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A Bridge Theory for Block Toeplitz Operators

Lecture Notes
Operator Winter School 2013

November 14, 2013

Foreword

Toeplitz operators arise naturally in several fields of mathematics and in a variety of problems in physics. Also the theory of hyponormal and subnormal operators is an extensive and highly developed area, which has made important contributions to a number of problems in functional analysis, operator theory, and mathematical physics. Thus, it becomes of central significance to describe in detail hyponormality and subnormality for Toeplitz operators. In this sense, the following question is challenging and interesting:

Which Toeplitz operators are hyponormal or subnormal ?

While the precise relation between normality and subnormality has been extensively studied, as have been the classes of subnormal and hyponormal operators, the relative position of the class of subnormals inside the classes of hyponormals is still far from being well understood. We call it a "bridge theory" for operators to explore a bridge between hyponormality and subnormality for bounded linear operators acting on an infinite dimensional complex Hilbert (or Banach) space. In this lecture I will try to provide a bridge theory for block Toeplitz operators. This is originated from Halmos's Problem 5 (in 1970):

Is every subnormal Toeplitz operator either normal or analytic ?

Even though Halmos's Problem 5 was, in 1984, answered in the negative by C. Cowen and J. Long, until now researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. In this lecture I attempt to set forth some of the recent developments that had taken place in the study of the subnormality of block Toeplitz operators acting on the vector-valued Hardy space $H^2_{\mathbb{C}^n}$ of the unit circle.

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Chapter 1
Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces, let $\mathscr{B}(\mathscr{H},\mathscr{K})$ be the set of bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$, and write $\mathscr{B}(\mathscr{H}) := \mathscr{B}(\mathscr{H},\mathscr{H})$. For an operator $T \in \mathscr{B}(\mathscr{H})$, we write $\ker T$ and $\operatorname{ran} T$ for the kernel and the range of T, respectively. For a set $\mathscr{M}$, $\operatorname{cl}\mathscr{M}$ and $\mathscr{M}^{\perp}$ denote the closure and the orthogonal complement of $\mathscr{M}$, respectively. For $A,B \in \mathscr{B}(\mathscr{H})$, we let $[A,B] := AB - BA$. For an operator $T \in \mathscr{B}(\mathscr{H})$, T^* denotes the adjoint of T. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if its self-commutator $[T^*,T]$ is positive semi-definite, and *quasinormal* if T commutes with T^*T. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *pure* if it has no nonzero reducing subspace on which it is normal. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *subnormal* if there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a normal operator N on $\mathscr{K}$ such that $N\mathscr{H} \subseteq \mathscr{H}$ and $T = N|_{\mathscr{H}}$. In this case, N is called a *normal extension* of T. For the general theory of subnormal and hyponormal operators, we refer to [Con] and [MP]. In general, it is quite difficult to examine whether such a normal extension exists for an operator. Of course, there are a couple of constructive methods for determining subnormality; one of them is the Bram-Halmos criterion of subnormality ([Br],[Con]), which states that an operator $T \in \mathscr{B}(\mathscr{H})$ is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$ for all finite collections $x_0, x_1, \cdots, x_k \in \mathscr{H}$. It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} [T^*,T] & [T^{*2},T] & \cdots & [T^{*k},T] \\ [T^*,T^2] & [T^{*2},T^2] & \cdots & [T^{*k},T^2] \\ \vdots & \vdots & \ddots & \vdots \\ [T^*,T^k] & [T^{*2},T^k] & \cdots & [T^{*k},T^k] \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{1.1}$$

In view of (1.1), between hyponormality and subnormality there exists a whole slew of increasingly stricter conditions, each expressible in terms of the joint hyponormality of the tuples $(I,T,T^2,\cdots,T^k)$. Given an n-tuple $\mathbf{T} = (T_1,\cdots,T_n)$ of operators on $\mathscr{H}$, we let $[\mathbf{T}^*,\mathbf{T}] \in \mathscr{B}(\mathscr{H}\oplus\cdots\oplus\mathscr{H})$ denote the *self-commutator* of $\mathbf{T}$, defined by

$$[\mathbf{T}^*,\mathbf{T}] := \begin{pmatrix} [T_1^*,T_1] & [T_2^*,T_1] & \cdots & [T_n^*,T_1] \\ [T_1^*,T_2] & [T_2^*,T_2] & \cdots & [T_n^*,T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*,T_n] & [T_2^*,T_n] & \cdots & [T_n^*,T_n] \end{pmatrix}.$$

This definition of self-commutator for n-tuples of operators on a Hilbert space was introduced by A. Athavale [At]. By analogy with the case $n = 1$, we shall say ([At], [CMX]) that $\mathbf{T}$ is *jointly hyponormal* (or simply, *hyponormal*) if $[\mathbf{T}^*,\mathbf{T}]$ is a positive operator on $\mathscr{H} \oplus \cdots \oplus \mathscr{H}$. $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and every T_i is a normal operator, and *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, the normality, subnormality or hyponormality of an n-tuple requires as a necessary condition that every coordinate in the tuple be normal, subnormal or hyponormal, respectively. Normality and subnormality require that the coordinates commute, but hyponormality does not. When the notion of "joint hyponormality" was first formally introduced by A. Athavale [At] in 1988, he conceived joint hyponormality as a notion at least as strong as requiring that the linear span of the operator coordinates consist of hyponormal operators. Recall ([Ath],[CMX],[CoS]) that $T \in \mathscr{L}(\mathscr{H})$ is said to be *weakly k-hyponormal* if

$$LS(T,T^2,\cdots,T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1,\cdots,\alpha_k) \in \mathbf{C}^k \right\}$$

consists entirely of hyponormal operators. If $k = 2$ then T is called *quadratically hyponormal*, and if $k = 3$ then T is said to be *cubically hyponormal*. Similarly, $T \in \mathscr{L}(\mathscr{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbf{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general. The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality (cf. [Cu1], [Cu2], [CuF], [CuL1], [CuL2], [CMX], [DPY], [McCP]). Joint hyponormality and weak joint hyponormality have been studied by A. Athavale [At], J. Conway and W. Szymanski [CS], R. Curto [Cu], R. Curto and W.Y. Lee [CuL1], R. Curto, P. Muhly, and J. Xia [CMX], R. Douglas, V. Paulsen and K. Yan [DPY], R. Douglas and K. Yan [DY], D. Farenick and R. McEachin [FM], C. Gu [Gu2], S. McCullough and V. Paulsen [McCP1],[McCP2], D. Xia [Xi], and others. Joint hyponormality originated from questions about commuting normal extensions of commuting operators, and it has also been considered with an aim at understanding the gap between hyponormality and subnormality for single operators. Since a (2×2)-operator matrix $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ (with A invertible) is positive if and only if $A \geq o$, $C \geq 0$, and $B^*A^{-1}B \leq C$, we can rephrase (1.1) as follows:

$$\begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{1.2}$$

Thus the Bram-Halmos criterion can be stated as follows: T is subnormal if and only if the positivity condition (1.3) holds for all $k \geq 1$. But it may not still be possible to test the positivity condition (1.3) for *every* positive integer k, in general. Hence the following question is interesting and challenging:

Which operators are subnormal ?

The class of Toeplitz operators is a nice test ground for this question. On the other hand, condition (1.2) provides a measure of the gap between hyponormality and subnormality. In fact the positivity condition (1.2) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.2) for all k. For $k \geq 1$, an operator T is said to be *k-hyponormal* if T satisfies the positivity condition (1.2) for a fixed k. Thus the Bram-Halmos criterion can be stated as: T is subnormal if and only if T is k-hyponormal for all $k \geq 1$. The k-hyponormality has been considered by many authors with an aim at understanding a bridge between hyponormality and subnormality. For instance, the Bram-Halmos criterion on subnormality indicates that 2-hyponormality is generally far from subnormality. There are special classes of operators, however, for which these two notions are equivalent. A trivial example is given by the class of operators whose square is compact. Also it was shown in ([CuL1]) that if $W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ is the weighted shift whose weight sequence consists of the initial weight x followed by the weight sequence of the recursively generated subnormal weighted shift $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ with an initial segment of positive weights $\sqrt{a}, \sqrt{b}, \sqrt{c}$ (cf. [CuF1], [CuF2], [CuF3]), then W_α is subnormal if and only if the positivity condition (1.1) is satisfied with $k = 2$.

We review a few essential facts about (block) Toeplitz operators, and for that we will use [BS], [Do1], [Do2], [GGK], [MAR], [Ni], and [Pe]. Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be the unit circle. Recall that the Hilbert space $L^2 \equiv L^2(\mathbf{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbf{Z}$, and that the Hardy space $H^2 \equiv H^2(\mathbf{T})$ is the closed linear span of $\{e_n : n = 0, 1, \cdots\}$. An element $f \in L^2$ is said to be analytic if $f \in H^2$. Let $H^\infty \equiv H^\infty(\mathbf{T}) := L^\infty \cap H^2$, i.e., H^∞ is the set of bounded analytic functions on the open unit disk $\mathbf{D}$.

Given a bounded measurable function $\varphi \in L^\infty$, the *Toeplitz operator* T_φ and the *Hankel operator* H_φ with *symbol* φ on H^2 are defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi g := JP^\perp(\varphi g) \qquad (g \in H^2), \tag{1.3}$$

where P and $P^\perp$ denote the orthogonal projections that map from L^2 onto H^2 and $(H^2)^\perp$, respectively, and J denotes the unitary operator from L^2 onto L^2 defined by $J(f)(z) = \bar{z}f(\bar{z})$ for $f \in L^2$. To study hyponormality (resp. normality and subnormality) of the Toeplitz operator T_φ with symbol φ we may, without loss of generality, assume that $\varphi(0) = 0$; this is because hyponormality (resp. normality and subnormality) is invariant under translations by scalars. Normal Toeplitz operators were characterized by a property of their symbols in the early 1960's by A. Brown and P.R. Halmos [BH] and the exact nature of the relationship between the symbol

$\varphi \in L^\infty$ and the hyponormality of T_φ was understood via Cowen's Theorem [Co4] in 1988.

Theorem 1.1. (**Cowen's Theorem**) ([Co4], [NT]) *For each $\varphi \in L^\infty$, let*

$$\mathscr{E}(\varphi) \equiv \{k \in H^\infty : \ ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty\}.$$

Then a Toeplitz operator T_φ is hyponormal if and only if $\mathscr{E}(\varphi)$ is nonempty.

This elegant and useful theorem has been used in the works [CuL1], [CuL2], [FL1], [FL2], [Gu1], [Gu2], [GS], [HKL1], [HKL2], [HL1], [HL2], [HL3], [Le], [NT], [Zhu], and etc., which have been devoted to the study of hyponormality for Toeplitz operators on H^2. However it may not even be possible to find tractable necessary and sufficient condition for the hyponormality of T_φ in terms of the Fourier coefficients of the symbol φ unless certain assumptions are made about φ. Tractable criteria for the cases of trigonometric polynomial symbols and rational symbols and bounded type symbols were derived from a Carathéodory-Schur interpolation problem ([Zhu]), a tangential Hermite-Fejér interpolation problem ([Gu1]) or the classical Hermite-Fejér interpolation problem ([HL3]), respectively.

We introduce the notion of block Toeplitz operators. Let $M_{n\times r}$ denote the set of all $n \times r$ complex matrices and write $M_n := M_{n\times n}$. For $\mathscr{X}$ a Hilbert space, let $L^2_{\mathscr{X}} \equiv L^2_{\mathscr{X}}(\mathbf{T})$ be the Hilbert space of $\mathscr{X}$-valued norm square-integrable measurable functions on $\mathbf{T}$ and let $H^2_{\mathscr{X}} \equiv H^2_{\mathscr{X}}(\mathbf{T})$ be the corresponding Hardy space. We observe that $L^2_{\mathbf{C}^n} = L^2 \otimes \mathbf{C}^n$ and $H^2_{\mathbf{C}^n} = H^2 \otimes \mathbf{C}^n$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbf{T})$ $(= L^\infty \otimes M_n)$ then $T_\Phi : H^2_{\mathbf{C}^n} \to H^2_{\mathbf{C}^n}$ denotes a *block Toeplitz operator* with *symbol* Φ defined by

$$T_\Phi f := P_n(\Phi f) \quad \text{for } f \in H^2_{\mathbf{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbf{C}^n}$ onto $H^2_{\mathbf{C}^n}$. A *block Hankel operator* with *symbol* $\Phi \in L^\infty_{M_n}$ is the operator $H_\Phi : H^2_{\mathbf{C}^n} \to H^2_{\mathbf{C}^n}$ defined by

$$H_\Phi f := J_n P_n^\perp(\Phi f) \quad \text{for } f \in H^2_{\mathbf{C}^n},$$

where J_n denotes the unitary operator from $L^2_{\mathbf{C}^n}$ onto $L^2_{\mathbf{C}^n}$ given by $J_n(f)(z) := \overline{z} I_n f(\overline{z})$ for $f \in L^2_{\mathbf{C}^n}$, with I_n the $n \times n$ identity matrix. Note that $H^2_{\mathbf{C}^n}$ can be viewed either as

$$H^2_{\mathbf{C}^n} = \mathbf{C}^n \oplus \mathbf{C}^n \oplus \mathbf{C}^n \oplus \cdots,$$

the infinite direct sum of $\mathbf{C}^n$, or as,

$$H^2_{\mathbf{C}^n} = H^2 \oplus \cdots \oplus H^2,$$

where the number of copies of H^2 is n. If we write $\Phi = \sum_{n=-\infty}^{\infty} \Phi_n z^n \in L^\infty_{M_n}$ $(z = e^{i\theta})$, then with respect to the decomposition $H^2_{\mathbf{C}^n} = \mathbf{C}^n \oplus \mathbf{C}^n \oplus \mathbf{C}^n \oplus \cdots$, we can write

$$T_\Phi = \begin{pmatrix} \Phi_0 & \Phi_{-1} & \Phi_{-2} & \cdots \\ \Phi_1 & \Phi_0 & \Phi-1 & \ddots \\ \Phi_2 & \Phi_1 & \Phi_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad H_\Phi = \begin{pmatrix} \Phi_{-1} & \Phi_{-2} & \Phi_{-3} & \cdots \\ \Phi_{-2} & \Phi_{-3} & \Phi_{-4} & \ddots \\ \Phi_{-3} & \Phi_{-4} & & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Also if we write

$$\Phi = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{pmatrix} \in L^\infty_{M_n},$$

then with respect to the decomposition $H^2_{\mathbf{C}^n} = H^2 \oplus \cdots \oplus H^2$, we can write

$$T_\Phi = \begin{pmatrix} T_{\varphi_{11}} & \cdots & T_{\varphi_{1n}} \\ & \vdots & \\ T_{\varphi_{n1}} & \cdots & T_{\varphi_{nn}} \end{pmatrix} \quad \text{and} \quad H_\Phi = \begin{pmatrix} H_{\varphi_{11}} & \cdots & H_{\varphi_{1n}} \\ & \vdots & \\ H_{\varphi_{n1}} & \cdots & H_{\varphi_{nn}} \end{pmatrix}.$$

For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\bar{z}).$$

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is called *inner* if $\Theta^*\Theta = I_m$ almost everywhere on $\mathbf{T}$. The following basic relations can be easily derived:

$$T^*_\Phi = T_{\Phi^*}, \;\; H^*_\Phi = H_{\widetilde{\Phi}} \quad (\Phi \in L^\infty_{M_n}); \tag{1.4}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad (\Phi, \Psi \in L^\infty_{M_n}); \tag{1.5}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}, \;\; H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad (\Phi \in L^\infty_{M_n}, \Psi \in H^\infty_{M_n}); \tag{1.6}$$

$$H^*_\Phi H_\Phi - H^*_{\Theta\Phi} H_{\Theta\Phi} = H^*_\Phi H_{\Theta^*} H^*_{\Theta^*} H_\Phi \quad (\Theta \text{ inner}, \Phi \in L^\infty_{M_n}). \tag{1.7}$$

A function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are functions $\psi_1, \psi_2 \in H^\infty(\mathbf{D})$ such that $\varphi(z) = \psi_1(z)/\psi_2(z)$ for almost all z in $\mathbf{T}$. Evidently, rational functions are of bounded type. For a matrix-valued function $\Phi \equiv [\varphi_{ij}] \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry φ_{ij} is of bounded type, and we say that Φ is *rational* if each entry φ_{ij} is a rational function. A matrix-valued trigonometric polynomial $\Phi \in L^\infty_{M_n}$ is of the form

$$\Phi(z) = \sum_{j=-m}^{N} A_j z^j \; (A_j \in M_n),$$

where A_N and A_{-m} are called the *outer* coefficients of Φ. We recall that for matrix-valued functions $A := \sum_{j=-\infty}^{\infty} A_j z^j \in L^2_{M_n}$ and $B := \sum_{j=-\infty}^{\infty} B_j z^j \in L^2_{M_n}$, we define the inner product of A and B by

$$(A, B) := \int_{\mathbf{T}} \operatorname{tr}(B^*A)\, d\mu = \sum_{j=-\infty}^{\infty} \operatorname{tr}(B^*_j A_j),$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix and define $||A||_2 := (A,A)^{\frac{1}{2}}$. We also define, for $A \in L^\infty_{M_n}$,

$$||A||_\infty := \text{ess sup}_{t\in\mathbf{T}}||A(t)|| \quad (||\cdot|| \text{ denotes the spectral norm of a matrix}).$$

On the other hand, recently C. Gu, J. Hendricks and D. Rutherford [GHR] considered the hyponormality of block Toeplitz operators and characterized it in terms of their symbols. In particular they showed that if T_Φ is a hyponormal block Toeplitz operator on $H^2_{\mathbf{C}^n}$, then its symbol Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$. Their characterization for hyponormality of block Toeplitz operators resembles Cowen's Theorem except for an additional condition – the normality condition of the symbol.

Theorem 1.2. (**Hyponormality of Block Toeplitz Operators**) ([GHR]) *For each $\Phi \in L^\infty_{M_n}$, let*

$$\mathscr{E}(\Phi) := \left\{K \in H^\infty_{M_n} : \ ||K||_\infty \le 1 \ \textit{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n}\right\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathscr{E}(\Phi)$ is nonempty.

The hyponormality of the Toeplitz operator T_Φ with arbitrary matrix-valued symbol Φ, though solved in principle by Cowen's Theorem [Co4] and the criterion due to Gu, Hendricks and Rutherford [GHR], is in practice very complicated.

This lecture concerns the gap between hyponormality and subnormality of block Toeplitz operators with rational symbols. In [Hal3, Problem 209], it was shown that there exists a hyponormal operator whose square is not hyponormal, e.g., $U^* + 2U$ (U is the unilateral shift on ℓ^2), which is a *trigonometric* Toeplitz operator, i.e., $U^* + 2U \equiv T_{\bar{z}+2z}$. This example addresses the gap between hyponormality and subnormality for Toeplitz operators. This matter is closely related to Halmos's Problem 5 [Hal1], [Hal2]: Is every subnormal Toeplitz operator either normal or analytic ? The most interesting partial answer to Halmos's Problem 5 was given by M. Abrahamse [Ab]. M. Abrahamse gave a general sufficient condition for the answer to Halmos's Problem 5 to be affirmative. Abrahamse's Theorem can be then stated as follows: Let $\varphi = \overline{g} + f \in L^\infty$ $(f, g \in H^2)$ be such that φ or $\overline{\varphi}$ is of bounded type. If T_φ is subnormal then T_φ is normal or analytic. In this lecture, we consider the question: Which subnormal block Toeplitz operators are either normal or analytic ?

Chapter 2
Basic Theory and Preliminaries

2.1 Hyponormality and subormality of scalar Toeplitz operators

An elegant and useful theorem of C. Cowen [Co3] characterizes the hyponormality of a Toeplitz operator T_φ on the Hardy space $H^2(\mathbf{T})$ of the unit circle $\mathbf{T} \subset \mathbf{C}$ by properties of the symbol $\varphi \in L^\infty(\mathbf{T})$. This result makes it possible to answer an algebraic question coming from operator theory – namely, is T_φ hyponormal ? - by studying the function φ itself. Normal Toeplitz operators were characterized by a property of their symbol in the early 1960's by A. Brown and P.R. Halmos [BH], and so it is somewhat of a surprise that 25 years passed before the exact nature of the relationship between the symbol $\varphi \in L^\infty$ and the positivity of the selfcommutator $[T_\varphi^*, T_\varphi]$ was understood (via Cowen's theorem). As Cowen notes in his survey paper [Co2], the intensive study of subnormal Toeplitz operators in the 1970's and early 80's is one explanation for the relatively late appearance of the sequel to the Brown-Halmos work. The characterization of hyponormality via Cowen's theorem requires one to solve a certain functional equation in the unit ball of H^∞. However the case of arbitrary trigonometric polynomials φ, though solved in principle by Cowen's theorem, is in practice very complicated. Indeed it may not even be possible to find tractable necessary and sufficient conditions for the hyponormality of T_φ in terms of the Fourier coefficients of φ unless certain assumptions are made about φ. In this chapter we present some recent development in this research.

2.1.1 Cowen's Theorem

In this section we present Cowen's theorem. Cowen's method is to recast the operator-theoretic problem of hyponormality of Toeplitz operators into the problem of finding a solution of a certain functional equation involving its symbol. This approach has been put to use in the works [CLL, CuL1, CuL2, CuL3, FL1, FL2, Gu1, HKL1, HKL2, HL3, NT, Zhu] to study Toeplitz operators.

We begin with:

Theorem 2.1 (Brown-Halmos). [BH] *Normal Toeplitz operators are translations and rotations of hermitian Toeplitz operators i.e.,*

$$T_\varphi \text{ normal} \iff \exists\, \alpha,\beta \in \mathbf{C}, \text{ a real valued } \psi \in \mathbf{L}^\infty \text{ such that } T_\varphi = \alpha T_\psi + \beta 1.$$

The following are basic properties of Toeplitz and Hankel operators.

1. $H_\psi^* = H_{\psi^*}$;
2. $H_\psi U = U^* H_\psi$ (U is the unilateral shift);
3. $\mathrm{Ker} H_\psi = \{0\}$ or $\theta \mathbf{H}^2$ for some inner function θ (by Beurling's theorem);
4. $T_{\varphi\psi} - T_\varphi T_\psi = H_{\overline{\varphi}}^* H_\psi$;
5. $H_\varphi T_h = H_{\varphi h} = T_{h^*}^* H_\varphi$ $(h \in \mathbf{H}^\infty)$.
6. If U is the unilateral shift on $\mathbf{H}^2$ then $\mathrm{comm}(U) = \{T_\varphi : \varphi \in \mathbf{H}^\infty\}$.

D. Sarason [Sa] gave a generalization of the above property (6).

Theorem 2.2 (Sarason's Interpolation Theorem). [Sa] *Let*

(i) U *=the unilateral shift on* $\mathbf{H}^2$*;*
(ii) $\mathscr{K} := \mathbf{H}^2 \ominus \psi \mathbf{H}^2$ *(*ψ *is an inner function);*
(iii) $S := PU|_{\mathscr{K}}$*, where* P *is the projection of* $\mathbf{H}^2$ *onto* $\mathscr{K}$*.*

If $T \in \mathrm{comm}(S)$ *then there exists a function* $\varphi \in \mathbf{H}^\infty$ *such that* $T = T_\varphi|_{\mathscr{K}}$ *with* $||\varphi||_\infty = ||T||$.

We then have:

Theorem 2.3 (Cowen's Theorem). [Co4] *If* $\varphi \in \mathbf{L}^\infty$ *is such that* $\varphi = \overline{g} + f$ $(f, g \in \mathbf{H}^2)$*, then*

$$T_\varphi \text{ is hyponormal} \iff g = c + T_{\overline{h}} f$$

for some constant c *and some* $h \in \mathbf{H}^\infty(\mathbf{D})$ *with* $||h||_\infty \le 1$.

Sketch of proof. Let $\varphi = f + \overline{g}$ $(f, g \in \mathbf{H}^2)$. For every polynomial $p \in \mathbf{H}^2$, $\langle (T_\varphi^* T_\varphi - T_\varphi T_\varphi^*)p, p\rangle = ||H_{\overline{f}} p||^2 - ||H_{\overline{g}} p||^2$. Since polynomials are dense in $\mathbf{H}^2$,

$$T_\varphi \text{ hyponormal} \iff ||H_{\overline{g}} u|| \le ||H_{\overline{f}} u||, \ \ \forall u \in \mathbf{H}^2 \tag{2.1}$$

Write $\mathscr{K} := \mathrm{cl\,ran}(H_{\overline{f}})$ and let S be the compression of the unilateral shift U to $\mathscr{K}$. Since $\mathscr{K}$ is invariant for U^*, we have $S^* = U^*|_{\mathscr{K}}$. Suppose T_φ is hyponormal. Define A on $\mathrm{ran}(H_{\overline{f}})$ by

$$A(H_{\overline{f}} u) = H_{\overline{g}} u. \tag{2.2}$$

Then A is well defined and by (2.1), $||A|| \le 1$, so A has an extension to $\mathscr{K}$, which will also be denoted A. Then we can see that $SA^* = A^*S$. By Sarason's interpolation theorem,

$\exists\, k \in \mathbf{H}^\infty(\mathbf{D})$ with $||k||_\infty = ||A^*|| = ||A||$ such that $A^* =$ the compression of T_k to $\mathscr{K}$.

Since $T_k^* H_{\overline{f}} = H_{\overline{f}} T_{k^*}$, we have that $\mathscr{K}$ is invariant for $T_k^* = T_{\overline{k}}$, which means that A is the compression of $T_{\overline{k}}$ to $\mathscr{K}$ and by (2.2),

$$H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}}. \tag{2.3}$$

Conversely, if (2.3) holds for some $k \in \mathbf{H}^\infty(\mathbf{D})$ with $||k||_\infty \le 1$, then (2.1) holds for all u, and hence T_φ is hyponormal. Consequently, T_φ is hyponormal if and only if $H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}}$ and also $H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}}$ if and only if $g = c + T_{\overline{h}} f$ for $h = k^*$, which completes the proof. $\square$

Theorem 2.4 (Nakazi-Takahashi Variation of Cowen's Theorem). [NT] *For $\varphi \in \mathbf{L}^\infty$, put*

$$\mathscr{E}(\varphi) := \{k \in \mathbf{H}^\infty : ||k||_\infty \le 1 \textit{ and } \varphi - k\overline{\varphi} \in \mathbf{H}^\infty\}.$$

Then T_φ is hyponormal if and only if $\mathscr{E}(\varphi) \neq \emptyset$.

Proof. Let $\varphi = f + \overline{g} \in \mathbf{L}^\infty$ $(f, g \in \mathbf{H}^2)$. By Cowen's theorem, T_φ is hyponormal if and only if $g = c + T_{\overline{k}} f$ for some constant c and some $k \in \mathbf{H}^\infty$ with $||k||_\infty \le 1$. If $\varphi = k\overline{\varphi} + h$ $(h \in H^\infty)$ then $\varphi - k\overline{\varphi} = \overline{g} - k\overline{f} + f - kg \in H^\infty$. Thus $\overline{g} - k\overline{f} \in \mathbf{H}^2$, so that $P(g - \overline{k} f) = c$ ($c =$ a constant), and hence $g = c + T_{\overline{k}} f$ for some constant c. Thus T_φ is hyponormal. The argument is reversible.

2.1.2 Hyponormalityof trigonometric Toeplitz operators

In this section we consider the hyponormality of *trigonometric* Toeplitz operators, i.e., Toeplitz operators with trigonometric polynomial symbols. To do this we first review the dilation theory.

If $B = \begin{pmatrix} A & * \\ * & * \end{pmatrix}$, then B is called a *dilation* of A and A is called a *compression* of B. It was well-known that every contraction has a unitary dilation: indeed if $||A|| \le 1$, then

$$B \equiv \begin{pmatrix} A & (I - AA^*)^{\frac{1}{2}} \\ (I - A^*A)^{\frac{1}{2}} & -A^* \end{pmatrix}$$

is unitary. An operator B is called a *power* (or *strong*) *dilation* of A if B^n is a dilation of A^n for all $n = 1, 2, 3, \cdots$. So if B is a (power) dilation of A then B should be of the form $B = \begin{pmatrix} A & 0 \\ * & * \end{pmatrix}$. Sometimes, B is called a lifting of A and A is said to be lifted to B. It was also well-known that every contraction has a isometric (power) dilation. In fact, the minimal isometric dilation of a contraction A is given by

$$B \equiv \begin{pmatrix} A & 0 & 0 & 0 & \cdots \\ (I-A^*A)^{\frac{1}{2}} & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$

We then have:

Theorem 2.5 (Commutant Lifting Theorem). *Let A be a contraction and T be a minimal isometric dilation of A. If $BA = AB$ then there exists a dilation S of B such that*

$$S = \begin{pmatrix} B & 0 \\ * & * \end{pmatrix}, \quad ST = TS, \quad \textit{and} \quad ||S|| = ||B||.$$

Proof. See [GGK, p.658].

We next consider the following interpolation problem, called the Carathéodory-Schur Interpolation Problem (CSIP): Given $c_0, \cdots, c_{N-1}$ in $\mathbf{C}$, find an analytic function φ on $\mathbf{D}$ such that

(i) $\widehat{\varphi}(j) = c_j \ (j = 0, \cdots, N-1)$;
(ii) $||\varphi||_\infty \le 1$.

The following is a solution of CSIP.

Theorem 2.6. ([FF], [Ga], [GGK])

$$\textit{CSIP is solvable} \iff C \equiv \begin{pmatrix} c_0 & & & \mathbf{O} & \\ c_1 & c_0 & & & \\ c_2 & c_1 & c_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{pmatrix} \textit{ is a contraction.}$$

Moreover, φ is a solution if and only if T_φ is a contractive lifting of C which commutes with the unilateral shift.

Proof. ($\Rightarrow$) Assume that we have a solution φ. Then the condition (ii) implies that T_φ is a contraction because $||T_\varphi|| = ||\varphi||_\infty \le 1$. So the compression of T_φ is also contractive. In particular, C must have norm less than or equal to 1 for all n. Therefore if CSIP is solvable, then $||C|| \le 1$.

($\Leftarrow$) Let $||C|| \le 1$ and let

$$A := \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} : \mathbf{C}^N \to \mathbf{C}^N.$$

Then A and C are contractions and $AC = CA$. Observe that the unilateral shift U is the minimal isometric dilation of A. By the Commutant Lifting Theorem, C can be lifted to a contraction S such that $SU = US$. But then S is an analytic Toeplitz operator, i.e., $S = T_\varphi$ with $\varphi \in \mathbf{H}^\infty$. Since S is a lifting of C we must have $\widehat{\varphi}(j) = c_j$ $(j = 0, 1, \cdots, N-1)$. Since S is a contraction, it follows that $||\varphi||_\infty = ||T_\varphi|| \leq 1$. □

Now suppose φ is a trigonometric polynomial of the form

$$\varphi(z) = \sum_{n=-N}^{N} a_n z^n \ (a_N \neq 0).$$

If a function $k \in \mathbf{H}^\infty$ satisfies $\varphi - k\overline{\varphi} \in \mathbf{H}^\infty$ then k necessarily satisfies

$$k \sum_{n=1}^{N} \overline{a_n} z^{-n} - \sum_{n=1}^{N} a_{-n} z^{-n} \in \mathbf{H}^2. \tag{2.4}$$

From (2.4) one compute the Fourier coefficients $\widehat{k}(0), \cdots, \widehat{k}(N-1)$ to be $\widehat{k}(n) = c_n$ $(n = 0, 1, \cdots, N-1)$, where $c_0, c_1, \cdots, c_{N-1}$ are determined uniquely from the coefficients of φ by the following relation

$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} \overline{a_1} & \overline{a_2} & \overline{a_3} & \cdots & \overline{a_N} \\ \overline{a_2} & \overline{a_3} & \cdots & \cdot & \\ \overline{a_3} & \cdots & \cdots & & \\ \vdots & \cdots & & \mathbf{O} & \\ \overline{a_N} & & & & \end{pmatrix}^{-1} \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-N} \end{pmatrix}. \tag{2.5}$$

Thus if $k(z) = \sum_{j=0}^{\infty} c_j z^j$ is a function in $\mathbf{H}^\infty$ then $\varphi - k\overline{\varphi} \in \mathbf{H}^\infty$ if and only if $c_0, c_1, \cdots, c_{N-1}$ are given by (2.5). Thus by Cowen's theorem, if $c_0, c_1, \cdots, c_{N-1}$ are given by (2.5) then the hyponormality of T_φ is equivalent to the existence of a function $k \in \mathbf{H}^\infty$ such that $\widehat{k}(j) = c_j$ $(j = 0, \cdots, N-1)$ and $||k||_\infty \leq 1$, which is precisely the formulation of CSIP. Therefore we have:

Theorem 2.7. ([Zhu], [CCL]) *If $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_N \neq 0$ and if $c_0, c_1, \cdots, c_{N-1}$ are given by (2.5) then*

$$T_\varphi \text{ is hyponormal} \iff C \equiv \begin{pmatrix} c_0 & & & & \\ c_1 & c_0 & & \mathbf{O} & \\ c_2 & c_1 & c_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{pmatrix} \text{ is a contraction.}$$

2.1.3 Subnormality of scalar Toeplitz operators

If $\varphi \in L^\infty$, we write

$$\varphi_+ \equiv P\varphi \in H^2 \quad \text{and} \quad \varphi_- \equiv \overline{P^\perp \varphi} \in zH^2.$$

Let BMO denote the set of functions of bounded mean oscillation in L^1. Then $L^\infty \subseteq BMO \subseteq L^2$. It is well-known that if $f \in L^2$, then H_f is bounded on H^2 whenever $P^\perp f \in BMO$ (cf. [Pe]). If $\varphi \in L^\infty$, then $\overline{\varphi_-}, \overline{\varphi_+} \in BMO$, so that $H_{\overline{\varphi_-}}$ and $H_{\overline{\varphi_+}}$ are well understood.

Throughout this section we assume that both φ and $\overline{\varphi}$ are of bounded type. We recall [Ab, Lemma 3] that if $\varphi \in L^\infty$, then

$$\varphi \text{ is of bounded type} \iff \ker H_\varphi \neq \{0\}. \tag{2.6}$$

From the Beurling's Theorem, $\ker H_{\overline{\varphi_-}} = \theta_0 H^2$ and $\ker H_{\overline{\varphi_+}} = \theta_+ H^2$ for some inner functions θ_0, θ_+. We thus have $b := \overline{\varphi_-}\theta_0 \in H^2$, and hence we can write $\varphi_- = \theta_0 \overline{b}$ and similarly $\varphi_+ = \theta_+ \overline{a}$ for some $a \in H^2$. In particular, if T_φ is hyponormal then since $[T_\varphi^*, T_\varphi] = H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi = H_{\overline{\varphi_+}}^* H_{\overline{\varphi_+}} - H_{\overline{\varphi_-}}^* H_{\overline{\varphi_-}}$, it follows that $||H_{\overline{\varphi_+}} f|| \geq ||H_{\overline{\varphi_-}} f||$ for all $f \in H^2$, and hence $\theta_+ H^2 = \ker H_{\overline{\varphi_+}} \subseteq \ker H_{\overline{\varphi_-}} = \theta_0 H^2$, which implies that θ_0 divides θ_+, i.e., $\theta_+ = \theta_0 \theta_1$ for some inner function θ_1. We write, for an inner function θ,

$$\mathscr{H}(\theta) := H^2 \ominus \theta H^2.$$

Note that if $f = \theta \overline{a} \in L^2$, then $f \in H^2$ if and only if $a \in H(z\theta)$; in particular, if $f(0) = 0$ then $a \in H(\theta)$. Thus, if $\varphi = \overline{\varphi_-} + \varphi_+ \in L^\infty$ is such that φ and $\overline{\varphi}$ are of bounded type such that $\varphi_+(0) = 0$ and T_φ is hyponormal, then we can write

$$\varphi_+ = \theta_0 \theta_1 \bar{a} \quad \text{and} \quad \varphi_- = \theta_0 \bar{b},$$

where $a \in \mathscr{H}(\theta_0\theta_1)$ and $b \in \mathscr{H}(\theta_0)$. By Kronecker's Lemma [Ni, p. 183], if $f \in H^\infty$ then $\overline{f}$ is a rational function if and only if $\operatorname{rank} H_{\overline{f}} < \infty$, which implies that

$$\overline{f} \text{ is rational} \iff f = \theta \overline{b} \text{ with a finite Blaschke product } \theta. \tag{2.7}$$

Also, from the scalar-valued case of (1.6), we can see that if $k \in \mathscr{E}(\varphi)$ then

$$[T_\varphi^*, T_\varphi] = H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi = H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_{k\overline{\varphi}}^* H_{k\overline{\varphi}} = H_{\overline{\varphi}}^* (1 - T_{\widetilde{k}} T_{\widetilde{k}}^*) H_{\overline{\varphi}}. \tag{2.8}$$

The present section concerns the question: *Which Toeplitz operators are subnormal?* Recall that a Toeplitz operator T_φ is called analytic if φ is in $\mathbf{H}^\infty$, that is, φ is a bounded analytic function on $\mathbf{D}$. These are easily seen to be subnormal: $T_\varphi h = P(\varphi h) = \varphi h = M_\varphi h$ for $h \in \mathbf{H}^2$, where M_φ is the normal operator of multiplication by φ on $\mathbf{L}^2$. P.R. Halmos raised the following problem, so-called the *Halmos's Problem 5* in his 1970 lectures "Ten Problems in Hilbert Space" [Hal1],

[Hal2]:

Is every subnormal Toeplitz operator either normal or analytic ?

The question is natural because the two classes, the normal and analytic Toeplitz operators, are fairly well understood and are obviously subnormal.

In 1976, M. Abrahamse [Ab] gave a general sufficient condition for the answer to the Halmos's Problem 5 to be affirmative.

Theorem 2.8 (Abrahamse's Theorem). ([Ab]) *If*

(i) *T_φ is hyponormal;*
(ii) *φ or $\overline{\varphi}$ is of bounded type;*
(iii) $\ker[T_\varphi^*, T_\varphi]$ *is invariant for T_φ,*

then T_φ is normal or analytic.

Observe that if S is a subnormal operator on $\mathscr{H}$ and if $N := \text{mne}\,(S)$ then

$$\ker[S^*,S] = \{f : < f, [S^*,S]f >= 0\} = \{f : \; ||S^*f|| = ||Sf||\} = \{f : \; N^*f \in \mathscr{H}\}.$$

Therefore, $S(\ker[S^*,S]) \subseteq \ker[S^*,S]$.

By Theorem 2.8 and the preceding remark we get:

Corollary 2.1. *If T_φ is subnormal and if φ or $\overline{\varphi}$ is of bounded type, then T_φ is normal or analytic.*

One may ask whether a weighted shift is unitarily equivalent to a Toepltz operator. The following is the first observation for this question.

Proposition 2.1. ([Ab]) *If A is a weighted shift with weights $a_0, a_1, a_2, \cdots$ such that*

$$0 \le a_0 \le a_1 \le \cdots < a_N = a_{N+1} = \cdots = 1,$$

then A is not unitarily equivalent to any Toeplitz operator.

Recall that the Bergman shift (whose weights are given by $\sqrt{\frac{n+1}{n+2}}$) is subnormal. The following question arises naturally:

$$\textit{Is the Bergman shift unitarily equivalent to a Toeplitz operator ?} \tag{2.9}$$

An affirmative answer to the question (2.9) gives a negative answer to Halmos's Problem 5. To see this, assume that the Bergman shift S is unitarily equivalent to T_φ, then ess-ran$(\varphi) \subseteq \; \sigma_e(T_\varphi) = \sigma_e(S) = \mathbf{T}$. Thus φ is unimodular. Since S is not an isometry it follows that φ is not inner. Therefore T_φ is not an analytic Toeplitz operator.

1983, S. Sun [Sun] has answered (2.9) in the negative.

Theorem 2.9 (Sun's Theorem). ([Sun]) *Let T be a weighted shift with a strictly increasing weight sequence $\{a_n\}_{n=0}^{\infty}$. If T is unitarily equivalent to T_φ then*

$$a_n = \sqrt{1-\alpha^{2n+2}}\,||T_\varphi|| \quad (0<\alpha<1).$$

Moreover we can say more:

Corollary 2.2. ([CoL]) *If T_φ is unitarily equivalent to a weighted shift, then T_φ is subnormal.*

Proof. This follows at once from Theorem 2.9 together with the observation that the weighted shift $T \equiv W_\alpha$ with weights $\alpha_n \equiv (1-\alpha^{2n+2})^{\frac{1}{2}}$ $(0<\alpha<1)$ is subnormal: indeed, if we write $r_n := \alpha_0^2\alpha_1^2\cdots\alpha_{n-1}^2$ for the moment of W and define a discrete measure μ on $[0,1]$ by

$$\mu(z) = \begin{cases} \Pi_{j=1}^{\infty}(1-\alpha^{2j}) & (z=0) \\ \Pi_{j=1}^{\infty}(1-\alpha^{2j})\frac{\alpha^{2k}}{(1-\alpha^2)\cdots(1-\alpha^{2k})} & (z=\alpha^k; k=1,2,\cdots) \end{cases},$$

then $r_n = \int_0^1 t^n d\mu$, which by Berger's theorem, T is subnormal. $\square$

Now If T_φ is unitarily equivalent to a weighted shift, what is the form of φ ? A careful analysis of the proof of Theorem 2.9 shows that

$$\psi = \varphi - \alpha\overline{\varphi} \in \mathbf{H}^\infty.$$

But

$$\begin{aligned} T_\psi = T_\varphi - \alpha T_\varphi^* &= \begin{pmatrix} 0 & -\alpha a_0 & & & \\ a_0 & 0 & -\alpha a_1 & & \\ & a_1 & 0 & -\alpha a_2 & \\ & & a_2 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \\ &\cong \begin{pmatrix} 0 & -\alpha & & & \\ 1 & 0 & -\alpha & & \\ & 1 & 0 & -\alpha & \\ & & 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} + K \quad (K \text{ is compact}) \\ &\cong T_{z-\alpha\bar{z}} + K. \end{aligned}$$

Thus $\mathrm{ran}\,(\psi) = \sigma_e(T_\psi) = \sigma_e(T_{z-\alpha\bar{z}}) = \mathrm{ran}(z-\alpha\bar{z})$. Thus ψ is a conformal mapping of $\mathbf{D}$ onto the interior of the ellipse with vertices $\pm i(1+\alpha)$ and passing through $\pm(1-\alpha)$. On the other hand, $\psi = \varphi - \alpha\overline{\varphi}$. So $\alpha\overline{\psi} = \alpha\overline{\varphi} - \alpha^2\varphi$, which implies

$$\varphi = \frac{1}{1-\alpha^2}(\psi + \alpha\overline{\psi}).$$

1984, C. Cowen and J. Long [CoL] have announced to answer Halmos's Problem 5 in the negative.

Theorem 2.10 (Cowen and Long's Theorem). [CoL] *For* $0 < \alpha < 1$*, let* ψ *be a conformal map of* $\mathbf{D}$ *onto the interior of the ellipse with vertices* $\pm i(1-\alpha)^{-1}$ *and passing through* $\pm(1+\alpha)^{-1}$*. Then* $T_{\psi+\alpha\overline{\psi}}$ *is a subnormal weighted shift that is neither analytic nor normal.*

However, Cowen and Long's idea does not give any general connection between subnormality and Toeplitz operators. Thus we would like to ask:

Which Toeplitz operators are subnormal ?

2.2 Bounded type functions and coprime-ness

M. Abrahamse [Ab, Lemma 6] showed that if T_φ is hyponormal, if $\varphi \notin H^\infty$, and if φ or $\overline{\varphi}$ is of bounded type then both φ and $\overline{\varphi}$ are of bounded type. Its proof given in [Ab] is somewhat intricate. However via Cowen's theorem we can easily see it: indeed, if T_φ is hyponormal and $\varphi \notin H^\infty$ then there exists nonzero $k \in H^\infty$ such that $\varphi - k\overline{\varphi} \in H^\infty$, so that by (1.6), $H_\varphi = H_{k\overline{\varphi}} = H_{\overline{\varphi}}T_k$, which implies that $\ker H_\varphi \neq \{0\}$ if and only if $\ker H_{\overline{\varphi}} \neq \{0\}$, and therefore if φ or $\overline{\varphi}$ is of bounded type then both φ and $\overline{\varphi}$ are of bounded type. However, by contrast to the scalar case, Φ^* may not be of bounded type even though T_Φ is hyponormal, $\Phi \notin H^\infty_{M_n}$ and Φ is of bounded type. For example, let $f \in H^\infty$ be such that $\overline{f}$ is of bounded type, let $g \in H^\infty$ be such that $\overline{g}$ is not of bounded type and let

$$\Phi := \begin{pmatrix} f+\overline{f} & 0 \\ 0 & g \end{pmatrix}.$$

Then Φ is not analytic and is of bounded type, but Φ^* is not of bounded type. Further since Φ is diagonal and hence it is normal, and

$$\Phi - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi^* = \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix},$$

which by Theorem 1.2, implies that T_Φ is hyponormal. But we have a one-way implication: if T_Φ is hyponormal and Φ^* is of bounded type then Φ is also of bounded type (see [GHR, Corollary 3.5 and Remark 3.6]). Thus whenever we deal with hyponormal Toeplitz operators T_Φ with symbols Φ satisfying that both Φ and Φ^* are of bounded type (e.g., Φ is a matrix-valued rational function), it suffices to assume that only Φ^* is of bounded type. In spite of this, for convenience, we will assume that Φ and Φ^* are of bounded type whenever we deal with bounded type symbols.

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$

$(m \le n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We note that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$: indeed, if $\Phi = \Delta A$ with $\Delta \in H^2_{M_{n\times r}}$ $(r < n)$ then for almost all $z \in \mathbf{T}$, $\operatorname{rank} \Phi(z) \le \operatorname{rank} \Delta(z) \le r < n$, so that $\det \Phi(z) = 0$ for almost all $z \in \mathbf{T}$. If $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$.

The *shift* operator S on $H^2_{\mathbf{C}^n}$ is defined by $S := T_{zI_n}$.

The following fundamental result will be useful in the sequel.

The Beurling-Lax-Halmos Theorem. *A nonzero subspace M of $H^2_{\mathbf{C}^n}$ is invariant for the shift operator S on $H^2_{\mathbf{C}^n}$ if and only if $M = \Theta H^2_{\mathbf{C}^m}$, where Θ is an inner matrix function in $H^\infty_{M_{n\times m}}$ $(m \le n)$. Furthermore, Θ is unique up to a unitary constant right factor; that is, if $M = \Delta H^2_{\mathbf{C}^r}$ (for Δ an inner function in $H^\infty_{M_{n\times r}}$), then $m = r$ and $\Theta = \Delta W$, where W is a (constant in z) unitary matrix mapping $\mathbf{C}^m$ onto $\mathbf{C}^m$.*

We traditionally assume that two matrix-valued functions A and B are *equal* if they are equal up to a unitary constant right factor. Observe by (1.6) that for $\Phi \in L^\infty_{M_n}$, $H_\Phi S = H_\Phi T_{zI_n} = H_{\Phi \cdot zI_n} = H_{zI_n \cdot \Phi} = T^*_{\overline{z}I_n} H_\Phi$, which implies that the kernel of a block Hankel operator H_Φ is an invariant subspace of the shift operator on $H^2_{\mathbf{C}^n}$. Thus, if $\ker H_\Phi \ne \{0\}$, then by the Beurling-Lax-Halmos Theorem,

$$\ker H_\Phi = \Theta H^2_{\mathbf{C}^m}$$

for some inner matrix function Θ. We note that Θ need not be a square matrix. For example, let θ_i $(i = 0, 1, 2)$ be a scalar inner function such that θ_1 and θ_2 are coprime and let $q \in L^\infty$ be such that $\ker H_q = \{0\}$. Define

$$\Theta := \frac{1}{\sqrt{2}} \begin{pmatrix} \theta_0 \theta_1 \\ \theta_0 \theta_2 \end{pmatrix} \quad \text{and} \quad \Phi := \begin{pmatrix} \overline{\theta_0 \theta_1} & \overline{\theta_0 \theta_2} \\ q\theta_2 & -q\theta_1 \end{pmatrix}.$$

Then a straightforward calculation shows that $\ker H_\Phi = \Theta H^2$ (cf. [GHR, Example 2.9]). But it was known [GHR, Theorem 2.2] that for $\Phi \in L^\infty_{M_n}$, Φ is of bounded type if and only if $\ker H_\Phi = \Theta H^2_{\mathbf{C}^n}$ for some square inner matrix function Θ.

Let $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ be a family of inner matrix functions. Then the greatest common left inner divisor Θ_d and the least common left inner multiple Θ_m of the family $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ are the inner functions defined by

$$\Theta_d H^2_{\mathbf{C}^p} = \bigvee_{i \in J} \Theta_i H^2_{\mathbf{C}^n} \quad \text{and} \quad \Theta_m H^2_{\mathbf{C}^q} = \bigcap_{i \in J} \Theta_i H^2_{\mathbf{C}^n}.$$

The greatest common right inner divisor Θ_d' and the least common right inner multiple Θ_m' of the family $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ are the inner functions defined by

$$\widetilde{\Theta}_d' H^2_{\mathbf{C}^r} = \bigvee_{i\in J} \widetilde{\Theta}_i H^2_{\mathbf{C}^n} \quad \text{and} \quad \widetilde{\Theta}_m' H^2_{\mathbf{C}^s} = \bigcap_{i\in J} \widetilde{\Theta}_i H^2_{\mathbf{C}^n}.$$

The Beurling-Lax-Halmos Theorem guarantees that Θ_d and Θ_m are unique up to a unitary constant right factor, and Θ_d' and Θ_m' are unique up to a unitary constant left factor. We write

$$\Theta_d = \mathrm{GCD}_\ell\{\Theta_i : i \in J\}, \quad \Theta_m = \mathrm{LCM}_\ell\{\Theta_i : i \in J\},$$
$$\Theta_d' = \mathrm{GCD}_r\{\Theta_i : i \in J\}, \quad \Theta_m' = \mathrm{LCM}_r\{\Theta_i : i \in J\}.$$

If $n = 1$, then $\mathrm{GCD}_\ell\{\cdot\} = \mathrm{GCD}_r\{\cdot\}$ (simply denoted $\mathrm{GCD}\{\cdot\}$) and $\mathrm{LCM}_\ell\{\cdot\} = \mathrm{LCM}_r\{\cdot\}$ (simply denoted $\mathrm{LCM}\{\cdot\}$). In general, it is not true that $\mathrm{GCD}_\ell\{\cdot\} = \mathrm{GCD}_r\{\cdot\}$ and $\mathrm{LCM}_\ell\{\cdot\} = \mathrm{LCM}_r\{\cdot\}$.

However, we have:

Lemma 2.1. *Let $\Theta_i := \theta_i I_n$ for an inner function θ_i $(i \in J)$.*

(a) $\mathrm{GCD}_\ell\{\Theta_i : i \in J\} = \mathrm{GCD}_r\{\Theta_i : i \in J\} = \theta_d I_n$, *where* $\theta_d = \mathrm{GCD}\{\theta_i : i \in J\}$.
(b) $\mathrm{LCM}_\ell\{\Theta_i : i \in J\} = \mathrm{LCM}_r\{\Theta_i : i \in J\} = \theta_m I_n$, *where* $\theta_m = \mathrm{LCM}\{\theta_i : i \in J\}$.

Proof. This follows from at once from the definition. □

In view of Lemma 3.1, if $\Theta_i = \theta_i I_n$ for an inner function θ_i $(i \in I)$, we can define the greatest common inner divisor Θ_d and the least common inner multiple Θ_m of the Θ_i by

$$\Theta_d \equiv \mathrm{GCD}\{\Theta_i : i \in J\} := \mathrm{GCD}_\ell\{\Theta_i : i \in J\} = \mathrm{GCD}_r\{\Theta_i : i \in J\};$$
$$\Theta_m \equiv \mathrm{LCM}\{\Theta_i : i \in J\} := \mathrm{LCM}_\ell\{\Theta_i : i \in J\} = \mathrm{LCM}_r\{\Theta_i : i \in J\} :$$

they are both diagonal matrices.

For $\Phi \in L^\infty_{M_n}$ we write

$$\Phi_+ := P_n(\Phi) \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \left[P_n^\perp(\Phi)\right]^* \in H^2_{M_n}.$$

Thus we can write $\Phi = \Phi_-^* + \Phi_+$. Suppose $\Phi = [\varphi_{ij}] \in L^\infty_{M_n}$ is such that Φ^* is of bounded type. Then we may write $\varphi_{ij} = \theta_{ij}\overline{b}_{ij}$, where θ_{ij} is an inner function and θ_{ij} and b_{ij} are coprime. Thus if θ is the least common inner multiple of θ_{ij}'s then we can write

$$\Phi = [\varphi_{ij}] = [\theta_{ij}\overline{b}_{ij}] = [\theta\overline{a}_{ij}] = \Theta A^* \quad (\Theta = \theta I_n,\ A \equiv [a_{ji}] \in H^2_{M_n}). \tag{2.10}$$

We note that in the factorization (2.10), $A(\alpha)$ is nonzero whenever $\theta(\alpha) = 0$. Let $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then in view of (2.10) we can write

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*,$$

where $\Theta_i = \theta_i I_n$ with an inner function θ_i for $i = 1,2$ and $A, B \in H^2_{M_n}$. In particular, if $\Phi \in L^\infty_{M_n}$ is rational then the θ_i can be chosen as finite Blaschke products, as we observed in (2.7).

By contrast with scalar-valued functions, in (2.10) Θ and A need not be (right) coprime: for instance, if $\Phi := \begin{pmatrix} z & z \\ z & z \end{pmatrix}$ then we can write

$$\Phi = \Theta A^* = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

but $\Theta := \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ and $A := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are not right coprime because $\frac{1}{\sqrt{2}}\begin{pmatrix} z & -z \\ 1 & 1 \end{pmatrix}$ is a common right inner divisor, i.e.,

$$\Theta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z \\ -1 & z \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} z & -z \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A = \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} z & -z \\ 1 & 1 \end{pmatrix}. \tag{2.11}$$

On the other hand, the condition "(left/right) coprime factorization" is not so easy to check in general. For example, consider a simple case: $\Phi_- := \begin{pmatrix} z & z \\ z & z \end{pmatrix}$. One is tempted to write

$$\Phi_- := \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^*.$$

But $\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are not right coprime as we have seen in the Introduction. On the other hand, observe that

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \equiv \Delta B^*,$$

where

$$\Delta := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z \\ -1 & z \end{pmatrix} \text{ is inner and } B := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2z \\ 0 & 2z \end{pmatrix}.$$

Again, Δ and B are not right coprime because $\ker H_{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} = H^2_{\mathbb{C}^2}$. Thus we might choose

$$\Phi_- = \left(z I_2 \Delta \right) \cdot B^* \quad \text{or} \quad \Phi_- = \Delta \cdot \left(\overline{z} I_2 B \right)^*.$$

A straightforward calculation show that $\ker H_{\Phi_-^*} = \Delta H^2_{\mathbb{C}^2}$. Hence the latter of the above factorizations is the desired factorization: i.e., Δ and $\overline{z} I_2 B$ are right coprime.

If $\Omega = \mathrm{GCD}_\ell \{A, \Theta\}$ in the representation (2.10):

$$\Phi = \Theta A^* = A^* \Theta \quad (\Theta \equiv \theta I_n \text{ for an inner function } \theta),$$

then $\Theta = \Omega\Omega_\ell$ and $A = \Omega A_\ell$ for some inner matrix Ω_ℓ (where $\Omega_\ell \in H^2_{M_n}$ because $\det\Theta$ is not identically zero) and some $A_l \in H^2_{M_n}$. Therefore if $\Phi^* \in L^\infty_{M_n}$ is of bounded type then we can write

$$\Phi = {A_\ell}^*\Omega_\ell, \quad \text{where } A_\ell \text{ and } \Omega_\ell \text{ are left coprime.} \tag{2.12}$$

$A_\ell^*\Omega_\ell$ is called the *left coprime factorization* of Φ; similarly, we can write

$$\Phi = \Omega_r A_r^*, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime.} \tag{2.13}$$

In this case, $\Omega_r A_r^*$ is called the *right coprime factorization* of Φ.

Remark 2.1. ([GHR, Corollary 2.5]) As a consequence of the Beurling-Lax-Halmos Theorem, we can see that

$$\Phi = \Omega_r A_r^* \text{ (right coprime factorization) } \Longleftrightarrow \ker H_{\Phi^*} = \Omega_r H^2_{\mathbf{C}^n}. \tag{2.14}$$

In fact, if $\Phi = \Omega_r A_r^*$ (right coprime factorization) then it is evident that $\ker H_{\Phi^*} \supseteq \Omega_r H^2_{\mathbf{C}^n}$. From the Beurling-Lax-Halmos Theorem, $\ker H_{\Phi^*} = \Theta H^2_{\mathbf{C}^n}$, for some inner function Θ, and hence $(I-P)(\Phi^*\Theta) = 0$, i.e., $\Phi^* = D\Theta^*$, for some $D \in H^2_{\mathbf{C}^n}$. We want to show that $\Omega_r = \Theta$ up to a unitary constant right factor. Since $\Theta H^2_{\mathbf{C}^n} \supseteq \Omega_r H^2_{\mathbf{C}^n}$, we have (cf. [FF, p.240]) that $\Omega_r = \Theta\Delta$ for some square inner function Δ. Thus, $D\Theta^* = \Phi^* = A_r\Omega_r^* = A_r\Delta^*\Theta^*$, which implies $A_r = D\Delta$, so that Δ is a common right inner factor of both A_r and Ω_r. But since A_r and Ω_r are right coprime, Δ must be a unitary constant. The proof of the converse implication is entirely similar. □

From now on, for notational convenience we write

$$I_\omega := \omega I_n \ \ (\omega \in H^2) \quad \text{and} \quad H^2_0 := I_z H^2_{M_n}.$$

It is not easy to check the condition "B and Θ are coprime" in the factorization $F = B^*\Theta$ ($\Theta \equiv I_\theta$ is inner and $B \in H^2_{M_n}$). But if F is rational (and hence Θ is given in a form $\Theta \equiv I_\theta$ with a finite Blaschke product θ) then we can obtain a more tractable criterion. To see this, we need to recall the notion of finite Blaschke-Potapov product. Let $\lambda \in \mathbf{D}$ and write

$$b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z},$$

which is called a *Blaschke factor*. If M is a closed subspace of $\mathbf{C}^n$ then the matrix function of the form

$$b_\lambda P_M + (I - P_M) \quad (P_M := \text{the orthogonal projection of } \mathbf{C}^n \text{ onto } M)$$

is called a *Blaschke-Potapov factor*; an $n \times n$ matrix function D is called *a finite Blaschke-Potapov product* if D is of the form

$$D = \nu \prod_{m=1}^{M} \Big(b_m P_m + (I - P_m) \Big),$$

where ν is an $n \times n$ unitary constant matrix, b_m is a Blaschke factor, and P_m is an orthogonal projection in $\mathbf{C}^n$ for each $m = 1, \cdots, M$. In particular, a scalar-valued function D reduces to a finite Blaschke product $D = \nu \prod_{m=1}^{M} b_m$, where $\nu = e^{i\omega}$. It is also known (cf. [Po]) that an $n \times n$ matrix function D is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product.

Write $\mathscr{Z}(\theta)$ for the set of zeros of an inner function θ. We then have:

Lemma 2.2. ([CHL1]) *Let $B \in H^\infty_{M_n}$ be rational and $\Theta = I_\theta$ with a finite Blaschke product θ. Then the following statements are equivalent:*

(a) $B(\alpha)$ is invertible for each $\alpha \in \mathscr{Z}(\theta)$;
(b) B and Θ are right coprime;
(c) B and Θ are left coprime.

Lemma 2.2 may be proved by a various way. An elementary proof of Lemma 2.2 is accomplished by using an interpolation problem as in [CHL1, Lemma 3.10].

The equivalence (b)⇔(c) in Lemma fails if Θ is not a *constant* diagonal matrix. To see this, suppose a rational function $B \in H^\infty_{M_n}$ has a non-constant right inner factor. Then we may write

$$B = G \begin{pmatrix} b_\lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M \\ M^\perp \end{pmatrix},$$

where $G \in H^\infty_{M_n}$, $\lambda \in \mathbf{D}$ and M is a non-zero closed subspace of $\mathbf{C}^n$. Thus $\ker B(\lambda) \supseteq M \neq \{0\}$, so that if two rational functions B_1 and B_2 in $H^\infty_{M_n}$ have a common (non-constant) right inner factor then

$$\ker B_1(\lambda) \cap \ker B_2(\lambda) \neq \{0\} \quad \text{for some } \lambda \in \mathbf{D}.$$

We now let

$$\Theta_1 := \begin{pmatrix} b_\alpha & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Theta_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} z & -z \\ 1 & 1 \end{pmatrix}.$$

Since

$$(\Theta_1\Theta_2)(\alpha) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \Theta_1(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we have $\ker(\Theta_1\Theta_2)(\alpha) = \{(x,y)^t : x + y = 0\}$ and $\ker \Theta_1(\alpha) = \{(x,y)^t : y = 0\}$. Therefore $\ker(\Theta_1\Theta_2)(\beta) \cap \ker \Theta_1(\beta) = \{0\}$ for any $\beta \in \mathbf{D}$. Thus by the above remark, $\Theta_1\Theta_2$ and Θ_1 are right coprime even though evidently, they are not left coprime.

2.3 Pseudo-hyponormality of block Toeplitz operators

If $\Phi \in L^\infty_{M_n}$, then by (3.2),

$$[T_\Phi^*, T_\Phi] = H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi + T_{\Phi^*\Phi - \Phi\Phi^*}\,.$$

Since the normality of Φ is a necessary condition for the hyponormality of T_Φ, the positivity of $H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi$ is an essential condition for the hyponormality of T_Φ. Thus, we isolate this property as a new notion, weaker than hyponormality. The reader will notice at once that this notion is meaningful for non-scalar symbols.

Definition 2.1. Let $\Phi \in L^\infty_{M_n}$. The *pseudo-selfcommutator* of T_Φ is defined by

$$[T_\Phi^*, T_\Phi]_p := H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi.$$

T_Φ is said to be *pseudo-hyponormal* if $[T_\Phi^*, T_\Phi]_p$ is positive semidefinite.

As in the case of hyponormality of scalar Toeplitz operators, we can see that the pseudo-hyponormality of T_Φ is independent of the constant matrix term $\Phi(0)$. Thus whenever we consider the pseudo-hyponormality of T_Φ we may assume that $\Phi(0) = 0$. Observe that if $\Phi \in L^\infty_{M_n}$ then $[T_\Phi^*, T_\Phi] = [T_\Phi^*, T_\Phi]_p + T_{\Phi^*\Phi - \Phi\Phi^*}$. Thus T_Φ is hyponormal if and only if T_Φ is pseudo-hyponormal and Φ is normal and (via Theorem 3.3 of [GHR]) T_Φ is pseudo-hyponormal if and only if $\mathscr{E}(\Phi) \neq \emptyset$.

For $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$, we write

$$\mathscr{C}(\Phi) := \Big\{ K \in H^\infty_{M_n} :\ \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

Thus if $\Phi \in L^\infty_{M_n}$ then $K \in \mathscr{E}(\Phi)$ if and only if $K \in \mathscr{C}(\Phi)$ and $||K||_\infty \leq 1$. Also if $K \in \mathscr{C}(\Phi)$ then $H_{\Phi_-^*} = H_{K\Phi_+^*} = T_{\widetilde{K}}^* H_{\Phi_+^*}$, which gives a necessary condition for the nonempty-ness of $\mathscr{C}(\Phi)$ (and hence the hyponormality of T_Φ): in other words,

$$K \in \mathscr{C}(\Phi) \implies \ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}\,. \tag{2.15}$$

We then have:

Proposition 2.2. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Thus we may write*

$$\Phi_+ = \Theta_1 A^* \quad \textit{and} \quad \Phi_- = \Theta_2 B^*,$$

where $\Theta_i = I_{\theta_i}$ for an inner function θ_i $(i = 1, 2)$ and $A, B \in H^2_{M_n}$. If $\mathscr{C}(\Phi) \neq \emptyset$, then Θ_2 is an inner divisor of Θ_1, i.e., *$\Theta_1 = \Theta_0\Theta_2$ for some inner function Θ_0.*

Proposition 2.2 shows that the hyponormality of T_φ with scalar-valued rational symbol φ implies $\deg(\varphi_-) \leq \deg(\varphi_+)$, which is a generalization of the well-known result for the cases of the trigonometric Toeplitz operators, i.e., if $\varphi = \sum_{n=-m}^N a_n z^n$ is such that T_φ is hyponormal then $m \leq N$ (cf. [FL1]).

We conclude this section with the pull-back property on the symbols of pseudo-hyponormal block Toeplitz operators.

Proposition 2.3. (Pull-back symbols) *Let $\Phi \in L^\infty_{M_n}$ be of the form*

$$\Phi_+ = \Theta_0\Theta_1 A^* \quad \textit{and} \quad \Phi_- = \Theta_0 B^* \ \text{(right coprime factorizations)},$$

where Θ_0 is an inner matrix function, $\Theta_1 := I_{\theta_1}$ with a finite Blaschke product θ_1, and $A, B \in H^2_{M_n}$. If Ω is an inner divisor of Θ_1, then

$$\mathscr{C}(\Phi) = \Big\{ K\Omega : K \in \mathscr{C}(\Phi^{1,\Omega}) \Big\}, \tag{2.16}$$

where $\Phi^{1,\Omega} := \Phi_-^* + P_{H_0^2}(\Phi_+\Omega^*)$. In particular,

$$T_\Phi \ \textit{is pseudo-hyponormal} \Longleftrightarrow T_{\Phi^{1,\Omega}} \ \textit{is pseudo-hyponormal}. \tag{2.17}$$

Proposition 2.3 guarantees that the analytic part of the symbol Φ can be "pulled back" to a function having the same inner part of the factorization as that of the co-analytic part without losing the pseudo-hyponormality. In particular, if Φ is a rational function (so the θ_i are finite Blaschke products) then Proposition 2.3 says that the analytic part of the symbol Φ can be pulled back to a matrix-valued rational function with the same degree as that of the co-analytic part. This generalizes the cases where Φ is a scalar-valued trigonometric polynomial: If $\Phi \equiv \overline{f} + g$, where f and g are analytic polynomials of degrees m and N ($m \le N$), respectively and if $\Psi \equiv \overline{f} + T_{\overline{z}^{N-m}} g$ then T_Φ is hyponormal if and only if T_Ψ is hyponormal (cf. [CuL1, Lemma 1.5]).

Chapter 3
Subnormality of Rational Toeplitz operators

3.1 Halmos's Problem 5

The notion of subnormality was introduced by P.R. Halmos in 1950 and the study of subnormal operators has been highly successful and fruitful (we refer to the book [Con] for details). Indeed, the theory of subnormal operators has made significant contributions to a number of problems in functional analysis, operator theory, mathematical physics, and other fields. Oddly however, the question Which operators are subnormal ? is difficult to answer. In general, it is quite intricate to examine whether a normal extension exists for an operator. On the other hand, Toeplitz operators arise in a variety of problems in several fields of mathematics and physics, and nowadays the theory of Toeplitz operators is a very wide area. Thus it is natural and significant to elucidate the subnormality of Toeplitz operators. In 1970, P.R. Halmos addressed a problem on the subnormality of Toeplitz operators T_φ on the Hardy space $H^2 \equiv H^2(\mathbf{T})$ of the unit circle $\mathbf{T}$ in the complex plane $\mathbf{C}$. This is the so-called Halmos's Problem 5, presented in his lectures, *Ten problems in Hilbert space* [Hal1], [Hal2]:

Halmos's Problem 5. Is every subnormal Toeplitz operator either normal or analytic ?

A Toeplitz operator T_φ is called *analytic* if $\varphi \in H^\infty$. Any analytic Toeplitz operator is easily seen to be subnormal: indeed, $T_\varphi h = P(\varphi h) = \varphi h = M_\varphi h$ for $h \in H^2$, where M_φ is the normal operator of multiplication by φ on L^2. The question is natural because the two classes, the normal and analytic Toeplitz operators, are fairly well understood and are subnormal. Halmos's Problem 5 has been partially answered in the affirmative by many authors (cf. [Ab], [AIW], [CuL1], [CuL2], [NT], and etc). In 1984, Halmos's Problem 5 was answered in the negative by C. Cowen and J. Long [CoL]: they found an analytic function ψ for which $T_{\psi+\alpha\overline{\psi}}$ $(0 < \alpha < 1)$ is subnormal - in fact, this Toeplitz operator is unitarily equivalent to a subnormal weighted shift W_β with weight sequence $\beta \equiv \{\beta_n\}$, where $\beta_n = (1-\alpha^{2n+2})^{\frac{1}{2}}$ for $n = 0, 1, 2, \cdots$. Unfortunately, Cowen and Long's construction does not provide an intrinsic connection between subnormality and the theory of Toeplitz operators. Until now researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. On the other hand, surprisingly, as C. Cowen notes in [Co1] and [Co2], some analytic Toeplitz operators are unitarily equivalent to non-analytic Toeplitz operators; i.e., the analyticity of Toeplitz operators is not invariant under unitary equivalence. In this sense, we might ask whether Cowen and Long's non-analytic subnormal Toeplitz operator is unitarily equivalent to an analytic Toeplitz operator. To this end, we have:

Observation. *Cowen and Long's non-analytic subnormal Toeplitz operator T_φ is not unitarily equivalent to any analytic Toeplitz operator.*

Proof. Assume to the contrary that T_φ is unitarily equivalent to an analytic Toeplitz operator T_f. Then by the above remark, T_f is unitarily equivalent to the subnormal weighted shift W_β with weight sequence $\beta \equiv \{\beta_n\}$, where $\beta_n = (1-\alpha^{2n+2})^{\frac{1}{2}}$ $(0 < \alpha < 1)$ for $n = 0, 1, 2, \cdots$; i.e., there exists a unitary operator V such that $V^* T_f V = W_\beta$. Thus if $\{e_n\}$ is the canonical orthonormal basis for ℓ^2 then $V^* T_f V e_j = W_\beta e_j = \beta_j e_{j+1}$ for $j = 0, 1, 2, \cdots$. We thus have $\big(V^* T_{|f|^2} V\big) e_j = W_\beta^* W_\beta e_j = \beta_j^2 e_j$, and hence, $T_{|f|^2-\beta_j^2}(V e_j) = 0$ for $j = 0, 1, 2, \cdots$. Fix $j \geq 0$ and observe that $V e_j \in \ker\big(T_{|f|^2-\beta_j^2}\big)$. By Coburn's Theorem [Cob], if $|f|^2 - \beta_j^2$ is nonzero then either $T_{|f|^2-\beta_j^2}$ or $T^*_{|f|^2-\beta_j^2}$ is one-one. It follows that $|f|^2 = \beta_j^2$ for $j = 0, 1, 2, \cdots$. This readily implies that $\beta_0 = \beta_1 = \beta_2 = \cdots$, a contradiction. □

Consequently, even if we interpret "*is*" in Halmos Problem 5 as "*is up to unitary equivalence*," the answer to Halmos Problem 5 is still negative.

We would like to reformulate Halmos's Problem 5 as follows:

Halmos's Problem 5 reformulated. *Which Toeplitz operators are subnormal ?*

Directly connected with Halmos's Problem 5 is the following question:

$$\text{Which subnormal Toeplitz operators are normal or analytic ?} \tag{3.1}$$

Partial answers to question (3.1) have been obtained by many authors (cf. [Ab], [AIW], [Co1], [CoL], [CHL1], [CHL2], [CuL1], [CuL2], [CuL3], [ItW], [NT]). The best answers are obtained in one of two ways: (i) by strengthening the assumption of "subnormality," and (ii) by restricting the symbol to a special class of L^∞. Indeed, in 1975, I. Amemiya, T. Ito and T.K. Wong showed that the answer to Halmos's Problem 5 is affirmative for quasinormal operators ([AIW]):

Amemiya, Ito and Wong's Theorem ([AIW, Theorem]). Every quasinormal Toeplitz operator is either normal or analytic.

On the other hand, a function $\varphi \in L^\infty$ is said to be of bounded type if there are analytic functions $\psi_1, \psi_2 \in H^\infty$ such that $\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$ for almost all $z \in \mathbf{T}$. Evidently, rational functions are of bounded type. In 1976, M.B. Abrahamse showed that the answer is affirmative for Toeplitz operators with bounded type symbols ([Ab]):

Abrahamse's Theorem ([Ab, Theorem]). Let $\varphi \in L^\infty$ be such that φ or $\overline{\varphi}$ is of bounded type. If

(i) T_φ is hyponormal;
(ii)$\ker[T_\varphi^*, T_\varphi]$ is invariant for T_φ,

then T_φ is normal or analytic.

Consequently, since $\ker[T^*, T]$ is invariant for every subnormal operator T, it follows that if $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type, then every subnormal Toeplitz operator T_φ must be either normal or analytic.

We say that a block Toeplitz operator T_Φ is *analytic* if $\Phi \in H^\infty_{M_n}$. Evidently, any analytic block Toeplitz operator with a normal symbol is subnormal because the multiplication operator M_Φ is a normal extension of T_Φ. As a first inquiry in the above reformulation of Halmos's Problem 5 the following question can be raised:

$$\text{Which subnormal Toeplitz operators are normal or analytic ?} \tag{3.2}$$

In particular, we examine to what extent Abrahamse's Theorem and Amemiya, Ito and Wong's Theorem remain valid for Toeplitz operators with matrix-valued symbols.

3.2 Subnormality of rational Toeplitz operators

Let $\lambda \in \mathbf{D}$ and write

$$b_\lambda(z) := \xi \frac{z-\lambda}{1-\overline{\lambda} z} \quad (\xi \in \mathbf{T})\,;$$

b is called a *Blaschke factor*. If M is a nonzero closed subspace of $\mathbf{C}^n$ then the matrix function of the form

$$b_\lambda P_M + (I - P_M) \quad (P_M := \text{the orthogonal projection of } \mathbf{C}^n \text{ onto } M)$$

is called a *Blaschke-Potapov factor*; an $n \times n$ matrix function D is called *a finite Blaschke-Potapov product* if D is of the form

$$D = \nu \prod_{m=1}^{d} \Big(b_m P_m + (I - P_m) \Big), \tag{3.3}$$

where ν is an $n \times n$ unitary constant matrix, b_m is a Blaschke factor, and P_m is an orthogonal projection in $\mathbf{C}^n$ for each $m = 1, \cdots, d$. In particular, a scalar-valued function D reduces to a finite Blaschke product $D = \nu \prod_{m=1}^{d} b_m$, where $\nu = e^{i\omega}$. It is known (cf. [Po]) that an $n \times n$ matrix function D is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product. Thus if $\Phi \in L^\infty_{M_n}$ is rational then Θ_1 and Θ_2 can be chosen as finite Blaschke-Potapov products in the right coprime factorizations of (2.10).

The condition "(left/right) coprime" for two matrix-valued functions is not easy to check in general. However, if one of them is a rational function whose determinant is not identically zero then we can obtain a more tractable criterion on their (left/right) coprime-ness. To see this, we first observe:

Lemma 3.1. *If $F \in H^2_{M_n}$ and M is a non-zero closed subspace of $\mathbf{C}^n$ then*

$$b_\lambda P_M + (I - P_M) \text{ is a right inner divisor of } F \iff M \subseteq \ker F(\lambda). \tag{3.4}$$

Proof. Immediate from a direct calculation. □

Corollary 3.1. *If $A, B \in H^2_{M_n}$ and B is a rational function such that $\det B$ is not identically zero then*

$$A \text{ and } B \text{ are right coprime} \iff \ker A(\alpha) \cap \ker B(\alpha) = \{0\} \text{ for any } \alpha \in \mathbf{D}.$$

From Corollary 3.1, we can see that if $\Theta = \theta I_n$ for a finite Blaschke product θ, then for any $A \in H^2_{M_n}$,

$$A \text{ and } \Theta \text{ are right coprime} \iff A(\alpha) \text{ is invertible for each zero } \alpha \text{ of } \theta \tag{3.5}$$

(cf. [CHL2, Lemma 3.3]).

For an operator $T \in \mathscr{B}(\mathscr{H})$, the *essential norm* $||T||_e$ is defined by

$$||T||_e := \inf\big\{||T - K|| : K \text{ is compact}\big\}.$$

It is known (cf. [Pe, Theorem I.5.3]) that if $\varphi \in L^\infty$ then the essential norm of a Hankel operator H_φ can be computed from the formula

$$||H_\varphi||_e = \mathrm{dist}_{L^\infty}\Big(\varphi, H^\infty + C\Big),$$

where C is the set of continuous functions on $\mathbf{T}$. In particular, since H_φ is compact if and only if $\varphi \in H^\infty + C$, it follows that $||H_\varphi||_e = 0$ if $\varphi \in H^\infty + C$.

The following proposition provides important information on $\mathscr{E}(\Phi)$ if Φ is a matrix-valued rational function such that T_Φ is hyponormal.

Proposition 3.1. ([CHKL]) *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function such that T_Φ is hyponormal. Then $\mathscr{E}(\Phi)$ contains an inner matrix function.*

We now introduce the notion of a "matrix pole" for matrix-valued rational functions.

To do so, we first consider a representation for poles of scalar-valued rational functions. Let $\varphi \in L^\infty$ be a rational function. Then we may write

$$\varphi_- = \theta \overline{a} \quad \text{(coprime factorization),}$$

where θ is a nonconstant finite Blaschke product and $a \in H^2$. Since $\varphi = \frac{a}{\theta} + \varphi_+$, it follows that $\varphi(z)$ has a pole at $z = \alpha \in \mathbf{D}$ if and only if θ has a zero at $z = \alpha$ if and only if the Blaschke factor b_α is an inner divisor of θ. Observe that $\ker H_\varphi = \ker H_{\overline{\varphi_-}} = \theta H^2$ and that $(z-\alpha)H^2 = b_\alpha H^2$ because $1 - \overline{\alpha} z$ is an outer function, and hence $(1-\overline{\alpha}z)H^2 = H^2$. We thus have

$$\varphi(z) \text{ has a pole at } z = \alpha \iff \ker H_\varphi \subseteq (z-\alpha)H^2. \tag{3.6}$$

On the other hand, block Hankel operators have been extensively exploited when considering properties of matrix-valued functions in $L^\infty_{M_n}$ (e.g., matrix-valued versions of Neharis Theorem, Hartmans Theorem and Kroneckers Lemma). In particular, if $\Phi \in L^\infty_{M_n}$ is a matrix-valued rational function, then it is known (cf. [Pe, p. 81]) that rank H_Φ is equal to the McMillan degree of Φ_-.

For the definition of matrix poles for matrix-valued rational functions, we will adopt the idea in (3.6).

Definition 3.1. Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function. Then we say that Φ has a *matrix pole* at $\alpha \in \mathbf{D}$ if

$$\ker H_\Phi \subseteq (z-\alpha)H^2_{\mathbf{C}^n}\,.$$

We shall say that an inner matrix function $\Theta \in H^\infty_{M_n}$ is *diagonal-constant* if Θ is of the form θI_n, where θ is an inner function. We then have:

Lemma 3.2. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function. Thus in view of (2.10), we may write*

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization).}$$

Then Φ has a matrix pole if and only if Θ has a nonconstant diagonal-constant inner divisor.

Remark 3.1. (i) Recall that if $\Phi \in L^\infty_{M_n}$ is a matrix-valued rational function then Φ is said to have a *pole* at $\alpha \in \mathbf{D}$ if some entry of $\Phi(z)$ has a pole at $z = \alpha$. We now claim that for $\alpha \in \mathbf{D}$,

$$\alpha \text{ is a matrix pole of } \Phi \implies \alpha \text{ is a pole of } \Phi\,. \tag{3.7}$$

Towards (5.1) we write

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization).}$$

Suppose α is a matrix pole of Φ. Then by Lemma 3.2, $\Theta = b_\alpha I_n \Theta_1$ for some inner function Θ_1. Thus by (3.5), $B(\alpha)$ is invertible. Since $\Phi \equiv \Phi_-^* + \Phi_+ = (B + \Phi_+\Theta)\Theta^*$ and $\det\Theta$ is inner, we have

$$\det \Phi = \frac{\det(B + \Phi_+\Theta)}{\det \Theta} = \frac{\det(B + \Phi_+\Theta)}{b_\alpha^n \det\Theta_1}.$$

But since $(B + \Phi_+\Theta)(\alpha) = B(\alpha)$ is invertible, it follows that α is a pole of $\det\Phi$, which implies that some entry of $\Phi(z)$ has a pole at $z = \alpha$. This proves (5.1). However the converse of (5.1) is not true. For example if

$$\Phi := \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix},$$

then Φ has a pole at $z = 0$. But since

$$\Phi_- = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^* \quad \text{(right coprime factorization)}$$

and $\Theta \equiv \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$ has no inner divisor of the form $b_\alpha I_2$, it follows that Φ has no matrix pole. Of course, by definition, if $n = 1$ then a matrix pole reduces to a pole.

(ii) From the viewpoint of scalar-valued rational functions, we are tempted to guess that if a matrix-valued rational function $\Phi \in L^\infty_{M_n}$ has a matrix pole at $z = \alpha \in \mathbf{D}$, then Φ can be written as

$$\Phi(z) = \sum_{k=-N}^{\infty} A_k (z - \alpha)^k \quad (N \geq 1; A_{-N} \text{ is invertible}), \tag{3.8}$$

where "nonzero" in the scalar-valued case is interpreted as "invertible" in the matrix-valued case. But this is not true. For example, consider the function

$$\Phi(z) = \begin{pmatrix} \frac{1}{z^2} + z^2 & 0 \\ 0 & \frac{1}{z} + z \end{pmatrix}.$$

Then since $\Phi_-(z) = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}$, it follows from Lemma 3.2 that Φ has a matrix pole at $z = 0$, while $\Phi(z) = \sum_{k=-2}^{2} A_k z^k$ with $A_{-2} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ non-invertible. However we can easily check that (3.8) is a sufficient condition for Φ to have a matrix pole at $z = \alpha$. □

Lemma 3.3. *Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type of the form*

$$\Phi_+ = A^*\Theta_1 \quad \textit{and} \quad \Phi_- = B^*\Theta_2 = B_\ell^*\Omega_2 \text{ (left coprime factorization)},$$

where $\Theta_i := I_{\theta_i}$ *with an inner function* θ_i $(i = 1,2)$. *If* T_Φ *is hyponormal, then* Ω_2 *is a right inner divisor of* Θ_1.

In the sequel, when we consider the symbol $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$, which is such that Φ and Φ^* are of bounded type, we will, in view of Lemma 3.3, assume that

$$\Phi_+ = A^*\Omega_1\Omega_2 \quad \text{and} \quad \Phi_- = B_\ell^*\Omega_2 \text{ (left coprime factorization)},$$

where $\Omega_1\Omega_2 = \Theta \equiv I_\theta$. We also note that $\Omega_2\Omega_1 = \Theta$: indeed, if $\Omega_1\Omega_2 = \Theta \equiv I_\theta$, then $(I_{\overline{\theta}}\Omega_1)\Omega_2 = I_n$, so that $\Omega_1(I_{\overline{\theta}}\Omega_2) = I_n$, which implies that $(I_{\overline{\theta}}\Omega_2)\Omega_1 = I_n$, and hence $\Omega_2\Omega_1 = I_\theta \equiv \Theta$.

Lemma 3.4. *Suppose* $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ *is such that* Φ *and* Φ^* *are of bounded type of the form*

$$\Phi_+ = \Delta_1 A_r^* \text{ (right coprime factorization)}$$

and

$$\Phi_- = \Delta_2 B_r^* \text{ (right coprime factorization)}.$$

If T_Φ *is hyponormal, then* Δ_2 *is a left inner divisor of* Δ_1, *i.e.,* $\Delta_1 = \Delta_2\Delta_0$ *for some* Δ_0. *Hence, in particular,*

$$\Delta_2\mathscr{H}(\Delta_0) \subseteq \operatorname{cl}\operatorname{ran}[T_\Phi^*, T_\Phi]. \tag{3.9}$$

Lemma 3.5. *Let* $\Phi, \Psi \in L^\infty_{M_n}$. *If* $\Phi = I_\varphi$ *or* $\Psi = I_\psi$ *for some* $\varphi, \psi \in L^\infty$, *then*

$$H_{\Phi\Psi} = T^*_{\widetilde{\Phi}} H_\Psi + H_\Phi T_\Psi. \tag{3.10}$$

In general, question (3.2) is more difficult to answer, in comparison with the scalar-valued case. Indeed, Abrahamse's Theorem does not hold for block Toeplitz operators (even with matrix-valued *trigonometric polynomial* symbol): For instance, if

$$\Phi := \begin{pmatrix} z+\overline{z} & 0 \\ 0 & z \end{pmatrix},$$

then

$$T_\Phi = \begin{pmatrix} U_+ + U_+^* & 0 \\ 0 & U_+ \end{pmatrix} \quad (U_+ := \text{the unilateral shift on } H^2)$$

is neither normal nor analytic, although T_Φ is evidently subnormal. We believe this is due to the absence of a "matrix pole" in the symbol Φ (see Definition 3.1). That is, once we assume that a rational symbol has a matrix pole, we can get a version of Abrahamse's Theorem (Theorem 3.1 below). This concept is different from the classical notion of "pole" for matrix-valued rational functions (i.e., some entry in the matrix has a pole). The two notions coincide for scalar-valued rational functions.

Theorem 3.1. [CHKL] *Let* $\Phi \in L^\infty_{M_n}$ *be a matrix-valued rational function having a "matrix pole," i.e., there exists* $\alpha \in \mathbf{D}$ *for which* $\ker H_\Phi \subseteq (z-\alpha)H^2_{\mathbb{C}^n}$. *If*

(i) T_Φ *is hyponormal;*
(ii) $\ker[T_\Phi^*, T_\Phi]$ *is invariant for* T_Φ,

then T_Φ *is normal. Hence in particular, if* T_Φ *is subnormal then* T_Φ *is normal.*

Remark. The assumption "Φ has a matrix pole" in Theorem 3.1 is automatically satisfied if Φ is scalar-valued (i.e., when $n = 1$). Thus, if $n = 1$, Theorem 3.1 is a special case of [Ab, Theorem].

On the other hand, in [CuL1, Theorem 3.2], it was shown that 2-hyponormality and subnormality coincide for Toeplitz operators T_φ with trigonometric polynomial symbols $\varphi \in L^\infty$. Also 2-hyponormality and subnormality enjoy some common properties. The following is one of them.

Lemma 3.6. [CuL2] *If* $T \in \mathscr{B}(\mathscr{H})$ *is 2-hyponormal then* $ker[T^*, T]$ *is invariant for* T.

In view of Lemma 3.6, Theorem 3.1 can be rephrased as:

Corollary 3.2. *Let* $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ *be a matrix-valued rational function. Assume* Φ *has a matrix pole, or equivalently, if we write*

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization)},$$

then Θ *has a nonconstant diagonal-constant inner divisor. Then the following are equivalent:*

1. T_Φ *is 2-hyponormal;*
2. T_Φ *is subnormal;*
3. T_Φ *is normal.*

In particular, [CuL1, Theorem 3.2] can be generalized to the matrix-valued case, as follows.

Corollary 3.3. *Let* $\Phi \in L^\infty_{M_n}$ *be a matrix-valued trigonometric polynomial whose co-analytic outer coefficient is invertible. Then the 2-hyponormality and the normality of* T_Φ *coincide.*

Proof. Using the notation of Corollary 3.2, write $\Phi_- := \sum_{j=1}^m B_{-j} z^j$, where B_{-m} is invertible. We have

$$\Theta := z^m I_n \quad \text{and} \quad B = B_{-m}^* + B_{-m+1}^* z + \cdots + B_{-1}^* z^{m-1},$$

and hence by (3.5) and our assumption, Θ and B are right coprime. The assertion now follows at once from Corollary 3.2.

Since a nonzero coefficient in $\mathbf{C}$ is trivially *invertible*, Corollary 3.3 reduces to [CuL1, Theorem 3.2] if $n = 1$.

Example 3.1. Consider the following matrix-valued trigonometric polynomial

$$\Phi := \begin{pmatrix} 2z^3+\overline{z} & -2z^3-\overline{z} \\ 2z^2+\overline{z}^2 & 2z^2+\overline{z}^2 \end{pmatrix}. \tag{3.11}$$

Then

$$\Phi_- = \begin{pmatrix} z & z^2 \\ -z & z^2 \end{pmatrix} \quad \text{and} \quad \Phi_+ = 2\begin{pmatrix} z^3 & -z^3 \\ z^2 & z^2 \end{pmatrix}.$$

A straightforward calculation shows that $\Phi^*\Phi = \Phi\Phi^*$. If

$$K := \frac{1}{4}\begin{pmatrix} -z+z^2 & -z-z^2 \\ 1+z & 1-z \end{pmatrix},$$

then $||K||_\infty \leq 1$ and $\Phi_-^* = K\Phi_+^*$. Thus by Lemma 1.2, T_Φ is hyponormal. But a direct calculation shows that T_Φ is not normal. We note that

$$\Phi_- \equiv \begin{pmatrix} z & z^2 \\ -z & z^2 \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} z & z^2 \\ -z & z^2 \end{pmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)^*, \tag{3.12}$$

where $\Theta \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} z & z^2 \\ -z & z^2 \end{pmatrix}$ and $B \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are right coprime by Corollary 3.1. However, $\Theta \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} z & z^2 \\ -z & z^2 \end{pmatrix}$ has a nonconstant diagonal inner divisor of the form zI_2, so that Φ has a matrix pole. But since T_Φ is not normal, it follows from Theorem 3.1 that T_Φ is not subnormal.

Remark 3.2. Theorem 3.1 may fail if we drop the assumption "Φ has a matrix pole", or equivalently, "Θ has a nonconstant diagonal-constant inner divisor" in the right coprime factorization $\Phi_- = \Theta B^*$. To see this we again consider the function (3.13):

$$\Phi \equiv \begin{pmatrix} \overline{z} & \overline{z}+2z \\ \overline{z}+2z & \overline{z} \end{pmatrix}.$$

We then have

$$\Phi_- = \begin{pmatrix} z & z \\ z & z \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & z \\ -1 & z \end{pmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}\right)^*,$$

where

$\Theta \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & z \\ -1 & z \end{pmatrix}$ and $B \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$ are right coprime (by Corollary 3.1).

As we saw in (3.14), T_Φ is quasinormal, and hence subnormal. But clearly, T_Φ is neither normal nor analytic. Here we note that Θ does not have any nonconstant diagonal-constant inner divisor of the form θI_n with a Blaschke factor θ.

3.3 Quasinormality of rational Toeplitz operators

Amemiya, Ito and Wong's Theorem does not hold for the cases of matrix-valued symbols: indeed, if

$$\Phi \equiv \begin{pmatrix} \bar{z} & \bar{z}+2z \\ \bar{z}+2z & \bar{z} \end{pmatrix}, \tag{3.13}$$

then a straightforward calculation shows that

$$T_\Phi = \begin{pmatrix} U_+^* & U_+^* + 2U_+ \\ U_+^* + 2U_+ & U_+^* \end{pmatrix} \tag{3.14}$$

commutes with $T_\Phi^* T_\Phi$, i.e., T_Φ is quasinormal, but T_Φ is neither normal nor analytic. However if $W := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, then W is unitary and

$$W^* T_\Phi W = 2\begin{pmatrix} U_+^* + U_+ & 0 \\ 0 & -U_+ \end{pmatrix}, \tag{3.15}$$

which says that T_Φ is unitarily equivalent to a direct sum of a normal operator, say $2(U_+^* + U_+)$ and an analytic Toeplitz operator, say $-2U_+$. This phenomenon is not an accident.

Since the self-commutator measures a form of deviation from normality, one might expect that subnormal operators with finite rank self-commutators are well behaved. Particular attention has been paid to the case of rank-one self-commutators. For example, B. Morrel [Mo] showed that every pure subnormal operator with rank-one self-commutator is unitarily equivalent to a linear function of the unilateral shift. Subnormal operators with finite rank self-commutators have been much investigated by many authors. Recently, D. Yakubovich [Ya] gave a nice characterization of subnormal operators with finite rank self-commutators under an assumption on their normal extensions. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to *have no point masses* if it has a normal extension N that has no nonzero eigenvectors.

Yakubovich's Theorem ([Ya, Theorem 2]) If $T \in \mathscr{B}(\mathscr{H})$ is a pure subnormal operator with finite rank self-commutator and without point masses then it is unitarily equivalent to a Toeplitz operator with a matrix-valued analytic rational symbol.

By using Yakubovich's Theorem, we first prove the following:

Theorem 3.2. [CHKL] *Every pure quasinormal operator with finite rank self-commutator is unitarily equivalent to a Toeplitz operator with a matrix-valued analytic rational symbol.*

Theorem 3.3. *(Amemiya, Ito and Wong's Theorem for Matrix-Valued Rational Symbols)* [CHKL] *Every pure quasinormal Toeplitz operator with a matrix-valued rational symbol is unitarily equivalent to an analytic Toeplitz operator.*

Proof. Let $\Phi \in L^\infty_{M_n}$ be a rational function and suppose T_Φ is quasinormal (and hence hyponormal). Thus Φ is normal (cf. [GHR]), and hence we have

$$[T_\Phi^*, T_\Phi] = H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi. \tag{3.16}$$

On the other hand, since Φ is rational, it follows from the Kronecker's lemma that H_{Φ^*} and H_Φ are of finite rank. Thus by (3.16), T_Φ has finite rank self-commutator. Now the theorem follows at once from Theorem 3.2. □

3.4 Square-hyponormality of rational Toeplitz operators

The study on the square-hyponormality was originated from [Hal3, Problem 209]. It is easy to see that every power of a normal operator is normal and the same statement is true for every subnormal operator. How about hyponormal operators? Problem 209 of [Hal3] shows that there exists a hyponormal operator whose square is not hyponormal (e.g., $U^* + 2U$ for the unilateral shift U). However, as we remarked in the preceding, there exist special classes of operators that square-hyponormality and subnormality coincide. For those classes of operators, it suffices to check the square-hyponormality to show the subnormality. This certainly gives a nice answer to Halmos's Problem 5 reformulated. Indeed, in [CuL1], it was shown that every hyponormal trigonometric Toeplitz operator whose square is hyponormal must be either normal or analytic, and hence it is subnormal. In this section we extend this result to the block Toeplitz operators whose symbols are matrix-valued rational functions.

Theorem 3.4. [CHL1] *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function. Then we may write*

$$\Phi_- = B^*\Theta\,,$$

where $B \in H^2_{M_n}$ and $\Theta := I_\theta$ with a finite Blaschke product θ. Suppose B and Θ are coprime. If both T_Φ and T_Φ^2 are hyponormal then T_Φ is either normal or analytic.

Corollary 3.4. *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued trigonometric polynomial whose co-analytic outer coefficient is invertible. If T_Φ and T_Φ^2 are hyponormal then T_Φ is normal.*

Remark 3.3. In Theorem 3.4, the "coprime" condition is essential. To see this, let

$$T_\Phi := \begin{pmatrix} T_b + T_b^* & 0 \\ 0 & T_b \end{pmatrix} \quad (b \text{ is a finite Blaschke product}).$$

Since $T_b + T_b^*$ is normal and T_b is analytic, it follows that T_Φ and T_Φ^2 are both hyponormal. Obviously, T_Φ is neither normal nor analytic. Note that $\Phi_- \equiv \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^* \cdot I_b$, where $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and I_b are not coprime. □

On the other hand, we have not been able to determine whether this phenomenon is quite accidental. In fact we would guess that if $\Phi \in L^\infty_{M_n}$ is a matrix-valued rational function such that T_Φ is subnormal then T_Φ is unitarily equivalent to $T_A \oplus T_B$, where T_A is normal and T_B is analytic.

Chapter 4
Subnormality of Bounded Type Toeplitz operators

This chapter focuses on subnormality for Toeplitz operators with matrix-valued bounded type symbols.

4.1 A connection between left coprime-ness and right coprime-ness

We first need to observe a connection between left coprime-ness and right coprime-ness. We first recall the representation (2.12), and for $\Psi \in L^\infty_{M_n}$ such that Ψ^* is of bounded type, write $\Psi = \Theta_2 B^* = B^* \Theta_2$. Let Ω be the greatest common left inner divisor of B and Θ_2. Then $B = \Omega B_\ell$ and $\Theta_2 = \Omega\Omega_2$ for some $B_\ell \in H^2_{M_n}$ and some inner matrix Ω_2. Therefore we can write

$$\Psi = B_\ell^* \Omega_2, \quad \text{where } B_\ell \text{ and } \Omega_2 \text{ are left coprime:} \tag{4.1}$$

in this case, $B_\ell^* \Omega_2$ is called a *left coprime factorization* of Ψ. Similarly,

$$\Psi = \Delta_2 B_r^*, \quad \text{where } B_r \text{ and } \Delta_2 \text{ are right coprime:} \tag{4.2}$$

in this case, $\Delta_2 B_r^*$ is called a *right coprime factorization* of Ψ.

In the sequel, when we consider the symbol $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$, which is such that Φ and Φ^* are of bounded type and for which T_Φ is hyponormal, we will, in view of Proposition 2.2, assume that

$$\Phi_+ = A^* \Omega_1 \Omega_2 \quad \text{and} \quad \Phi_- = B_\ell^* \Omega_2 \text{ (left coprime factorization)}, \tag{4.3}$$

where $\Omega_1\Omega_2 = \Theta = \theta I_n$. We also note that $\Omega_2\Omega_1 = \Theta$: indeed, if $\Omega_1\Omega_2 = \Theta = \theta I_n$, then $(\overline{\theta} I_n \Omega_1)\Omega_2 = I_n$, so that $\Omega_1(\overline{\theta} I_n \Omega_2) = I_n$, which implies that $(\overline{\theta} I_n \Omega_2)\Omega_1 = I_n$, and hence $\Omega_2\Omega_1 = \theta I_n = \Theta$.

We recall the inner-outer factorization of vector-valued functions. If D and E are Hilbert spaces and if F is a function with values in $\mathscr{B}(E,D)$ such that $F(\cdot)e \in H^2_D(\mathbf{T})$ for each $e \in E$, then F is called a strong H^2-function. The strong H^2-function F is called an *inner* function if $F(\cdot)$ is an isometric operator from D into E. Write $\mathscr{P}_E$ for the set of all polynomials with values in E, i.e., $p(\zeta) = \sum_{k=0}^n \widehat{p}(k)\zeta^k$, $\widehat{p}(k) \in E$. Then the function $Fp = \sum_{k=0}^n F\widehat{p}(k)z^k$ belongs to $H^2_D(\mathbf{T})$. The strong H^2-function F is called *outer* if

$$\mathrm{cl}\, F \cdot \mathscr{P}_E = H^2_D(\mathbf{T}).$$

Note that every $F \in H^2_{M_n}$ is a strong H^2-function. We then have an analogue of the scalar Inner-Outer Factorization Theorem.

Inner-Outer Factorization. (cf. [Ni]) Every strong H^2-function F with values in $\mathscr{B}(E,D)$ can be expressed in the form

$$F = F^i F^e,$$

where F^e is an outer function with values in $\mathscr{B}(E,D')$ and F^i is an inner function with values in $\mathscr{B}(D',D)$ for some Hilbert space D'.

We introduce a key idea which provides a connection between left coprime-ness and right coprime-ness.

Definition 4.1. If $\Delta \in H^\infty_{M_n}$ is an inner function, we define

$$D(\Delta) := \mathrm{GCD}\left\{\theta I_n : \ \theta \text{ is inner and } \Delta \text{ is a (left) inner divisor of } \theta I_n\right\},$$

where $\mathrm{GCD}(\cdot)$ denotes the greatest common inner divisor.

Lemma 4.1. *If $\Delta \in H^\infty_{M_n}$ is an inner function then*

$$D(\Delta) = \delta I_n \quad \textit{for some inner funtion } \delta. \tag{4.4}$$

Note that $D(\Delta)$ is unique up to a diagonal-constant inner function of the form $e^{i\xi} I_n$.

If one of two inner functions is diagonal-constant then the "left" coprime-ness and the "right" coprime-ness between them coincide.

Lemma 4.2. *Let $\Delta \in H^\infty_{M_n}$ be inner and $\Theta := \theta I_n$ for some inner function θ. Then the following are equivalent:*

(a) *Θ and Δ are left coprime;*
(b) *Θ and Δ are right coprime;*
(c) *Θ and $D(\Delta)$ are coprime.*

Lemma 4.3. *Let $A \in H^2_{M_n}$ be such that $\det A$ is not identically zero and $\Theta := \theta I_n$ for some inner function θ. Then the following are equivalent:*

(a) *Θ and A are left coprime;*
(b) *Θ and A are right coprime.*

In Lemma 4.3, if θ is given as a finite Blaschke product then the "determinant" assumption may be dropped.

Lemma 4.4. *Let $A \in H^2_{M_n}$ and $\Theta := \theta I_n$ for a finite Blaschke product θ. Then the following are equivalent:*

(a) *Θ and A are left coprime;*
(b) *Θ and A are right coprime;*
(c) *$A(\alpha)$ is invertible for each zero α of θ.*

Proof. Immediate from Lemma 4.3 and (3.5).

4.2 A matrix-valued version of Abrahamse's Theorem

As a first inquiry in the reformulation of Halmos's Problem 5 the following question can be raised:

Is Abrahamse's Theorem valid for block Toeplitz operators ?

In this section we provide two matrix-valued version of Abrahamse's Theorem. We first observe:

Lemma 4.5. *Let θ_0 be a nonconstant inner function. Then $\mathscr{H}_{\theta_0}$ contains an outer function that is invertible in H^∞.*

We are ready for:

Theorem 4.1. (Abrahamse's Theorem for matrix-valued symbols, Version I) [CHL3] *Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type and $\det \Phi_+$ and $\det \Phi_-$ are not identically zero. Then in view of (4.1), we may write*

$$\Phi_- = B^* \Theta \quad \textit{(left coprime factorization)}.$$

Assume that Θ is a diagonal inner matrix function (which is not necessarily diagonal-constant) and that Θ has a nonconstant diagonal-constant inner divisor $\Omega \equiv \omega I_n$ (ω inner) such that Ω and $\Theta\Omega^*$ are coprime. If

(i) *T_Φ is hyponormal; and*
(ii) $\ker [T^*_\Phi, T_\Phi]$ *is invariant under T_Φ ,*

then T_Φ is either normal or analytic. Hence, in particular, if T_Φ is subnormal then it is either normal or analytic.

In Theorem 4.1, if Θ has a nonconstant diagonal-constant inner divisor of the form ωI_n with a Blaschke factor ω, then we can strengthen Theorem 4.1 by dropping the "determinant" assumption.

Corollary 4.1. *Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type. Then in view of (4.1), we may write*

$$\Phi_- = B^* \Theta \quad \textit{(left coprime factorization)}, \tag{4.5}$$

where Θ is a diagonal inner matrix function. Assume that Θ has a nonconstant diagonal-constant inner divisor $\Omega \equiv \omega I_n$ with a finite Blaschke product ω such that Ω and $\Theta\Omega^*$ are coprime. If

(i) *T_Φ is hyponormal; and*
(ii) $\ker[T_\Phi^*, T_\Phi]$ *is invariant under T_Φ,*

then T_Φ is either normal or analytic. Hence, in particular, if T_Φ is subnormal then it is either normal or analytic.

Corollary 4.2. (Abrahamse's Theorem for matrix-valued symbols, Version II) [CHL3] *Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type and $\det \Phi_+$ and $\det \Phi_-$ are not identically zero. Then in view of (4.3), we may write*

$$\Phi_+ = A^* \Theta_0 \Theta_2 \quad \textit{and} \quad \Phi_- = B^* \Theta_2,$$

where $\Theta_0 \Theta_2 = \theta I_n$ with an inner function θ. Assume that A, B and Θ_2 are left coprime and Θ_2 has a nonconstant diagonal-constant inner divisor $\Omega \equiv \omega I_n$ (ω inner). If

(i) *T_Φ is hyponormal; and*
(ii) $\ker[T_\Phi^*, T_\Phi]$ *is invariant under T_Φ,*

then T_Φ is either normal or analytic. Hence, in particular, if T_Φ is subnormal then it is either normal or analytic.

Chapter 5
A Subnormal Toeplitz Completion

Given a partially specified operator matrix with some known entries, the problem of finding suitable operators to complete the given partial operator matrix so that the resulting matrix satisfies certain given properties is called a *completion problem*. Dilation problems are special cases of completion problems: in other words, the dilation of T is a completion of the partial operator matrix $\begin{pmatrix} T & ? \\ ? & ? \end{pmatrix}$. In recent years, operator theorists have been interested in the subnormal completion problem for block Toeplitz matrices. In this chapter, we solve this completion problem.

A *partial block Toeplitz matrix* is simply an $n \times n$ matrix some of whose entries are specified Toeplitz operators and whose remaining entries are unspecified. A *subnormal completion* of a partial operator matrix is a particular specification of the unspecified entries resulting in a subnormal operator. For example

$$\begin{pmatrix} T_z & 1 - T_z T_{\overline{z}} \\ 0 & T_{\overline{z}} \end{pmatrix} \tag{5.1}$$

is a subnormal (even unitary) completion of the 2×2 partial operator matrix

$$\begin{pmatrix} T_z & ? \\ ? & T_{\overline{z}} \end{pmatrix}.$$

A *subnormal Toeplitz completion* of a partial block Toeplitz matrix is a subnormal completion whose unspecified entries are Toeplitz operators. Then the following question (the most simple case) comes up at once: Does there exist a subnormal Toeplitz completion of $\begin{pmatrix} T_z & ? \\ ? & T_{\overline{z}} \end{pmatrix}$? Evidently, (5.1) is not such a completion. To answer this question, let

$$\Phi \equiv \begin{pmatrix} z & \varphi \\ \psi & \overline{z} \end{pmatrix} \quad (\varphi, \psi \in L^\infty).$$

If T_Φ is hyponormal then by Theorem 1.2, Φ should be normal. Thus a straightforward calculation shows that $|\varphi| = |\psi|$ and $\overline{z}(\varphi + \overline{\psi}) = z(\varphi + \overline{\psi})$, which implies that

$\varphi = -\overline{\psi}$. Thus a direct calculation shows that

$$[T_\Phi^*, T_\Phi] = \begin{pmatrix} * & * \\ * & T_z T_{\overline{z}} - 1 \end{pmatrix},$$

which is not positive semi-definite because $T_z T_{\overline{z}} - 1$ is not. Therefore, there are no hyponormal Toeplitz completions of $\begin{pmatrix} T_z & ? \\ ? & T_{\overline{z}} \end{pmatrix}$. However the following problem is a nontrivial: Complete the unspecified Toeplitz entries of the partial block Toeplitz matrix $A := \begin{pmatrix} T_{\overline{z}} & ? \\ ? & T_{\overline{z}} \end{pmatrix}$ to make A subnormal. Unexpectedly, the answer to this problem is very difficult and complicated. In this chapter we are interested in the following problem which is a more general version:

Problem 5.1. Let b_λ be a Blaschke factor of the form $b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z}$ ($\lambda \in \mathbf{D}$). Complete the unspecified Toeplitz entries of the partial block Toeplitz matrix

$$A := \begin{pmatrix} T_{\overline{b}_\alpha} & ? \\ ? & T_{\overline{b}_\beta} \end{pmatrix} \quad (\alpha, \beta \in \mathbf{D})$$

to make A subnormal.

We begin with:

Lemma 5.1. *Let*

$$\Phi \equiv \begin{pmatrix} \overline{b}_\alpha & \varphi \\ \psi & \overline{b}_\beta \end{pmatrix} \quad (\varphi, \psi \in L^\infty)$$

be such that T_Φ is hyponormal. Then $\alpha = \beta$.

Theorem 5.1. [CHL3] *Let $\varphi, \psi \in L^\infty$ and consider*

$$A := \begin{pmatrix} T_{\overline{b}_\alpha} & T_\varphi \\ T_\psi & T_{\overline{b}_\beta} \end{pmatrix} \quad (\alpha, \beta \in \mathbf{D}),$$

where b_λ is a Blaschke factor of the form $b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z}$ ($\lambda \in \mathbf{D}$). The following statements are equivalent.

(a) A is normal.
(b) A is subnormal.
(c) A is 2-hyponormal.
(d) $\alpha = \beta$ and one of the following conditions holds:

1. $\varphi = e^{i\theta} b_\alpha + \zeta$ *and* $\psi = e^{i\omega}\varphi$ $(\zeta \in \mathbf{C};\ \theta, \omega \in [0, 2\pi))$;
2. $\varphi = \mu \overline{b}_\alpha + e^{i\theta}\sqrt{1+|\mu|^2}\, b_\alpha + \zeta$ *and* $\psi = e^{i(\pi - 2\arg\mu)}\varphi$ $(\mu, \zeta \in \mathbf{C},\ \mu \neq 0,\ |\mu| \neq 1,\ \theta \in [0, 2\pi))$,

except in the following special case:

$$\varphi_- = b_\alpha \theta_0' \overline{a} \text{ and } \psi_- = b_\alpha \theta_1' \overline{b} \text{ with } (ab)(\alpha) = (\theta_0' \theta_1')(\alpha) \neq 0. \tag{5.2}$$

However, if we also know that $\varphi, \psi \in L^\infty$ *are rational functions having the same number of poles then either* (2) *holds for* $|\mu| = 1$ *or*

$$\varphi = e^{i\theta} \overline{b}_\alpha + 2e^{i\omega} b_\alpha + \zeta \quad \text{and} \quad \psi = e^{-2i\theta} \varphi \quad (\theta, \omega \in [0, 2\pi),\ \zeta \in \mathbf{C}): \tag{5.3}$$

in this case, $A + e^{-i\theta}\zeta$ *is quasinormal.*

Remark 5.1. We would also ask whether there is a subnormal *non*-Toeplitz completion of $\begin{pmatrix} T_{\overline{z}} & ? \\ ? & T_{\overline{z}} \end{pmatrix}$. Unexpectedly, there is a normal non-Toeplitz completion of $\begin{pmatrix} T_{\overline{z}} & ? \\ ? & T_{\overline{z}} \end{pmatrix}$. To see this, let B be a selfadjoint operator and put

$$T = \begin{pmatrix} T_{\overline{z}} & T_z + B \\ T_z + B & T_{\overline{z}} \end{pmatrix}.$$

Then

$$[T^*, T] = \begin{pmatrix} T_{\overline{z}}B + BT_z - (T_zB + BT_{\overline{z}}) & T_zB + BT_{\overline{z}} - (T_{\overline{z}}B + BT_z) \\ BT_{\overline{z}} + T_zB - (BT_z + T_{\overline{z}}B) & T_{\overline{z}}B + BT_z - (T_zB + BT_{\overline{z}}) \end{pmatrix},$$

so that T is normal if and only if

$$T_{\overline{z}}B + BT_z = T_zB + BT_{\overline{z}}, \text{ i.e., } [T_z, B] = [T_{\overline{z}}, B]. \tag{5.4}$$

We define

$$\alpha_1 := 0 \quad \text{and} \quad \alpha_n := -\frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right) \quad \text{for } n \geq 2.$$

Let $D \equiv \operatorname{diag}(\alpha_n)$, i.e., a diagonal operator whose diagonal entries are α_n ($n = 1, 2, \cdots$) and for each $n = 1, 2\cdots$, let B_n be defined by

$$B_n = -\frac{1}{2^{n-1}} \operatorname{diag}(\alpha_{n-1}) T^*_{z^{2n}}.$$

Then

$$||B_n|| \leq \frac{1}{2^{n-1}} \sup\{\alpha_{n-1}\} < \frac{1}{2^{n-1}},$$

which implies that

$$\left|\left|\sum_{n=1}^{\infty} B_n\right|\right| \leq 2.$$

We define C by

$$C := \sum_{n=1}^{\infty} B_n.$$

Then C looks like:

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2^2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2^2} & 0 & \frac{1}{2^3} & \cdots \\ 0 & 0 & 0 & 0 & \frac{3}{2^2} & 0 & \frac{3}{2^3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{2^3} & 0 & \frac{5}{2^4} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{11}{2^4} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{21}{2^5} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that C is bounded. If we define B by

$$B := D + C + C^*,$$

then a straightforward calculation shows that B satisfies equation (5.4). Therefore the operator

$$T = \begin{pmatrix} T_{\bar{z}} & T_z + B \\ T_z + B & T_{\bar{z}} \end{pmatrix}$$

is normal. We note that $T_z + B$ is not a Toeplitz operator. $\square$

Chapter 6
Unsolved problems

6.1 The scalar case

Even though C. Cowen and J. Long have answered Halmos's Problem 5 in the negative, we don't know any more knowledge on the subnormality of Toeplitz operators. Indeed, Cowen and Long's subnormal Toeplitz operator $T_{f+\lambda\overline{f}}$ $(0<\lambda<1$, $f\equiv\psi$ -so called the ellipse map- is given by Theorem 2.10) is the (originally) only non-analytic and non-normal subnormal Toeplitz operator. Thus the following two problems are interesting and the affirmative answers for these problems provide a good chance to understand the subnormality of Toeplitz operators.

Problem 6.1. For which $f\in\mathbf{H}^\infty$, is there λ $(0<\lambda<1)$ with $T_{f+\lambda\overline{f}}$ subnormal ?

Problem 6.2. Suppose ψ is as in Theorem 2.10 (i.e., the ellipse map). Are there $g\in\mathbf{H}^\infty$, $g\neq\lambda\psi+c$, such that $T_{\psi+\overline{g}}$ is subnormal ?

On the other hand, if T_φ is the subnormal operator in the Cowen and Long's Theorem then $\mathscr{E}(\varphi)=\{\lambda\}$ with $|\lambda|<1$. Also it is easily check that if T_φ is normal then $\mathscr{E}(\varphi)=\{e^{i\theta}\}$. Thus we are tempted to guess that if T_φ is subnormal and non-normal then $\mathscr{E}(\varphi)=\{\lambda\}$ with $|\lambda|<1$. However we were unable to decide whether or not it is true.

Problem 6.3. If T_φ is subnormal, does it follow that $\mathscr{E}(\varphi)=\{\lambda\}$ with $|\lambda|<1$?

If the answer to Problem 6.3 is affirmative then for $\varphi=\overline{g}+f$, the subnormality of T_φ implies $\overline{g}-\lambda\overline{f}\in\mathbf{H}^2$ with $|\lambda|<1$, so that $g=\overline{\lambda}f+c$ (c a constant), which says that the answer to Problem 6.2 is negative.

From Lemma 3.6, if $T\in\mathscr{B}(\mathscr{H})$ is 2-hyponormal then $\ker[T^*,T]$ is invariant for T. / Thus this fact together with Abrahamse's Theorem says that if T_φ is 2-hyponormal and if φ or $\bar{\varphi}$ is of bounded type then T_φ is subnormal, so that T_φ has a nontrivial invariant subspace. The following question is naturally raised:

Problem 6.4. Does every 2-hyponormal Toeplitz operator have a nontrivial invariant subspace ? More generally, does every 2-hyponormal operator have a nontrivial invariant subspace ?

Write $C(K)$ for the set of all complex-valued continuous functions on K and $R(K)$ for the uniform closure of all rational functions with poles off K in $C(K)$. It is well known ([Bro]) that if T is a hyponormal operator such that $R(\sigma(T)) \neq C(\sigma(T))$ then T has a nontrivial invariant subspace. But it remains still open whether every hyponormal operator with $R(\sigma(T)) = C(\sigma(T))$ (i.e., a *thin* spectrum) has a nontrivial invariant subspace. Recall that $T \in \mathscr{B}(\mathscr{H})$ is called a *von-Neumann operator* if $\sigma(T)$ is a spectral set for T, or equivalently, $f(T)$ is normaloid (i.e., norm equals spectral radius) for every rational function f with poles off $\sigma(T)$. B. Prunaru [Pru] has proved that polynomially hyponormal operators have nontrivial invariant subspaces. It was also known ([Ag]) that von-Neumann operators enjoy the same property. The following is a sub-question of Problem 6.4.

Problem 6.5. Is every 2-hyponormal operator with thin spectrum a von-Neumann operator ?

Although the existence of a non-subnormal polynomially hyponormal weighted shift was established in [CP1] and [CP2], it is still an open question whether the implication "polynomially hyponormal $\Rightarrow$ subnormal" can be disproved with a Toeplitz operator.

Problem 6.6. Does there exist a Toeplitz operator which is polynomially hyponormal but not subnormal ?

In [CuL2] it was shown that every pure 2-hyponormal operator with rank-one self-commutator is a linear function of the unilateral shift. McCarthy and Yang [McCYa] classified all rationally cyclic subnormal operators with finite rank self-commutators. However it remains still open what are the pure subnormal operators with finite rank self-commutators. Now the following question comes up at once:

Problem 6.7. If T_φ is a 2-hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that T_φ is analytic ?

For affirmativeness to Problem 6.7 we shall give a partial answer. To do this we recall Theorem 15 in [NT] which states that if T_φ is subnormal and $\varphi = q\bar{\varphi}$, where q is a finite Blaschke product then T_φ is normal or analytic. But from a careful examination of the proof of the theorem we can see that its proof uses subnormality assumption only for the fact that $\ker[T_\varphi^*, T_\varphi]$ is invariant under T_φ. Thus in view of Lemma 3.6, the theorem is still valid for "2–hyponormal" in place of "subnormal". We thus have:

Corollary 6.1. *If T_φ is 2–hyponormal and $\varphi = q\bar{\varphi}$, where q is a finite Blaschke product then T_φ is normal or analytic.*

Corollary 6.2. *If T_φ is 2-hyponormal and $\mathscr{E}(\varphi)$ contains at least two elements then T_φ is normal or analytic.*

Proof. This follows from Corollary 6.1 and the fact ([NT, Proposition 8]) that if $\mathscr{E}(\varphi)$ contains at least two elements then φ is of bounded type.

We now give a partial answer to Problem 6.7.

Proposition 6.1. *Suppose* $\log|\varphi|$ *is not integrable. If* T_φ *is a 2–hyponormal operator with nonzero finite rank self-commutator then* T_φ *is analytic.*

Proof. If T_φ is hyponormal such that $\log|\varphi|$ is not integrable then by an argument of [NT, Theorem 4], $\varphi = q\bar{\varphi}$ for some inner function q. Also if T_φ has a finite rank self-commutator then by [NT, Theorem 10], there exists a finite Blaschke product $b \in \mathscr{E}(\varphi)$. If $q \neq b$, so that $\mathscr{E}(\varphi)$ contains at least two elements, then by Corollary 6.2, T_φ is normal or analytic. If instead $q = b$ then by Corollary 6.1, T_φ is also normal or analytic. □

Proposition 6.1 reduces Problem 6.7 to the class of Toeplitz operators such that $\log|\varphi|$ is integrable. If $\log|\varphi|$ is integrable then there exists an outer function e such that $|\varphi| = |e|$. Thus we may write $\varphi = ue$, where u is a unimodular function. Since by the Douglas-Rudin theorem (cf. [Ga, p.192]), every unimodular function can be approximated by quotients of inner functions, it follows that if $\log|\varphi|$ is integrable then φ can be approximated by functions of bounded type. Therefore if we could obtain such a sequence ψ_n converging to φ such that T_{ψ_n} is 2–hyponormal with finite rank self-commutator for each n, then we would answer Problem J affirmatively. On the other hand, if T_φ attains its norm then by a result of Brown and Douglas [BD], φ is of the form $\varphi = \lambda\frac{\psi}{\theta}$ with $\lambda > 0$, ψ and θ inner. Thus φ is of bounded type. Therefore by Corollary 6.2, if T_φ is 2–hyponormal and attains its norm then T_φ is normal or analytic. However we were not able to decide that if T_φ is a 2–hyponormal operator with finite rank self-commutator then T_φ attains its norm.

6.2 The block case

6.2.1. Nakazi-Takahashi's Theorem for matrix-valued symbols. T. Nakazi and K. Takahashi [NT] have shown that if $\varphi \in L^\infty$ is such that T_φ is a hyponormal operator whose self-commutator $[T_\varphi^*, T_\varphi]$ is of finite rank then there exists a finite Blaschke product $b \in \mathscr{E}(\varphi)$ such that $\deg(b) = \text{rank}\,[T_\varphi^*, T_\varphi]$. What is the matrix-valued version of Nakazi and Takahashi's Theorem ? A candidate is as follows: If $\Phi \in L^\infty_{M_n}$ is such that T_Φ is a hyponormal operator whose self-commutator $[T_\Phi^*, T_\Phi]$ is of finite rank then there exists a finite Blaschke-Potapov product $B \in \mathscr{E}(\Phi)$ such that $\deg(B) = \text{rank}\,[T_\Phi^*, T_\Phi]$. We recall that the degree of the finite Blaschke-Potapov product B is defined by

$$\deg(B) := \dim \mathscr{H}(B) = \deg(\det B). \tag{6.1}$$

In this sense, we would pose the following question:

Problem 6.8. If $\Phi \in L^\infty_{M_n}$ is such that T_Φ is a hyponormal operator whose self-commutator $[T^*_\Phi, T_\Phi]$ is of finite rank, does there exist a finite Blaschke-Potapov product $B \in \mathscr{E}(\Phi)$ such that $\text{rank}\,[T^*_\Phi, T_\Phi] = \deg\Big(\det B\Big)$?

On the other hand, in [NT], it was shown that if $\varphi \in L^\infty$ is such that T_φ is subnormal and $\varphi = q\overline{\varphi}$, where q is a finite Blaschke product then T_φ is normal or analytic. We now we pose its block version:

Problem 6.9. If $\Phi \in L^\infty_{M_n}$ is such that T_Φ is subnormal and $\Phi = B\Phi^*$, where B is a a finite Blaschke-Potapov product, does it follow that T_Φ is normal or analytic ?

6.2.2. Subnormality of block Toeplitz operators. In Remark 3.3 we have shown that if the "coprime" condition of Theorem 3.4 is dropped, then Theorem 3.4 may fail. However we note that the example given in Remark 3.3 is unitarily equivalent to a direct sum of a normal Toeplitz operator and an analytic Toeplitz operator. Based on this observation, we have:

Problem 6.10. Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function. If T_Φ and T^2_Φ are hyponormal, but T_Φ is neither normal nor analytic, does it follow that T_Φ is unitarily equivalent to the form

$$\begin{pmatrix} T_A & 0 \\ 0 & T_B \end{pmatrix} \quad \text{(where } T_A \text{ is normal and } T_B \text{ is analytic)?}$$

From the view point that if $T \in \mathscr{B}(\mathscr{H})$ is subnormal then $\ker\,[T^*, T]$ is invariant under T, we might be tempted to guess that if the condition "T_Φ and T^2_Φ are hyponormal"is replaced by "T_Φ is hyponormal and $\ker\,[T^*_\Phi, T_\Phi]$ is invariant under T_Φ," then the answer to Problem 6.10 is affirmative. But this is not the case. Indeed, consider

$$T_\Phi = \begin{pmatrix} 2U + U^* & U^* \\ U^* & 2U + U^* \end{pmatrix}.$$

Then a straightforward calculation shows that T_Φ is hyponormal and $\ker\,[T^*_\Phi, T_\Phi]$ is invariant under T_Φ, but T_Φ is never normal. However, if the condition "T_Φ and T^2_Φ are hyponormal" is strengthened to "T_Φ is subnormal", what conclusion do you draw ?

6.2.3. Subnormal completion problem. Theorem 5.1 provides the subnormal Toeplitz completion of

$$\begin{pmatrix} U^* & ? \\ ? & U^* \end{pmatrix} \quad (U \text{ is the shift on } H^2). \tag{6.2}$$

Moreover Remark 5.1 shows that there is a normal non-Toeplitz completion of (6.2). However we were unable to find all subnormal completions of (6.2).

Problem 6.11. Let U be the shift on H^2. Complete the unspecified entries of the partial block matrix $\begin{pmatrix} U^* & ? \\ ? & U^* \end{pmatrix}$ to make it subnormal.

References

[Abd] A. Abdollahi, *Self-commutators of automatic composition operators on the Dirichlet space*, Proc. Amer. Math. Soc. **136(9)**(2008), 3185–3193.

[Ab] M.B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43** (1976), 597–604.

[Ag] J. Agler, *An invariant subspace problem*, J. Funct. Anal. **38**(1980), 315–323.

[AC] P.R. Ahern and D.N. Clark, *On functions orthogonal to invariant subspaces*, Acta Math. **124**(1970), 191–204.

[AIW] I. Amemiya, T. Ito, and T.K. Wong, *On quasinormal Toeplitz operators*, Proc. Amer. Math. Soc. **50** (1975), 254–258.

[Ap] C. Apostol, *The reduced minimum modulus*, Michigan Math. J. **32** (1985), 279–294.

[At] A. Athavale, *On joint hyponormality of operators*, Proc. Amer. Math. Soc. **103** (1988), 417–423.

[BS] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, Springer, Berline-Heidelberg, 2006.

[Br] J. Bram, *Sunormal operators*, Duke Math. J. **22** (1955), 75–94.

[BD] A. Brown and R.G. Douglas, *Partially isometric Toeplitz operators*, Proc. Amer. Math. Soc. **16**(1965), 681–682.

[BH] A. Brown and P.R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213**(1963/1964), 89–102.

[Bro] S. Brown, *Hyponormal operators with thick spectra have invariant subspaces*, Ann. of Math. **125**(1987), 93–103.

[Cob] L.A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13**(1966), 285–288.

[Con] J.B. Conway, *The Theory of Subnormal Operators*, Math Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.

[CS] J.B. Conway and W. Szymanski, *Linear combination of hyponormal operators*, Rocky Mountain J. Math. **18**(1988), 695–705.

[Co1] C. Cowen, *On equivalnce of Toeplitz operators*, J. Operator Theory **7**(1982), 167–172.

[Co2] C. Cowen, *More subnormal Toeplitz operators*, J. Reine Angew. Math. **367**(1986), 215–219.

[Co3] C. Cowen, *Hyponormal and subnormal Toeplitz operators*, Surveys of Some Recent Results in Operator Theory, I (J.B. Conway and B.B. Morrel, eds.), Pitman Research Notes in Mathematics, Volume 171, Longman, 1988, pp. (155-167).

[Co4] C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103**(1988), 809–812.

[CoL] C. Cowen and J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351**(1984), 216–220.

[Cu] R.E. Curto, *Joint hyponormality: A bridge between hyponormality and subnormality*, Operator Theory: Operator Algebras and Applications (Durham, NH, 1988) (W.B. Arveson and R.G. Douglas, eds.), Proc. Sympos. Pure Math., vol. 51, Part II, Amer. Math. Soc., Providence, 1990, 69–91.

[CCL] R.E. Curto, M. Cho and W.Y. Lee, *Triangular Toeplitz contractions and Cowen's sets for analytic polynomials*, Proc. Amer. Math. Soc. **130**(2002), 3597-3604.

[CuF1] R.E. Curto and L. A. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. **17**(1991), 603–635.

[CuF2] R.E. Curto and L. A. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, Integral Equations Operator Theory **17**(1993), 202–246.

[CuF3] R.E. Curto and L. A. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem II*, Integral Equations Operator Theory **18**(1994), 369–426.

[CHKL] R.E. Curto, I.S. Hwang, D. Kang and W.Y. Lee, *Subnormal and quasinormal Toeplitz operators with matrix-valued rational symbols*, prerpint, 2013.

[CHL1] R.E. Curto, I.S. Hwang and W.Y. Lee, *Hyponormality and subnormality of block Toeplitz operators*, Adv. Math. **230**(2012), 2094–2151.

[CHL2] R.E. Curto, I.S. Hwang and W.Y. Lee, *Which subnormal Toeplitz operators are either normal or analytic ?*, J. Funct. Anal. **263(8)**(2012), 2333–2354.

[CHL3] R.E. Curto, I.S. Hwang and W.Y. Lee, *Abrahamses Theorem for matrix-valued symbols and subnormal Toeplitz completions*, Preprint 2013.

[CJP] R.E. Curto, I.B. Jung and S.S. Park, *A characterization of k-hyponormality via weak subnormality*, J. Math. Anal. Appl. **279**(2003), 556–568.

[CLL] R.E. Curto, S.H. Lee and W.Y. Lee, *Subnormality and 2-hyponormality for Toeplitz operators*, Integral Equations Operator Theory, **44**(2002), 138–148.

[CuL1] R.E. Curto and W.Y. Lee, *Joint hyponormality of Toeplitz pairs*, Memoirs Amer. Math. Soc. **712**, Amer. Math. Soc., Providence, 2001.

[CuL2] R.E. Curto and W.Y. Lee, *Towards a model theory for* 2*–hyponormal operators*, Integral Equations Operator Theory **44**(2002), 290–315.

[CuL3] R.E. Curto and W.Y. Lee, *Subnormality and k-hyponormality of Toeplitz operators: A brief survey and open questions*, Operator Theory and Banach Algebras (Rabat, 1999), 73–81, Theta, Bucharest, 2003.

[CMX] R.E. Curto, P.S. Muhly and J. Xia, *Hyponormal pairs of commuting operators*, Contributions to Operator Theory and Its Applications (Mesa, AZ, 1987) (I. Gohberg, J.W. Helton and L. Rodman, eds.), Operator Theory: Advances and Applications, vol. 35, Birkhäuser, Basel–Boston, 1988, 1–22.

[CP1] R.E. Curto and M. Putinar, *Existence of non-subnormal polynomially hyponormal operators*, Bull. Amer. Math. Soc. (N.S.), **25**(1991), 373–378.

[CP2] R.E. Curto and M. Putinar, *Nearly subnormal operators and moment problems*, J. Funct. Anal. **115**(1993), 480–497.

[Do1] R.G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[Do2] R.G. Douglas, *Banach Algebra Techniques in the Theory of Toeplitz Operators*, CBMS 15, Providence, Amer. Math. Soc. 1973.

[DPY] R.G. Douglas, V.I. Paulsen, and K. Yan, *Operator theory and algebraic geometry*, Bull. Amer. Math. Soc. (N.S.) **20**(1989), 67–71.

[DY] R.G. Douglas and K. Yan, *A multi-variable Berger-Shaw theorem*, J. Operator Theory **27**(1992), 205–217.

[Fan] P. Fan, *Remarks on hyponormal trigonometric Toeplitz operators*, Rocky Mountain J. Math. **13**(1983), 489–493.

[FL1] D.R. Farenick and W.Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348**(1996), 4153–4174.

[FL2] D.R. Farenick and W.Y. Lee, *On hyponormal Toeplitz operators with polynomial and circulant-type symbols*, Integral Equation and Operator Theory **29**(1997), 202–210.

[FM] D.R. Farenick and R. McEachin, *Toeplitz operators hyponormal with the unilateral shift*, Integral Equations Operator Theory **22**(1995), 273–280.

[FF] C. Foiaş and A. Frazho, *The commutant lifting approach to interpolation problems*, Operator Theory: Adv. Appl. vol 44, Birkhäuser, Boston, 1993.

[Ga] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[GGK] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of Linear Operators, Vol II*, Basel, Birkhäuser, 1993.

[Gu1] C. Gu, *A generalization of Cowen's characterization of hyponormal Toeplitz operators*, J. Funct. Anal. **124**(1994), 135–148.

[Gu2] C. Gu, *On a class of jointly hyponormal Toeplitz operators*, Trans. Amer. Math. Soc. **354**(2002), 3275–3298.

[Gu3] C. Gu, *Non-subnormal k-hyponormal Toeplitz operators* (preprint).

[GHR] C. Gu, J. Hendricks and D. Rutherford, *Hyponormality of block Toeplitz operators*, Pacific J. Math. **223** (2006), 95–111.

[GK] C. Gu and D. Kang, *Normal Toeplitz and hankel operators with operator-valued symbols*, (preprint, 2013).

[GS] C. Gu and J.E. Shapiro, *Kernels of Hankel operators and hyponormality of Toeplitz operators*, Math. Ann. **319**(2001), 553–572.

[Hal1] P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76**(1970), 887–933.

[Hal2] P. R. Halmos, *Ten years in Hilbert space*, Integral Equations Operator Theory **2**(1979), 529–564.

[Hal3] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed. Springer, New York, 1982.

[Har] R. E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 109, Marcel Dekker, New York, 1988.

[HKL1] I. S. Hwang, I. H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols*, Math. Ann. **313(2)** (1999), 247-261.

[HKL2] I. S. Hwang, I. H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols: An extremal case*, Math. Nach. **231** (2001), 25-38.

[HL1] I. S. Hwang and W.Y. Lee, *Hyponormality of trigonometric Toeplitz operators*, Trans. Amer. Math. Soc. **354** (2002), 2461-2474.

[HL2] I. S. Hwang and W.Y. Lee, *Hyponormality of Toeplitz operators with rational symbols*, Math. Ann. **335**(2006), 405–414.

[HL3] I. S. Hwang and W.Y. Lee, *Hyponormal Toeplitz operators with rational symbols*, J. Operator Theory **56**(2006), 47–58.

[HL4] I. S. Hwang and W.Y. Lee, *Block Toeplitz Operators with rational symbols*, J. Phys. A: Math. Theor. **41(18)**(2008), 185207.

[HL5] I. S. Hwang and W.Y. Lee, *Block Toeplitz Operators with rational symbols (II)*, J. Phys. A: Math. Theor. **41(38)**(2008), 385206.

[HL6] I. S. Hwang and W.Y. Lee, *Joint hyponormality of rational Toeplitz pairs*, Integral Equations Operator Theory **65**(2009), 387–403.

[IC] Kh.D. Ikramov and V.N. Chugunov, *Normality conditions for a complex Toeplitz matrix*, Zh. Vychisl. Mat. i Mat. Fiz. **36**(1996), 3–10.

[It] T. Ito, *Every normal Toeplitz matrix is either of type(I) or type (II)*, SIAM J. Matrix Anal. Appl. **17**(1996), 998–1006.

[ItW] T. Ito and T.K. Wong, *Subnormality and quasinormality of Toeplitz operators*, Proc. Amer. Math. Soc. **34**(1972), 157–164.

[Le] W. Y. Lee, *Cowen sets for Toeplitz operators with finite rank selfcommutators*, J. Operator Theory **54(2)**(2005), 301-307.

[MaZ] J. Ma and S. Zhou, *A necessary and sufficient condition for an operator to be subnormal*, Nanjing Daxue Xuebao (Chinese) **2**(1985), 258–267.

[MP] B.D. MacCluer and M.A. Pons, *Automatic composition operators on Hardy and Bergman spaces in the ball*, Houston J. Math. **32(4)**(2006), 1121–1132.

[MP] M. Martin and M. Putinar, *Lectures on Hyponormal Operators*, Operator Theory: Adv. Appl. vol 39, Birkhäuser, Verlag, 1989.

[MAR] R.A. Martínez-Avendaño and P. Rosenthal, *An Introduction to Operators on the Hardy-Hilbert Space*, Springer, New York, 2007.

[McCYa] J.E. McCarthy and L. Yang, *Subnormal operators and quadrature domains*, Adv. Math. **127**(1997), 52–72.

[McCP1] S. McCullough and V. Paulsen, *A note on joint hyponormality*, Proc. Amer. Math. Soc. **107**(1989), 187–195.

[McCP2] S. McCullough and V. Paulsen, *k-hyponormality of weighted shifts*, Proc. Amer. Math. Soc. **116**(1992), 165–169.

[Mo] B.B. Morrel, *A decomposition for some operators*, Indiana Univ. Math. J. **23**(1973), 497–511.

[NT] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338**(1993), 753–769.

[Ni] N. K. Nikolskii, *Treatise on the Shift Operator*, Springer, New York, 1986.

[Pe] V. V. Peller, *Hankel Operators and Their Applications*, Springer, New York, 2003.

[Po] V.P. Potapov, *On the multiplicative structure of J-nonexpansive matrix functions*, Tr. Mosk. Mat. Obs. (1955), 125-236 (in Russian); English trasl. in: Amer. Math. Soc. Transl. (2) 15(1966), 131-243.

[Pru] B. Prunaru, *Invariant subspaces for polynomially hyponormal operators*, Proc. Amer. Math. Soc. **125**(1997), 1689–1691.

[Sa] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127**(1967), 179–203.

[Sch] I. Schur, *Über Potenzreihen die im Innern des Einheitskreises beschrānkt sind*, J. Reine Angew. Math. **147**(1917), 205–232.

[Smu] J.L. Smul'jan, *An operator Hellinger integral (*Russian*)*, Mat. Sb. (N.S.) **91**(1959), 381–430.

[Spi] I. M. Spitkovskii *A criterion for normality of operators in Hilbert space*, Funct. Anal. Appl. **16**(1982), 367–379.

[Sun] S. Sun, *Bergman shift is not unitarily equivalent to a Toeplitz operator*, Kexue Tongbao(English Ed.) **28**(1983), 1027–1030.

[Ta] S. Takenaka, *On the orthogonal functions and a new formula of interpolation*, Japan J. Math. **2**(1925), 129–145.

[Xi] D. Xia, *On the semi-hyponormal n-tuple of operators*, Integral Equations Operator Theory, **6**(1983), 879–898.

[Ya] D.V. Yakubovich, *Real separated algebraic curves, quadrature domains, Ahlfors type functions and operator theory*, J. Funct. Anal. **236**(2006), 25–58.

[Zhu] K. Zhu, *Hyponormal Toeplitz operators with polynomial symbols*, Integral Equations Operator Theory **21**(1996), 376–381

[Zo] N. Zorboska, *Closed range essentially normal composition operators*, Acta Sci. Math.(Szeged) **65**(1999), 287–292.