand, by (5.26),

$$\begin{aligned}
\psi e_1 &= \left(\varphi - \overline{r_0}\,\overline{r}\sqrt{1-a_0^2}\,\overline{\varphi}\right) e_1 \\
&= a_1 e_2 + \overline{r_1}\sqrt{1-a_1^2}e_{-2} - \overline{r_0}\,\overline{r}\sqrt{1-a_0^2}\left(a_0 e_0 - \sqrt{1-a_0^2}\,\overline{r_0}\,e_{-2}\right) \\
&= a_1 e_2 - \overline{r_0}\,\overline{r} a_0 \sqrt{1-a_1^2}\,e_0 + \left(\overline{r_1}\sqrt{1-a_1^2} + \overline{r}\,\overline{r_0}^2\,(1-a_1^2)\right) e_{-2}.
\end{aligned}$$

Thus we have

$$\overline{r_1}\sqrt{1-a_1^2} + \overline{r}\,\overline{r_0}^2(1-a_0^2) = 0$$

So, $\sqrt{1-a_1^2} = 1-a_0^2$, i.e., $a_1 = \sqrt{1-(1-a_0^2)^2}$. If we put $\alpha^2 \equiv 1-a_0^2$, i.e., $a_0 = (1-\alpha^2)^{\frac{1}{2}}$ then $a_1 = (1-\alpha^4)^{\frac{1}{2}}$. Inductively, we get $a_n = (1-\alpha^{2n+2})^{\frac{1}{2}}$. ☐

Corollary 5.3.8. *The Bergman shift is not unitarily equivalent to any Toeplitz operator.*

Proof. $\frac{n+1}{n+2} \neq 1-\alpha^{2n+2}$ for any $\alpha > 0$. ☐

Lemma 5.3.9. *The weighted shift $T \equiv W_\alpha$ with weights $\alpha_n \equiv (1-\alpha^{2n+2})^{\frac{1}{2}}$ $(0 < \alpha < 1)$ is subnormal.*

Proof. Write $r_n := \alpha_0^2\alpha_1^2\cdots\alpha_{n-1}^2$ for the moment of W. Define a discrete measure μ on $[0,1]$ by

$$\mu(z) = \begin{cases} \Pi_{j=1}^{\infty}(1-\alpha^{2j}) & (z=0) \\ \Pi_{j=1}^{\infty}(1-\alpha^{2j})\frac{\alpha^{2k}}{(1-\alpha^2)\cdots(1-\alpha^{2k})} & (z=\alpha^k; k=1,2,\cdots). \end{cases}$$

Then $r_n = \int_0^1 t^n d\mu$. By Berger's theorem, T is subnormal. ☐

Corollary 5.3.10. *If $T_\varphi \cong$ a weighted shift, then T_φ is subnormal.*

Remark 5.3.11. If $T_\varphi \cong$ a weighted shift, what is the form of φ ? A careful analysis of the proof of Theorem 5.3.7 shows that

$$\psi = \varphi - \alpha\overline{\varphi} \in \mathbf{H}^\infty .$$

But

$$\begin{aligned} T_\psi = T_\varphi - \alpha T_\varphi^* &= \begin{bmatrix} 0 & -\alpha a_0 & & & \\ a_0 & 0 & -\alpha a_1 & & \\ & a_1 & 0 & -\alpha a_2 & \\ & & a_2 & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\alpha & & & \\ 1 & 0 & -\alpha & & \\ & 1 & 0 & -\alpha & \\ & & 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} + K \quad (K \text{ compact}) \\ &\cong T_{z-\alpha\overline{z}} + K. \end{aligned}$$

Thus $\operatorname{ran}(\psi) = \sigma_e(T_\psi) = \sigma_e(T_{z-\alpha\overline{z}}) = \operatorname{ran}(z-\alpha\overline{z})$. Thus ψ is a conformal mapping of $\mathbb{D}$ onto the interior of the ellipse with vertices $\pm i(1+\alpha)$ and passing through $\pm(1-\alpha)$. On the other hand, $\psi = \varphi - \alpha\overline{\varphi}$. So $\alpha\overline{\psi} = \alpha\overline{\varphi} - \alpha^2\varphi$, which implies

$$\varphi = \frac{1}{1-\alpha^2}(\psi + \alpha\overline{\psi}).$$

We now have:

Theorem 5.3.12 (Cowen and Long's Theorem). *For $0 < \alpha < 1$, let ψ be a conformal map of $\mathbb{D}$ onto the interior of the ellipse with vertices $\pm i(1-\alpha)^{-1}$ and passing through $\pm(1+\alpha)^{-1}$. Then $T_{\psi+\alpha\overline{\psi}}$ is a subnormal weighted shift that is neither analytic nor normal.*

Proof. Let $\varphi = \psi + \alpha\overline{\psi}$. Then φ is a continuous map of $\mathbb{D}$ onto $\mathbb{D}$ with $\operatorname{wind}(\varphi) = 1$. Let

$$K := 1 - T_{\overline{\varphi}}T_\varphi = T_{\overline{\varphi}\varphi} - T_{\overline{\varphi}}T_\varphi = H_\varphi^* H_\varphi,$$

which is compact since φ is continuous. Now $\varphi - \alpha\overline{\varphi} = (1-\alpha^2)\psi \in \mathbf{H}^\infty$, so $H_\psi = 0$ and hence, $H_\varphi = \alpha H_{\overline{\varphi}}$. Thus

$$K = H_\varphi^* H_\varphi = \alpha^2 H_{\overline{\varphi}}^* H_{\overline{\varphi}} = \alpha^2(1 - T_\varphi T_{\overline{\varphi}}),$$

so that

$$K T_\varphi = \alpha^2(1 - T_\varphi T_{\overline{\varphi}})T_\varphi = \alpha^2 T_\varphi(1 - T_{\overline{\varphi}}T_\varphi) = \alpha^2 T_\varphi K.$$

By Coburn's theorem, $\ker T_\varphi = \{0\}$ or $\ker T_{\overline{\varphi}} = \{0\}$. But since

$$\operatorname{ind}(T_\varphi) = -\operatorname{wind}(\varphi) = -1,$$

it follows

$$\ker T_\varphi = \{0\} \text{ and } \dim \ker T_{\overline{\varphi}} = 1.$$

Let $e_0 \in \ker T_{\overline{\varphi}}$ and $||e_0|| = 1$. Write

$$e_{n+1} := \frac{T_\varphi e_n}{||T_\varphi e_n||}.$$

We claim that $Ke_n = \alpha^{2n+2}e_n$: indeed, $Ke_0 = \alpha^2(1 - T_\varphi T_{\overline{\varphi}})e_0 = \alpha^2 e_0$ and if we assume $Ke_j = \alpha^{2j+2}e_j$ then

$$Ke_{j+1} = ||T_\varphi e_j||^{-1}(KT_\varphi e_j) = ||T_\varphi e_j||^{-1}(\alpha^2 T_\varphi K e_j) = ||T_\varphi e_j||^{-1}(\alpha^{2j+4}T_\varphi e_j) = \alpha^{2j+4}e_{j+1}.$$

Thus we can see that

$$\begin{cases} \alpha^2, \alpha^4, \alpha^6, \cdots \text{ are eigenvalues of } K\,; \\ \{e_n\}_{n=0}^\infty \text{ is an orthonormal set since } K \text{ is self-adjoint.} \end{cases}$$

We will then prove that $\{e_n\}$ forms an orthonormal basis for $\mathbf{H}^2$. Observe

$$\operatorname{tr}(H_\varphi^* H_\varphi) = \text{the sum of its eigenvalues.}$$

Thus

$$\sum_{n=0}^\infty \alpha^{2n+2} \le \operatorname{tr}(H_\varphi^* H_\varphi) = ||H_\varphi||_2^2 \quad (||\cdot||_2 \text{ denotes the Hilbert-Schmidt norm}). \tag{5.31}$$

Since $\psi \in \mathbf{H}^\infty$, we have

$$\begin{aligned} ||H_\varphi||_2^2 = ||H_\psi + \alpha H_{\overline{\psi}}||_2^2 = \alpha^2 ||H_{\overline{\psi}}||_2^2 = \alpha^2 \operatorname{tr}(H_{\overline{\psi}}^* H_{\overline{\psi}}) = \alpha^2 \operatorname{tr}\,[T_{\overline{\psi}}, T_\psi] \\ \le \frac{\alpha^2}{\pi}\mu(\sigma(T_\psi)) = \frac{\alpha^2}{\pi}\mu(\psi(\mathbb{D})) = \frac{\alpha^2}{1-\alpha^2}, \end{aligned}$$

which together with (5.31) implies that

$$\sum \alpha^{2n+2} \le ||H_\varphi||_2^2 \le \frac{\alpha^2}{1-\alpha^2} = \sum_{n=0}^\infty \alpha^{2n+2},$$

so $\operatorname{tr}(H_\varphi^* H_\varphi) = \sum_{n=0}^\infty \alpha^{2n+2}$, which say that $\{\alpha^{2n+2}\}_{n=0}^\infty$ is a complete set of nonzero eigenvalues for $K \equiv H_\varphi^* H_\varphi$ and each has multiplicity one. Now, by Beurling's theorem,

$$\ker K = \ker H_\varphi^* H_\varphi = \ker H_\varphi = b\mathbf{H}^2, \text{ where } b \text{ is inner or } b = 0.$$

Since $KT_\varphi = \alpha^2 T_\varphi K$, we see that

$$f \in \ker K \;\Rightarrow\; T_\varphi f \in \ker K$$

So, since $b \in \ker K$, it follows

$$T_\varphi b = b\varphi - H_\varphi b = b\varphi \in \ker K,$$

which means that $b\varphi = bh$ for some $h \in \mathbf{H}^2$. Since $\varphi \notin \mathbf{H}^2$ it follows that $b = 0$ and $\ker K = 0$. Thus 0 is not an eigenvalue. Therefore $\{e_n\}_{n=0}^\infty$ is an onthonormal basis for $\mathbf{H}^2$. Remember that $T_\varphi e_n = ||T_\varphi e_n||e_{n+1}$. So we can see that T_φ is a weighted shift with weights $\{||T_\varphi e_n||\}$. Since

$$\alpha^{2n+2} e_n = K e_n = (1 - T_{\overline{\varphi}} T_\varphi) e_n,$$

we have

$$(1 - \alpha^{2n+2}) e_n = T_{\overline{\varphi}} T_\varphi e_n,$$

so that

$$1 - \alpha^{2n+2} = \langle (1 - \alpha^{2n+2}) e_n, e_n \rangle = \langle T_{\overline{\varphi}} T_\varphi e_n,\ e_n \rangle = ||T_\varphi e_n||^2.$$

Thus the weights are $(1 - \alpha^{2n+2})^{\frac{1}{2}}$. By Lemma 5.3.9, T_φ is subnormal. Evidently, $\varphi \notin \mathbf{H}^\infty$ and T_φ is not normal since ran(φ) is not contained in a line segment. ☐

Corollary 5.3.13. *If $\varphi = \psi + \alpha\overline{\psi}$ is as in Theorem 5.3.12, then neither φ nor $\overline{\varphi}$ is bounded type.*

Proof. From Abrahamse's theorem and Theorem 5.3.12. ☐

We will present a couple of open problems which are related to the subnormality of Toeplitz operators. They are of particular interest in operator theory.

Problem 5.2. *For which $f \in \mathbf{H}^\infty$, is there λ $(0 < \lambda < 1)$ with $T_{f+\lambda\overline{f}}$ subnormal ?*

Problem 5.3. *Suppose ψ is as in Theorem 5.3.12 (i.e., the ellipse map). Are there $g \in \mathbf{H}^\infty$, $g \neq \lambda\psi + c$, such that $T_{\psi+\overline{g}}$ is subnormal ?*

Problem 5.4. *More generally, if $\psi \in \mathbf{H}^\infty$, define*

$$\mathcal{S}(\psi) := \{g \in \mathbf{H}^\infty :\ T_{\psi+\overline{g}} \text{ is subnormal } \}.$$

Describe $\mathcal{S}(\psi)$. For example, for which $\psi \in \mathbf{H}^\infty$, is it balanced?, or is it convex?, or is it weakly closed? What is $\operatorname{ext}\mathcal{S}(\psi)$? For which $\psi \in \mathbf{H}^\infty$, is it strictly convex ?, i.e., $\partial\mathcal{S}(\psi) \subset \operatorname{ext}\mathcal{S}(\psi)$?

In general, $\mathcal{S}(\psi)$ is not convex. In the below (Theorem 5.3.14), we will show that if ψ is as in Theorem 5.3.12 then $\{\lambda :\ T_{\psi+\lambda\overline{\psi}}$ is subnormal$\}$ is a non-convex set.

C. Cowen gave an interesting remark with no demonstration in [Cow3]: *If T_φ is subnormal then $\mathcal{E}(\varphi) = \{\lambda\}$ with $|\lambda| < 1$.* However we were unable to decide whether or not it is true. By comparison, if T_φ is normal then $\mathcal{E}(\varphi) = \{e^{i\theta}\}$.

Problem 5.5. *Is the above Cowen's remark true ? That is, if T_φ is subnormal, does it follow that $\mathcal{E}(\varphi) = \{\lambda\}$ with $|\lambda| < 1$?*

If the answer to Problem 5.5 is affirmative, i.e., the Cowen's remark is true then for $\varphi = \overline{g} + f$,

$$T_\varphi \text{ is subnormal } \Longrightarrow \overline{g} - \lambda\overline{f} \in \mathbf{H}^2 \text{ with } |\lambda| < 1 \Longrightarrow g = \overline{\lambda} f + c \ (c \text{ a constant}),$$

which says that the answer to Problem 5.3 is negative.

When ψ is as in Theorem 5.3.12, we examine the question: For which λ, is $T_{\psi+\lambda\overline{\psi}}$ subnormal ?

We then have:

Theorem 5.3.14. *Let $\lambda \in \mathbb{C}$ and $0 < \alpha < 1$. Let ψ be the conformal map of the disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ passing through $\pm(1-\alpha)$. For $\varphi = \psi + \lambda\overline{\psi}$, T_φ is subnormal if and only if $\lambda = \alpha$ or $\lambda = \frac{\alpha^k e^{i\theta}+\alpha}{1+\alpha^{k+1}e^{i\theta}}$ $(-\pi < \theta \leq \pi)$.*

To prove Theorem 5.3.14, we need an auxiliary lemma:

Proposition 5.3.15. *Let T be the weighted shift with weights*

$$w_n^2 = \sum_{j=0}^{n} \alpha^{2j}.$$

Then $T + \mu T^*$ is subnormal if and only if $\mu = 0$ or $|\mu| = \alpha^k$ $(k = 0, 1, 2, \cdots)$.

Proof. See [CoL]. ☐

Proof of Theorem 5.3.14. By Theorem 5.3.12, $T_{\psi+\alpha\overline{\psi}} \cong (1-\alpha^2)^{\frac{3}{2}} T$, where T is a weighted shift of Proposition 5.3.15. Thus $T_\psi \cong (1-\alpha^2)^{\frac{1}{2}}(T - \alpha T^*)$, so

$$T_\varphi = T_\psi + \lambda T_\psi^* \cong (1-\alpha^2)^{\frac{1}{2}}(1-\lambda\alpha)\left(T + \frac{\lambda-\alpha}{1-\lambda\alpha}T^*\right).$$

Applying Proposition 5.3.15 with $\frac{\lambda-\alpha}{1-\lambda\alpha}$ in place of μ gives that for $k = 0, 1, 2, \cdots$,

$$\begin{aligned} \left|\frac{\lambda-\alpha}{1-\lambda\alpha}\right| = \alpha^k \iff & \frac{\lambda-\alpha}{1-\lambda\alpha} = \alpha^k e^{i\theta} \\ \iff & \lambda - \alpha = \alpha^k e^{i\theta} - \lambda\alpha^{k+1}e^{i\theta} \\ \iff & \lambda(1+\alpha^{k+1}e^{i\theta}) = \alpha + \alpha^k e^{i\theta} \\ \iff & \lambda = \frac{\alpha + \alpha^k e^{i\theta}}{1+\alpha^{k+1}e^{i\theta}} \ (-\pi < \theta \leq \pi) \end{aligned}$$

☐

However we find that, surprisingly, some analytic Toeplitz operators are unitarily equivalent to some non-analytic Toeplitz operators. So C. Cowen noted that subnormality of Toeplitz operators may not be the *wrong* question to be studying.

Example 5.3.16. *Let ψ be the ellipse map as in the example of Cowen and Long. Then*

$$T_\psi \cong T_\varphi \text{ with } \varphi = \frac{i\, e^{-\frac{i\theta}{2}}(1+\alpha^2 e^{i\theta})}{1-\alpha^2}\left(\psi \;+\; \frac{\alpha e^{i\theta}+\alpha}{1+\alpha^2 e^{i\theta}}\;\overline{\psi}\right) \qquad (-\pi < \theta \le \pi)$$

Proof. Note that

$$T \cong e^{\frac{i\theta}{2}} T \quad \text{and} \quad T + \lambda T^* \cong e^{\frac{i\theta}{2}} T + \lambda e^{-\frac{i\theta}{2}} T^*.$$

Thus we have

$$\begin{aligned}
T_\psi &\cong (1-\alpha^2)^{\frac{1}{2}}(T-\alpha T^*)\\
&\cong (1-\alpha^2)^{\frac{1}{2}} i (T+\alpha T^*)\\
&\cong (1-\alpha^2)^{\frac{1}{2}} i e^{-\frac{i\theta}{2}}(T+\alpha e^{i\theta} T^*)\\
&\cong (1-\alpha^2)^{-1} i e^{-\frac{i\theta}{2}}\Big(T_\psi + \alpha T_{\overline{\psi}} + \alpha e^{i\theta}(T_{\overline{\psi}} + \alpha T_\psi)\Big)\\
&\cong (1-\alpha^2)^{-1} i\, e^{-\frac{i\theta}{2}} T_{(1+\alpha^2 e^{i\theta})\psi + \alpha(1+e^{i\theta})\overline{\psi}} \qquad (-\pi < \theta < \pi)\\
&\cong \frac{i\, e^{-\frac{i\theta}{2}}(1+\alpha^2 e^{i\theta})}{1-\alpha^2} T_{\psi + \frac{\alpha e^{i\theta}+\alpha}{1+\alpha^2 e^{i\theta}}\overline{\psi}} \qquad (-\pi < \theta \le \pi).
\end{aligned}$$

□

Problem 5.6. *Let ψ be the ellipse map as in the example of Cowen and Long. Is $T_{\psi+\alpha\overline{\psi}} \cong T_\zeta$ for some $\zeta \in \mathbf{H}^\infty$?*

If the answer to Problem 5.6 would be affirmative then we could say that Halmos's Problem 5 remains still open. In this case we have a reformulation of Halmos's Problem 5:

If T_φ is a non-normal subnormal Toeplitz operator, does it follow that

$$T_\varphi \cong T_\psi \quad \textit{for some } \psi \in \mathbf{H}^\infty\,?$$

5.3.2 Weak Subnormality

Now it seems to be interesting to understand the gap between k-hyponormality and subnormality for Toeplitz operators. As a candidate for the first question in this line we posed the following ([CuL1]):

Problem 5.7. *Is every 2-hyponormal Toeplitz operator subnormal?*

In [CuL1], the following was shown:

Theorem 5.3.17. [CuL1] *Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic. Hence, in particular, every 2-hyponormal trigonometric Toeplitz operator is subnormal.*

It is well known ([Cu1]) that there is a gap between hyponormality and 2–hyponormality for weighted shifts. Theorem 5.3.17 also shows that there is a big gap between hyponormality and 2-hyponormality for Toeplitz operators. For example, if

$$\varphi(z) = \sum_{n=-m}^{N} a_n z^n \quad (m < N)$$

is such that T_φ is hyponormal then by Theorem 5.3.17, T_φ is never 2-hyponormal because T_φ is neither analytic nor normal (recall that if $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ is such that T_φ is normal then $m = N$ (cf. [FL1])).

We can extend Theorem 5.3.17 First of all we observe:

Proposition 5.3.18. [CuL2] If $T \in \mathcal{L}(\mathcal{H})$ is 2-hyponormal then

$$T\big(\ker [T^*, T]\big) \subseteq \ker [T^*, T]. \tag{5.32}$$

Proof. Suppose that $[T^*, T]f = 0$. Since T is 2-hyponormal, it follows that (cf. [CMX, Lemma 1.4])

$$|\langle [T^{*2}, T]g, f\rangle|^2 \le \langle [T^*, T]f, f\rangle \langle [T^{*2}, T^2]g, g\rangle \quad \text{for all } g \in \mathcal{H}.$$

By assumption, we have that for all $g \in \mathcal{H}$, $0 = \langle [T^{*2}, T]g, f\rangle = \langle g, [T^{*2}, T]^* f\rangle$, so that $[T^{*2}, T]^* f = 0$, i.e., $T^* T^2 f = T^2 T^* f$. Therefore,

$$[T^*, T]Tf = (T^*T^2 - TT^*T)f = (T^2T^* - TT^*T)f = T[T^*, T]f = 0,$$

which proves (5.32). □

Corollary 5.3.19. *If T_φ is 2-hyponormal and if φ or $\bar{\varphi}$ is of bounded type then T_φ is normal or analytic, so that T_φ is subnormal.*

Proof. This follows at once from Abrahamse's theorem and Proposition 5.3.18. □

Corollary 5.3.20. *If T_φ is a 2-hyponormal operator such that $\mathcal{E}(\varphi)$ contains at least two elements then T_φ is normal or analytic, so that T_φ is subnormal.*

Proof. This follows from Corollary 5.3.19 and the fact ([NaT, Proposition 8]) that if $\mathcal{E}(\varphi)$ contains at least two elements then φ is of bounded type. $\square$

From Corollaries 5.3.19 and 5.3.20, we can see that if T_φ is 2-hyponormal but not subnormal then φ is not of bounded type and $\mathcal{E}(\varphi)$ consists of exactly one element.

For a strategy to answer Problem 5.7 we will introduce the notion of "weak subnormality," which was introduced by R. Curto and W.Y. Lee [CuL2]. Recall that the operator T is subnormal if and only if there exist operators A and B such that $\widehat{T} := \begin{bmatrix} T & A \\ 0 & B \end{bmatrix}$ is normal, i.e.,

$$\begin{cases} [T^*, T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*, B] + A^*A = 0. \end{cases} \tag{5.33}$$

We now introduce:

Definition 5.3.21. [CuL2] An operator $T \in B(H)$ is said to be *weakly subnormal* if there exist operators $A \in L(H', H)$ and $B \in L(H')$ such that the first two conditions in (5.33) hold: $[T^*, T] = AA^*$ and $A^*T = BA^*$. The operator $\widehat{T}$ is said to be a *partially normal extension* of T.

Clearly,

$$\text{subnormal} \implies \text{weakly subnormal} \implies \text{hyponormal}. \tag{5.34}$$

The converses of both implications in (5.34) are not true in general. Moreover, we can easily see that the following statements are equivalent for $T \in B(H)$:

(a) T is weakly subnormal;
(b) There is an extension $\widehat{T}$ of T such that $\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f$ for all $f \in \mathcal{H}$;
(c) There is an extension $\widehat{T}$ of T such that $\mathcal{H} \subseteq \ker [\widehat{T}^*, \widehat{T}]$.

Weakly subnormal operators possess the following invariance properties:

(i) (Unitary equivalence) if T is weakly subnormal with a partially normal extension $\left(\begin{smallmatrix} T & A \\ 0 & B \end{smallmatrix}\right)$ then for every unitary U, $\left(\begin{smallmatrix} U^*TU & U^*A \\ 0 & B \end{smallmatrix}\right)$ $\left(= \left(\begin{smallmatrix} U^* & 0 \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} T & A \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} U & 0 \\ 0 & I \end{smallmatrix}\right)\right)$ is a partially normal extension of U^*TU, i.e., U^*TU is also weakly subnormal.

(ii) (Translation) if $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal then $T - \lambda$ is also weakly subnormal for every $\lambda \in \mathbb{C}$: indeed if T has a partially normal extension $\widehat{T}$ then $\widehat{T - \lambda} := \widehat{T} - \lambda$ satisfies the properties in Definition 5.3.21.

(iii) (Restriction) if $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal and if $\mathfrak{M} \in \operatorname{Lat} T$ then $T|_{\mathfrak{M}}$ is also weakly subnormal because for a partially normal extension $\widehat{T}$ of T, $\widehat{T|_{\mathfrak{M}}} := \widehat{T}$ still satisfies the required properties.

How does one find partially normal extensions of weakly subnormal operators? Since weakly subnormal operators are hyponormal, one possible solution of the equation $AA^* = [T^*, T]$ is $A := [T^*, T]^{\frac{1}{2}}$. Indeed this is the case.

Theorem 5.3.22. [CuL2] *If $T \in B(H)$ is weakly subnormal then T has a partially normal extension $\widehat{T}$ on $\mathcal{K}$ of the form*

$$\widehat{T} = \begin{bmatrix} T & [T^*, T]^{\frac{1}{2}} \\ 0 & B \end{bmatrix} \quad \text{on } \mathcal{K} := \mathcal{H} \oplus \mathcal{H}. \tag{3.2.6.1}$$

The proof of Theorem 5.3.22 will make use of the following elementary fact.

Lemma 5.3.23. *If T is weakly subnormal then*

$$T(\ker [T^*, T]) \subseteq \ker [T^*, T].$$

Proof. By definition, there exist operators A and B such that $[T^*, T] = AA^*$ and $A^*T = BA^*$. If $[T^*, T]f = 0$ then $AA^*f = 0$ and hence $A^*f = 0$. Therefore

$$[T^*, T]Tf = AA^*Tf = ABA^*f = 0,$$

as desired. □

Definition 5.3.24. Let T be a weakly subnormal operator on H and let $\widehat{T}$ be a partially normal extension of T on K. We shall say that $\widehat{T}$ is a *minimal partially normal extension* of T if K has no proper subspace containing H to which the restriction of $\widehat{T}$ is also a partially normal extension of T. We write $\widehat{T} :=$ m.p.n.e.(T).

Lemma 5.3.25. *Let T be a weakly subnormal operator on H and let $\widehat{T}$ be a partially normal extension of T on K. Then $\widehat{T} =$ m.p.n.e.(T) if and only if*

$$\mathcal{K} = \bigvee \{\widehat{T}^{*n} h : \ h \in H, \ n = 0, 1\}. \tag{5.35}$$

Proof. See [CuL2]. □

It is well known (cf. [Con2, Proposition II.2.4]) that if T is a subnormal operator on $\mathcal{H}$ and N is a normal extension of T then N is a minimal normal extension of T if and only if

$$\mathcal{K} = \bigvee \{\widehat{T}^{*n} h : \ h \in H, \ n \geq 0\}.$$

Thus if T is a subnormal operator then T may have a partially normal extension different from a normal extension. For, consider the unilateral (unweighted) shift U_+ acting on $\ell^2(\mathbb{Z}_+)$. Then m.n.e.$(U_+) = U$, the bilateral shift acting on $\ell^2(\mathbb{Z})$, with orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$. It is easy to verify that m.p.n.e.$(U_+) = U|_{\mathcal{L}}$, where $\mathcal{L} :=< e_{-1} > \oplus \ \ell^2(\mathbb{Z}_+)$.

Theorem 5.3.26. *Let $T \in B(H)$.*

(i) *If T is 2-hyponormal then $[T^*,T]^{\frac{1}{2}}T[T^*,T]^{-\frac{1}{2}}|_{\mathrm{Ran}[T^*,T]}$ is bounded;*

(ii) *T is $(k+1)$-hyponormal if and only if T is weakly subnormal and $\widehat{T} :=$ m.p.n.e.(T) is k-hyponormal.*

Proof. See [CJP, Theorems 2.7 and 3.2]. □

In 1966, Stampfli [Sta3] explicitly exhibited for a subnormal weighted shift A_0 its minimal normal extension

$$N := \begin{bmatrix} A_0 & B_1 & & 0 & \\ & A_1 & B_2 & & \\ & & A_2 & \ddots & \\ 0 & & & \ddots & \end{bmatrix}, \tag{5.36}$$

where A_n is a weighted shift with weights $\{a_0^{(n)}, a_1^{(n)}, \cdots\}$, $B_n := \mathrm{diag}\{b_0^{(n)}, b_1^{(n)}, \cdots\}$, and these entries satisfy:

(I) $(a_j^{(n)})^2 - (a_{j-1}^{(n)})^2 + (b_j^{(n)})^2 \ge 0$ ($b_j^{(0)} = 0$ for all j);

(II) $b_j^{(n)} = 0 \Longrightarrow b_{j+1}^{(n)} = 0$;

(III) there exists a constant M such that $|a_j^{(n)}| \le M$ and $|b_j^{(n)}| \le M$ for $n = 0, 1, \cdots$ and $j = 0, 1, \cdots$, where

$$b_j^{(n+1)} := [(a_j^{(n)})^2 - (a_{j-1}^{(n)})^2 + (b_j^{(n)})^2]^{\frac{1}{2}} \quad \text{and} \quad a_j^{(n+1)} := a_j^{(n)} \frac{b_{j+1}^{(n+1)}}{b_j^{(n+1)}}$$

(if $b_{j_0}^{(n)} = 0$, then $a_{j_0}^{(n)}$ is taken to be 0).

We will now discuss analogues of the preceding results for k-hyponormal operators. Our criterion on k-hyponormality follows:

Theorem 5.3.27. *An operator $A_0 \in B(\mathcal{H}_0)$ is k-hyponormal if and only if the following three conditions hold for all n such that $0 \le n \le k-1$:*

(I$_n$) *$D_n \ge 0$;*

(II$_n$) *$A_{n-1}(\ker D_{n-1}) \subseteq \ker D_{n-1}$ $(n \ge 1)$;*

(III$_n$) *$D_{n-1}^{\frac{1}{2}} A_{n-1} D_{n-1}^{-\frac{1}{2}}|_{\mathrm{Ran}\,(D_{n-1})}$ $(n \ge 1)$ is bounded,*

where

$$D_0 := [A_0^*, A_0], \quad D_{n+1} := D_n|_{\mathcal{H}_{n+1}} + [A_{n+1}^*, A_{n+1}], \quad \mathcal{H}_{n+1} := \overline{\mathrm{ran}\,(D_n)}$$

and A_{n+1} denotes the bounded extension of $D_n^{\frac{1}{2}} A_n D_n^{-\frac{1}{2}}$ to $\overline{\mathrm{ran}\,(D_n)} (= \mathcal{H}_{n+1})$ from Ran (D_n).

Proof. Suppose A_0 is k-hyponormal. We now use induction on k. If $k=2$ then A_0 is 2-hyponormal, and so $D_0 := [A_0^*, A_0] \geq 0$. By Theorem 5.3.26 (i), $D_0^{\frac{1}{2}} A_0 D_0^{-\frac{1}{2}}|_{\operatorname{ran}(D_0)}$ is bounded. Let A_1 be the bounded extension of $D_0^{\frac{1}{2}} A_0 D_0^{-\frac{1}{2}}$ from $\operatorname{Ran}(D_0)$ to $\mathcal{H}_1 := \overline{\operatorname{Ran}(D_0)}$ and $D_1 := D_0|_{\mathcal{H}_1} + [A_1^*, A_1]$. Writing $\widehat{A_0} := \begin{bmatrix} A_0 & D_0^{\frac{1}{2}} \\ 0 & A_1 \end{bmatrix}$, we have $\widehat{A_0} =$ m.p.n.e. (A_0), which is hyponormal by Theorem 5.3.26(ii). Thus

$$[\widehat{A_0}^*, \widehat{A_0}] = \begin{bmatrix} 0 & 0 \\ 0 & D_0|_{\mathcal{H}_1} + [A_1^*, A_1] \end{bmatrix} \geq 0.$$

and hence $D_1 \geq 0$. Also by [CuL2, Lemma 2.2], $A_0(\ker D_0) \subseteq \ker D_0$ whenever A_0 is 2-hyponormal. Thus (I_n), (II_n), and (III_n) hold for $n = 0, 1$. Assume now that if A_0 is k-hyponormal then (I_n),(II_n) and (III_n) hold for all $0 \leq n \leq k-1$. Suppose A_0 is $(k+1)$-hyponormal. We must show that (I_n),(II_n) and (III_n) hold for $n = k$. Define

$$S := \begin{bmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{k-2}^{\frac{1}{2}} \\ 0 & & & & A_{k-1} \end{bmatrix} : \bigoplus_{i=0}^{k-1} \mathcal{H}_i \longrightarrow \bigoplus_{i=0}^{k-1} \mathcal{H}_i.$$

By our inductive assumption, $D_{k-1} \geq 0$. Writing $\widehat{T}^{(n)} :=$ m.p.n.e.$(\widehat{T}^{(n-1)})$ when it exists, we can see by our assumption that $S = \widehat{A_0}^{(k-1)}$: indeed, if

$$S_l := \begin{bmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{l-2}^{\frac{1}{2}} \\ 0 & & & & A_{l-1} \end{bmatrix}$$

then since by assumption $[S_l^*, S_l] = 0 \oplus D_l$ and $A_l = D_{l-1}^{\frac{1}{2}} A_{l-1} D_{l-1}^{-\frac{1}{2}}|_{\operatorname{Ran}(D_{l-1})}$, it follows that S_l is the minimal partially normal extension of S_{l-1} $(1 \leq l \leq k-1)$. But since by our assumption A_0 is $(k+1)$-hyponormal, it follows from Lemma 5.3.26(ii) that S is 2-hyponormal. Thus by Theorem 5.3.26(i), $[S^*, S]^{\frac{1}{2}} S [S^*, S]^{-\frac{1}{2}}|_{\operatorname{Ran}([S^*,S])}$ is bounded, which says that $D_{k-1}^{\frac{1}{2}} A_{k-1} D_{k-1}^{-\frac{1}{2}}|_{\operatorname{Ran}(D_{k-1})}$ is bounded, proving (III_n) for $n = k$. Observe that A_k, $\mathcal{H}_k$ and D_k are well-defined. Writing $\widehat{S} := \begin{bmatrix} S & D_{k-1}^{\frac{1}{2}} \\ 0 & A_k \end{bmatrix}$, we can see that $\widehat{S} =$ m.p.n.e.(S), which is hyponormal, again by Theorem 5.3.26(ii). Thus, since $[\widehat{S}^*, \widehat{S}] = \begin{bmatrix} 0 & 0 \\ 0 & D_k \end{bmatrix} \geq 0$, we have $D_k \geq 0$, proving (I_n) for $n = k$. On the other hand, since S is 2-hyponormal, it follows that $S(\ker[S^*,S]) \subseteq \ker[S^*,S]$. Since $[S^*,S] = \begin{bmatrix} 0 & 0 \\ 0 & D_{k-1} \end{bmatrix}$, we have $\ker[S^*,S] = \bigoplus_{i=0}^{k-2} \mathcal{H}_i \bigoplus \ker(D_{k-1})$. Thus, since

$$\begin{bmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{k-2}^{\frac{1}{2}} \\ 0 & & & & A_{k-1} \end{bmatrix} \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{k-2} \\ \ker(D_{k-1}) \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{k-2} \\ \ker(D_{k-1}) \end{bmatrix},$$

we must have that $A_{k-1}(\ker(D_{k-1})) \subseteq \ker(D_{k-1})$, proving (II$_n$) for $n = k$. This proves the necessity condition.

Toward sufficiency, suppose that conditions (I$_n$), (II$_n$) and (III$_n$) hold for all n such that $0 \le n \le k-1$. Define

$$S_n := \begin{bmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{n-2}^{\frac{1}{2}} \\ 0 & & & & A_{n-1} \end{bmatrix} \quad (1 \le n \le k-1).$$

Then S_{k-2} is weakly subnormal and $S_{k-1} = \text{m.p.n.e.}\,(S_{k-2})$. Since, by assumption, $D_{k-1} \ge 0$, we have $[S_{k-1}^*, S_{k-1}] = \begin{bmatrix} 0 & 0 \\ 0 & D_{k-1} \end{bmatrix} \ge 0$. It thus follows from Theorem 5.3.26(ii) that S_{k-2} is 2-hyponormal. Note that $S_n = \text{m.p.n.e.}\,(S_{n-1})$ for $n = 1, \cdots, k-1$ ($S_0 := A_0$). Thus, again by Theorem 5.3.26(ii), S_{k-3} is 3-hyponormal. Now repeating this argument, we can conclude that $S_0 \equiv A_0$ is k-hyponormal. This completes the proof. ☐

Corollary 5.3.28. *An operator $A_0 \in B(\mathcal{H}_0)$ is subnormal if and only if the conditions* (I$_n$), (II$_n$), *and* (III$_n$) *hold for all $n \ge 0$. In this case, the minimal normal extension N of A_0 is given by*

$$N = \begin{bmatrix} A_0 & D_0^{\frac{1}{2}} & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & \\ & & A_2 & \ddots \\ 0 & & & \ddots \end{bmatrix} : \bigoplus_{i=0}^{\infty} \mathcal{H}_i \to \bigoplus_{i=0}^{\infty} \mathcal{H}_i.$$

5.3.3 Gaps between k-Hyponormality and Subnormality

We find gaps between subnormality and k-hyponormality for Toeplitz operators.

Theorem 5.3.29. [Gu2],[CLL] *Let $0 < \alpha < 1$ and let ψ be the conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. Let $\varphi = \psi + \lambda\bar{\psi}$ and let T_φ be the corresponding Toeplitz operator on H^2. Then T_φ is k-hyponormal if and only if λ is in the circle $\left|z - \frac{\alpha(1-\alpha^{2j})}{1-\alpha^{2j+2}}\right| = \frac{\alpha^j(1-\alpha^2)}{1-\alpha^{2j+2}}$ for $j = 0, 1, \cdots, k-2$ or in the closed disk $\left|z - \frac{\alpha(1-\alpha^{2(k-1)})}{1-\alpha^{2k}}\right| \le \frac{\alpha^{k-1}(1-\alpha^2)}{1-\alpha^{2k}}$.*

For $0 < \alpha < 1$, let $T \equiv W_\beta$ be the weighted shift with weight sequence $\beta = \{\beta_n\}_{n=0}^\infty$, where (cf. [Cow2, Proposition 9])

$$\beta_n := (\sum_{j=0}^{n} \alpha^{2j})^{\frac{1}{2}} \quad \text{for } n = 0, 1, \cdots. \tag{5.37}$$

Let D be the diagonal operator, $D = \operatorname{diag}(\alpha^n)$, and let $S_\lambda \equiv T + \lambda T^*$ ($\lambda \in \mathbb{C}$). Then we have that

$$[T^*, T] = D^2 = \operatorname{diag}(\alpha^{2n}) \quad \text{and} \quad [S_\lambda^*, S_\lambda] = (1-|\lambda|^2)[T^*, T] = (1-|\lambda|^2)D^2.$$

Define

$$A_l := \alpha^l\, T + \frac{\lambda}{\alpha^l}\, T^* \quad (l = 0, \pm1, \pm2, \cdots).$$

It follows that $A_0 = S_\lambda$ and

$$DA_l = A_{l+1}D \quad \text{and} \quad A_l^* D = DA_{l+1}^* \quad (l = 0, \pm1, \pm2, \cdots). \tag{5.38}$$

Theorem 5.3.30. *Let $0 < \alpha < 1$ and $T \equiv W_\beta$ be the weighted shift with weight sequence $\beta = \{\beta_n\}_{n=0}^\infty$, where*

$$\beta_n = (\sum_{j=0}^{n} \alpha^{2j})^{\frac{1}{2}} \quad \textit{for } n = 0, 1, \cdots.$$

Then $A_0 := T + \lambda T^*$ is k-hyponormal if and only if $|\lambda| \le \alpha^{k-1}$ or $|\lambda| = \alpha^j$ for some $j = 0, 1, \cdots, k-2$.

Proof. Observe that

$$\begin{aligned} [A_l^*, A_l] &= [\alpha^l T^* + \tfrac{\bar{\lambda}}{\alpha^l} T,\ \alpha^l T + \tfrac{\lambda}{\alpha^l} T^*] \\ &= \alpha^{2l}[T^*, T] - \tfrac{|\lambda|^2}{\alpha^{2l}}[T^*, T] = \left(\alpha^{2l} - \tfrac{|\lambda|^2}{\alpha^{2l}}\right) D^2. \end{aligned} \tag{5.39}$$

Since $\ker D = \{0\}$ and $DA_n = A_{n+1}D$, it follows that $\mathcal{H}_n = H$ for all n; if we use A_l for the operator A_n in Theorem 5.3.27 then we have, by (5.39) and the definition of D_j, that

$$\begin{aligned}D_j &= D_{j-1} + [A_j^*, A_j] = D_{j-2} + [A_{j-1}^*, A_{j-1}] + [A_j^*, A_j] = \cdots \\ &= [A_0^*, A_0] + [A_1^*, A_1] + \cdots + [A_j^*, A_j] = (1-|\lambda|^2)D^2 + \cdots + \left(\alpha^{2j} - \frac{|\lambda|^2}{\alpha^{2j}}\right) D^2 \\ &= \left(\frac{1-\alpha^{2(j+1)}}{1-\alpha^2}\right)\left(1 - \frac{|\lambda|^2}{\alpha^{2j}}\right) D^2.\end{aligned}$$

By Theorem 5.3.27, A_0 is k-hyponormal if and only if $D_{k-1} \geq 0$ or $D_j = 0$ for some j such that $0 \leq j \leq k-2$ (in this case A_0 is subnormal). Note that $D_j = 0$ if and only if $|\lambda| = \alpha^j$. On the other hand, if $D_j > 0$ for $j = 0, 1, \cdots, k-2$, then

$$D_{k-1} = \left(\frac{1-\alpha^{2k}}{1-\alpha^2}\right)\left(1 - \frac{|\lambda|^2}{\alpha^{2(k-1)}}\right) D^2 \geq 0$$

if and only if $|\lambda| \leq \alpha^{k-1}$. Therefore A_0 is k-hyponormal if and only if $|\lambda| \leq \alpha^{k-1}$ or $|\lambda| = \alpha^j$ for some j, $j = 0, 1, \cdots, k-2$. □

We are ready for:

Proof. of Theorem 5.3.29 It was shown in [CoL] that $T_{\psi+\alpha\bar{\psi}}$ is unitarily equivalent to $(1-\alpha^2)^{\frac{3}{2}}T$, where T is the weighted shift in Theorem 5.3.30. Thus T_ψ is unitarily equivalent to $(1-\alpha^2)^{\frac{1}{2}}(T - \alpha T^*)$, so T_φ is unitarily equivalent to

$$(1-\alpha^2)^{\frac{1}{2}}(1-\lambda\,\alpha)(T + \frac{\lambda - \alpha}{1-\lambda\,\alpha}T^*) \quad \text{(cf. [Cow1, Theorem 2.4]).}$$

Applying Theorem 5.3.30 with $\frac{\lambda-\alpha}{1-\lambda\,\alpha}$ in place of λ, we have that for $k = 0, 1, 2, \cdots$,

$$\begin{aligned}\left|\frac{\lambda-\alpha}{1-\lambda\,\alpha}\right| \leq \alpha^k &\iff |\lambda-\alpha|^2 \leq \alpha^{2k}|1-\lambda\,\alpha|^2 \\ &\iff |\lambda|^2 - \frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}(\lambda + \bar{\lambda}) + \frac{\alpha^2 - \alpha^{2k}}{1-\alpha^{2k+2}} \leq 0 \\ &\iff \left|\lambda - \frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}\right| \leq \frac{\alpha^k(1-\alpha^2)}{1-\alpha^{2k+2}}.\end{aligned}$$

This completes the proof. □

5.4 Comments and Problems

From Corollary 5.3.19 we can see that if T_φ is a 2-hyponormal operator such that φ or $\bar{\varphi}$ is of bounded type then T_φ has a nontrivial invariant subspace. The following question is naturally raised:

Problem 5.8. *Does every* 2*-hyponormal Toeplitz operator have a nontrivial invariant subspace? More generally, does every* 2*-hyponormal operator have a nontrivial invariant subspace?*

It is well known ([Bro]) that if T is a hyponormal operator such that $R(\sigma(T)) \neq C(\sigma(T))$ then T has a nontrivial invariant subspace. But it remains still open whether every hyponormal operator with $R(\sigma(T)) = C(\sigma(T))$ (i.e., a *thin* spectrum) has a nontrivial invariant subspace. Recall that $T \in \mathcal{B}(\mathcal{H})$ is called a *von-Neumann operator* if $\sigma(T)$ is a spectral set for T, or equivalently, $f(T)$ is normaloid (i.e., norm equals spectral radius) for every rational function f with poles off $\sigma(T)$. Recently, B. Prunaru [Pru] has proved that polynomially hyponormal operators have nontrivial invariant subspaces. It was also known ([Ag1]) that von-Neumann operators enjoy the same property. The following is a sub-question of Problem G.

Problem 5.9. *Is every 2-hyponormal operator with thin spectrum a von-Neumann operator?*

Although the existence of a non-subnormal polynomially hyponormal weighted shift was established in [CP1] and [CP2], it is still an open question whether the implication "polynomially hyponormal ⇒ subnormal" can be disproved with a Toeplitz operator.

Problem 5.10. *Does there exist a Toeplitz operator which is polynomially hyponormal but not subnormal?*

In [CuL2] it was shown that every pure 2-hyponormal operator with rank-one self-commutator is a linear function of the unilateral shift. McCarthy and Yang [McCYa] classified all rationally cyclic subnormal operators with finite rank self-commutators. However it remains still open what are the pure subnormal operators with finite rank self-commutators.

Now the following question comes up at once:

Problem 5.11. *If T_φ is a* 2*-hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that T_φ is analytic?*

For affirmativeness to Problem J we shall give a partial answer. To do this we recall Theorem 15 in [NaT] which states that if T_φ is subnormal and $\varphi = q\bar{\varphi}$, where q is a finite Blaschke product then T_φ is normal or analytic. But from a careful examination of the proof of the theorem we can see that its proof uses subnormality assumption only for the fact that $\ker [T_\varphi^*, T_\varphi]$ is invariant under T_φ. Thus in view of Proposition 3.2.2, the theorem is still valid for "2–hyponormal" in place of "subnormal". We thus have:

Theorem 5.4.1. *If T_φ is 2–hyponormal and $\varphi = q\bar{\varphi}$, where q is a finite Blaschke product then T_φ is normal or analytic.*

We now give a partial answer to Problem 5.11.

Theorem 5.4.2. *Suppose log $|\varphi|$ is not integrable. If T_φ is a 2–hyponormal operator with nonzero finite rank self-commutator then T_φ is analytic.*

Proof. If T_φ is hyponormal such that $\log|\varphi|$ is not integrable then by an argument of [NaT, Theorem 4], $\varphi = q\bar{\varphi}$ for some inner function q. Also if T_φ has a finite rank self-commutator then by [NaT, Theorem 10], there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$. If $q \neq b$, so that $\mathcal{E}(\varphi)$ contains at least two elements, then by Corollary 5.3.20, T_φ is normal or analytic. If instead $q = b$ then by Theorem 5.4.1, T_φ is also normal or analytic. ☐

Theorem 5.4.2 reduces Problem 5.11 to the class of Toeplitz operators such that $\log|\varphi|$ is integrable. If $\log|\varphi|$ is integrable then there exists an outer function e such that $|\varphi| = |e|$. Thus we may write $\varphi = ue$, where u is a unimodular function. Since by the Douglas-Rudin theorem (cf. [Ga, p.192]), every unimodular function can be approximated by quotients of inner functions, it follows that if $\log|\varphi|$ is integrable then φ can be approximated by functions of bounded type. Therefore if we could obtain such a sequence ψ_n converging to φ such that T_{ψ_n} is 2–hyponormal with finite rank self-commutator for each n, then we would answer Problem J affirmatively. On the other hand, if T_φ attains its norm then by a result of Brown and Douglas [BD], φ is of the form $\varphi = \lambda\frac{\psi}{\theta}$ with $\lambda > 0$, ψ and θ inner. Thus φ is of bounded type. Therefore by Corollary 5.3.20, if T_φ is 2–hyponormal and attains its norm then T_φ is normal or analytic. However we were not able to decide that if T_φ is a 2–hyponormal operator with finite rank self-commutator then T_φ attains its norm.

Chapter 6

A Brief Survey on the Invariant Subspace Problem

6.1 A Brief History

Let $\mathcal{H}$ be a separable complex Hilbert space. If $T \in \mathcal{L}(\mathcal{H})$ then T is said to have a nontrivial invariant subspace if there is a subspace $\mathfrak{M}$ of $\mathcal{H}$ such that $\{0\} \neq \mathfrak{M} \neq \mathcal{H}$ and $T\mathfrak{M} \subset \mathfrak{M}$. In this case we can represent T as

$$T = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \quad \text{on } \mathfrak{M} \oplus \mathfrak{M}^{\perp}.$$

Example 6.1.1. If T has aigenvalue λ, put

$$\mathfrak{M}_\lambda := \{x : Tx = \lambda x\} \equiv \text{the eigenspace corresponding to } \lambda.$$

Then evidently $T\mathfrak{M}_\lambda \subseteq \mathfrak{M}_\lambda$. If $T \neq \lambda$ then $\mathfrak{M}_\lambda$ is nontrivial.

Invariant Subspace Problem (1932, J. von Neumann) Let $\mathcal{X} \equiv$ a Banach space of dim ≥ 2 and $T \in \mathcal{B}(\mathcal{X})$. Does T have a nontrivial invariant subspace?

Let $\mathbf{K}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$. If $K \in \mathbf{K}(\mathcal{H})$ has a polar decomposition $K = U\,|T|$, where $|T| := (T^*T)^{\frac{1}{2}}$ and U is a partial isometry, then $|T| \in \mathbf{K}(\mathcal{H})$ and so has a diagonal matrix $\operatorname{diag}(\lambda_1, \lambda_2, \cdots)$ relative to some orthonormal basis for $\mathcal{H}$. For $p \geq 1$ we define

$$\mathcal{C}_p(\mathcal{H}) := \left\{ K \in \mathbf{K}(\mathcal{H}) : \sum_{n=1}^{\infty} \lambda_n^p < \infty \right\},$$

which is called the *Schatten p-ideal*. The ideal $\mathcal{C}_1(\mathcal{H})$ is known as the *trace class* and the ideal $\mathcal{C}_2(\mathcal{H})$ as the *Hilbert-Schmidt class*.

1984 C.J. Read answered ISP in the negative (for ℓ_1).

Comment. However ISP is still open for a separable Hilbert space.

1934 J. von Neumann (unpublished): $T \in \mathbf{K}(\mathcal{H}) \Longrightarrow T$ has n.i.s.

1954 N. Aronszajn and K. Smith (Ann. of Math.): $T \in \mathbf{K}(\mathcal{H}) \Longrightarrow T$ has n.i.s.

1966 A. Bernstein and A. Robinson (Pacific J. of Math.)

T is polynomially compact (i.e., $p(T)$ is compact for a polynomial p) $\Longrightarrow T$ has n.i.s.

1966 P. Halmos (Pacific J. of Math.) reproved Bernstein-Robinson theorem via analysis technique.

1973 K. Lomonosov (Funk. Anal. Pril.)

$$T (\neq \lambda) \text{ commutes with a nonzero compact operator} \Longrightarrow T \text{ has n.i.s.}$$

1978 S. Brown (Int. Eq. Op. Th.): T is subnormal $\Longrightarrow T$ has n.i.s.

1986 S. Brown, Chevreau, C. Pearcy (J. Funct. Anal.)

$$||T|| \leq 1,\ \sigma(T) \supseteq \mathbb{T} \ (= \text{the unit circle}) \Longrightarrow T \text{ has n.i.s.}$$

1987 S. Brown (Ann. of Math.)

$$T \text{ is hyponormal with } \operatorname{int}\sigma(T) \neq \emptyset \Longrightarrow T \text{ has n.i.s.}$$

Problem. Prove or disprove ISP for hyponormal operators.

6.2 Basic Facts

The *spectral picture* of $T \in B(H)$, $\mathcal{SP}(T)$, is the structure consisting of $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes.

If H_i $(i = 1, 2)$ is a separable Hilbert space and $T_i \in B(H_i)$, then T_1 and T_2 are said to be *compalent* (notation:$T_1 \sim T_2$) if there exists a unitary operator $W \in B(H_1, H_2)$ and a compact operator $K \in \mathbf{K}(H_2)$ such that $WT_1W^* + K = T_2$.

Proposition 6.2.1. *The relation of compalence on $B(H)$ is an equivalence relation and partitions $B(H)$ into equivalence classes.*

Definition 6.2.2. An operator $T \in B(H)$ is called *essentially normal* if $[T^*, T] \in \mathbf{K}(H)$, or equivalently, if $\pi(T)$ is normal in $B(H)/\mathbf{K}(H)$. We write $(EN)(H)$ for the set of all essentially normal operators in $B(H)$.

Theorem 6.2.3. (BDF Theorem) [BDF] *If $T \in (EN)(H_1)$ and $T_2 \in (EN)(H_2)$ then*

$$T_1 \sim T_2 \iff \mathcal{SP}(T_1) = \mathcal{SP}(T_2).$$

Suppose there exists a unitary operator W and a compact operator K such that $WT_1W^* + K = T_2$. If $||K|| < \epsilon$ then T_1 and T_2 are said to be ϵ-compalent. (Notation: $T_1 \sim T_2(\epsilon)$).

Theorem 6.2.4. [Ber] *If $N \in B(H)$ is normal then for any $\epsilon > 0$, there exists a diagonal operator D_ϵ such that $N \sim D_\epsilon(\epsilon)$.*

6.3 Quasitriangular operators

An operator $T \in B(H)$ is called *quasitriangular* if there exists a sequence $\{P_n\}$ of projections of finite rank that

$$P_n \to 1 \text{ weakly} \quad \text{and} \quad ||P_nTP_n - TP_n|| \to 0.$$

We write $QT(H)$ for the set of all quasitriangular operators in $B(H)$.

Compact operators are quasitriangular. Indeed, if P_n is a projection such that $P_n \to I$ weakly and K is compact then $||P_nKP_n - K|| \to 0$. So

$$||P_nKP_n - KP_n|| = ||P_n(KP_n)P_n - (KP_n)|| = ||P_nK'P_n - K'|| \to 0.$$

A trivial example of a quasitriangular operator is an upper triangular operator: indeed if P_n denotes the orthogonal projection onto $\bigvee\{e_1, \cdots, e_n\}$, then $TP_nH \subset P_nH$, so $P_nTP_n = TP_n$.

Definition 6.3.1. An operator $T \in B(H)$ is called *triangular* if there exists an orthonormal basis $\{e_n\}$ for H such that T is upper triangular.

Evidently, triangular $\Rightarrow$ quasitriangular.

Theorem 6.3.2. (P.Halmos, Quasitriangular operators, Acta Sci. Math. (Szeged) 29 (1968), 283–293)

$QT(H)$ is norm closed.

Theorem 6.3.3. *normal $\Rightarrow$ quasitriangular.*

Proof. By Theorem 6.2.4, if T is normal then for any $\epsilon > 0$,

$$T \sim D_\epsilon(\epsilon) \quad \text{with a diagonal } D_\epsilon,$$

i.e., $W_\epsilon T W_\epsilon^* = D_\epsilon + K_\epsilon$ with $||K_\epsilon|| < \epsilon$. So, $||T - W_n^* D_n W_n|| = \epsilon \to 0$. □

Theorem 6.3.4. (P.Halmos, Quasitriangular operators, Acta Sci. Math. (Szeged) 29 (1968), 283–293)

$$QT(H) = Triangular + Compact.$$

Theorem 6.3.5. (R. Douglas and C. Pearcy, A note on quasitriangular operators, Duke math. J. 37(1970), 177-188)

Similarity preserves quasitriangularity

Corollary 6.3.6. *Compalence preserves quasitriangularity.*

Theorem 6.3.7. (AFV Theorem) (Apostol, Foias, and Voiculescu, Some results on non-quasitriangular operators II, Rev. Roumaine Math. Pure Appl. 18(1973), 159–181) *If $T \in B(H)$ then T is quasitriangular if and only if $\mathcal{SP}(T)$ contains no hole or pseudohole associated with a negative number.*

Definition 6.3.8. An operator $T \in B(H)$ is said to have a *nontrivial hyperinvariant subspace* if there exists a nontrivial closed subspace $\mathfrak{M}$ such that

$$T'\mathfrak{M} \subset \mathfrak{M} \quad \text{for every } T' \text{ with } TT' = T'T.$$

Definition 6.3.9. An operator $T \in B(H)$ is called biquasitriangular if $T, T^* \in B(H)$. We write $(BQT)(H)$ for the set of all biquasitriangular operators on H.

Theorem 6.3.10. *If $T \notin (BQT)(H)$ then either T or T^* has an eigenvalue and so T has a nontrivial hyperinvariant subspace.*

Proof. We consider $T \notin (QT)(H)$. By the AFV theorem, there exists λ_0 such that $T - \lambda_0$ is semi-Fredholm with $-\infty \leq \text{index}\,(T - \lambda_0) < 0$. Thus $\dim\ker\,(T^* - \overline{\lambda_0}) > 0$, and hence $\overline{\lambda_0}$ is an eugenvalue for T^*. In fact, $\mathfrak{M} = \{x \in H : T^*x = \overline{\lambda}x\}$ is hyperinvariant for T^*. Since λ_0 cannot be nonquasitriangular, we have $\mathfrak{M} \neq H$. Thus $\mathfrak{M}^\perp$ is a nontrivial hyperinvariant subspace for T. □

1973 Berger-Shaw *If T is hyponormal and cyclic (i.e., there exists a vector e_0 such that $\mathcal{H} = cl\{p(T)e_0 : p = a\ polynomial\}$ then $[T^*, T] \in \mathcal{C}_1$.*

If T is not cyclic, and so

$$\mathfrak{M} \equiv \text{cl}\,\{p(T)e : \ e = \text{a vector}\} \neq \mathcal{H}$$

then $T\mathfrak{M} \subseteq \mathfrak{M}$.

1979 Voiculescu: Normal $\cong$ Diagonal normal $+\,\mathcal{C}_2$.

Sub-Conclusion. The only hyponormal operators without known n.i.s. belong to

$$T \cong \text{Diagonal normal + Compact with } [T^*, T] \in \mathcal{C}_1.$$

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