FINITE DIMENSIONAL WEAK HOPF ALGEBRAS AND THEIR APPLICATIONS

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1. Lecture 1. Introduction to weak Hopf algebras

Abstract.

We begin with an explanation why weak Hopf algebras are interesting and give basic definitions and examples of finite dimensional weak Hopf algebras. Then we discuss their most important properties and describe the fundamental results of the weak Hopf algebra theory. Motivated by applications, we especially focus on semisimple and C*-weak Hopf algebras.

We use Sweedler's leg notation for comultiplication : $\Delta(b) = b_{(1)} \otimes b_{(2)}$. Let k be a field. For simplicity, one can suppose $k = \mathbb{C}$.

Definitions.

Definition 1.0.1. [BNSz] A *finite quantum groupoid* or *a weak Hopf algebra* is a finite dimensional k-vector space H that has structures of an algebra $(H,\, m,\, 1)$ and a coalgebra $(H,\, \Delta,\, \varepsilon)$ related as follows :

Δ is a (not necessarily unit-preserving) homomorphism :

$$\Delta(hg) = \Delta(h)\Delta(g), \tag{1}$$

The unit and counit satisfy the identities :

$$\varepsilon(hgf) = \varepsilon(hg_{(1)})\varepsilon(g_{(2)}f) = \varepsilon(hg_{(2)})\varepsilon(g_{(1)}f), \tag{2}$$

$$(\Delta\otimes \mathrm{id})\Delta(1) = (\Delta(1)\otimes 1)(1\otimes\Delta(1)) = (1\otimes\Delta(1))(\Delta(1)\otimes 1), \tag{3}$$

There is a linear map $S : H \to H$, called an *antipode*, such that

$$m(\mathrm{id}\otimes S)\Delta(h) = (\varepsilon\otimes \mathrm{id})(\Delta(1)(h\otimes 1)), \tag{4}$$

$$m(S\otimes \mathrm{id})\Delta(h) = (\mathrm{id}\otimes\varepsilon)((1\otimes h)\Delta(1)), \tag{5}$$

$$m(m\otimes id)(S\otimes \mathrm{id}\otimes S)(\Delta\otimes id)\Delta(h) = S(h), \tag{6}$$

for all $h, g, f \in H$.

A *morphism* between quantum groupoids H_1 and H_2 is a map $\alpha : H_1 \to H_2$ which is both algebra and coalgebra homomorphism and which intertwines the antipodes of H_1 and H_2, i.e., $\alpha \circ S_1 = S_2 \circ \alpha$.

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The dual quantum groupoid

It is not hard to see that the set of axioms of Definition 1.0.1 is self-dual. This allows to define a natural quantum groupoid structure, $\hat{H}$, on the dual vector space $\hat{H}$ by the usual procedure of "reversing the arrows" :

$$\langle \phi\psi,\, h \rangle = \langle \phi \otimes \psi,\, \Delta(h) \rangle, \tag{7}$$

$$\langle \hat{\Delta}(\phi),\, h \otimes g \rangle = \langle \phi,\, hg \rangle, \tag{8}$$

$$\langle \hat{S}(\phi),\, h \rangle = \langle \phi,\, S(h) \rangle, \tag{9}$$

for all $\phi, \psi \in \hat{H}$, $h, g \in H$. The unit $\hat{1}$ of $\hat{H}$ is ε and counit $\hat{\varepsilon}$ is $\phi \mapsto \langle \phi,\, 1 \rangle$.

Counital maps and subalgebras

The linear maps defined in (4) and (5) are called *target* and *source counital maps* (see examples below for explanation of the terminology) and denoted ε_t and ε_s respectively :

$$\varepsilon_t(h) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)), \tag{10}$$

$$\varepsilon_s(h) = (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)). \tag{11}$$

In the next proposition we collect several useful properties of counital maps.

Proposition 1.0.2. *For all $h, g \in H$ we have*

(i) *Counital maps are idempotents in $End_k(H)$:*

$$\varepsilon_t(\varepsilon_t(h)) = \varepsilon_t(h), \qquad \varepsilon_s(\varepsilon_s(h) = \varepsilon_s(h),$$

(ii) *the relation between ε_t, ε_s and comultiplication is as follows*

$$(id \otimes \varepsilon_t)\Delta(h) = 1_{(1)}h \otimes 1_{(2)}, \qquad (\varepsilon_s \otimes id)\Delta(h) = 1_{(1)} \otimes h1_{(2)},$$

(iii) *the images of counital maps are characterized by*

$$h = \varepsilon_t(h) \text{ iff } \Delta(h) = 1_{(1)}h \otimes 1_{(2)}, \qquad h = \varepsilon_s(h) \text{ iff } \Delta(h) = 1_{(1)} \otimes h1_{(2)},$$

(iv) *$\varepsilon_t(H)$ and $\varepsilon_s(H)$ commute.*

(v) *we also have identities dual to (ii) :*

$$h\varepsilon_t(g) = \varepsilon(h_{(1)}g)h_{(2)}, \qquad \varepsilon_s(h)g = h_{(1)}\varepsilon(gh_{(2)}).$$

Proof. We prove the identities containing the target counital map, the proofs of their source counterparts are similar. Using the axioms (3) and (2) we compute

$$\varepsilon_t(\varepsilon_t(h)) = \varepsilon(1_{(1)}h)\varepsilon(1'_{(1)}1_{(2)})1'_{(2)} = \varepsilon(1_{(1)}h)\varepsilon(1_{(1)})1_{(3)} = \varepsilon_t(h),$$

where $1'$ stands for the second copy of the unit, proving (i). For (ii) we have

$$\begin{aligned} h_{(1)} \otimes \varepsilon_t(h_{(2)}) &= h_{(1)}\varepsilon(1_{(1)}h_{(2)}) \otimes 1_{(2)} \\ &= 1_{(1)}h_{(1)}\varepsilon(1_{(2)}h_{(2)}) \otimes 1_{(3)} = 1_{(1)}h \otimes 1_{(2)}. \end{aligned}$$

To prove (iii) we observe that

$$\Delta(\varepsilon_t(h)) = \varepsilon(1_{(1)}h)1_{(2)} \otimes 1_{(3)} = \varepsilon(1_{(1)}h)1'_{(1)}1_{(2)} \otimes 1'_{(2)} = 1'_{(1)}\varepsilon_t(h) \otimes 1'_{(2)},$$

on the other hand, applying $(\varepsilon \otimes \mathrm{id})$ to both sides of $\Delta(h) = 1_{(1)}h \otimes 1_{(1)}$, we get $h = \varepsilon_t(h)$. (iv) is immediate in view of the identity $1_{(1)} \otimes 1'_{(1)}1_{(2)} \otimes 1'_{(2)} = 1_{(1)} \otimes 1_{(2)}1'_{(1)} \otimes 1'_{(2)}$. Finally, we show (v) :

$$\begin{aligned}\varepsilon(h_{(1)}g)h_{(2)} &= \varepsilon(h_{(1)}g_{(1)})h_{(2)}\varepsilon_t(g_{(2)})\\ &= \varepsilon(h_{(1)}g_{(1)})h_{(2)}g_{(2)}S(g_{(3)})\\ &= hg_{(1)}S(g_{(2)}) = h\varepsilon_t(g),\end{aligned}$$

where the antipode axiom (4) is used. □

The images of counital maps

$$H_t = \varepsilon_t(H) = \{h \in H \mid \Delta(h) = 1_{(1)}h \otimes 1_{(2)}\}, \tag{12}$$

$$H_s = \varepsilon_s(H) = \{h \in H \mid \Delta(h) = 1_{(1)} \otimes h1_{(2)}\} \tag{13}$$

play the role of "non-commutative" bases of H. The next proposition summarizes their properties.

Proposition 1.0.3. *H_t (resp. H_s) is a left (resp. right) coideal subalgebra of H. These subalgebras commute with each other ; moreover*

$$\begin{aligned}H_t &= \{(\phi \otimes id)\Delta(1) \mid \phi \in \hat{H}\},\\ H_s &= \{(id \otimes \phi)\Delta(1) \mid \phi \in \hat{H}\},\end{aligned}$$

i.e., H_t (resp. H_s) is generated by the right (resp. left) tensorands of $\Delta(1)$.

Démonstration. Clearly, H_t and H_s are coideals by Proposition 1.0.2(iii), that commute by 1.0.2(iv).

We have : $H_t = \varepsilon(1_{(1)}H)1_{(2)} \subset \{(\phi \otimes \mathrm{id})\Delta(1) \mid \phi \in \hat{H}\}$, conversely $\phi(1_{(1)})1_{(2)} = \phi(1_{(1)})\varepsilon_t(1_{(2)}) \subset H_t$, therefore $H_t = \{(\phi \otimes \mathrm{id})\Delta(1) \mid \phi \in \hat{H}\}$. Too see that it is an algebra we note that $1 = \varepsilon_t(1) \in H_t$ and for all $h, g \in H$ compute, using Proposition 1.0.2(ii) and (v) :

$$\begin{aligned}\varepsilon_t(h)\varepsilon_t(g) &= \varepsilon(\varepsilon_t(h)_{(1)}g)\varepsilon_t(h)_{(2)}\\ &= \varepsilon(1_{(1)}\varepsilon_t(h)g)1_{(2)} = \varepsilon_t(\varepsilon_t(h)g) \in H_t.\end{aligned}$$

The statements about H_s are proven similarly. □

Definition 1.0.4. We will call H_t (resp. H_s) a target (resp. source) *counital subalgebra.*

Properties of the antipode

They are the same as the properties of the antipode of a finite-dimensional Hopf algebra.

Proposition 1.0.5. *The antipode S is unique and bijective. Also, it is both algebra and coalgebra anti-homomorphism.*

Proof. Let $f * g = m(f \otimes g)\Delta$ be the convolution of $f, g \in \mathrm{End}_k(H)$. Then $S * \mathrm{id} = \varepsilon_s$, $\mathrm{id} * S = \varepsilon_t$, and $S * \mathrm{id} * S = S$. If S' is another antipode of H then

$$S' = S' * \mathrm{id} * S' = S' * \mathrm{id} * S = S * \mathrm{id} * S = S.$$

To check that S is an algebra anti-homomorphism, we compute

$$\begin{aligned} S(1) &= S(1_{(1)})1_{(2)}S(1_{(3)}) = S(1_{(1)})\varepsilon_t(1_{(2)}) = \varepsilon_t(1) = 1, \\ S(hg) &= S(h_{(1)}g_{(1)})\varepsilon_t(h_{(2)}g_{(2)}) \\ &= S(h_{(1)}g_{(1)})h_{(2)}\varepsilon_t(g_{(2)})S(h_{(3)}) \\ &= \varepsilon_s(h_{(1)}g_{(1)})S(g_{(2)})S(h_{(2)}) \\ &= S(g_{(1)})\varepsilon_s(h_{(1)})\varepsilon_t(g_{(2)})S(h_{(2)}) = S(g)S(h), \end{aligned}$$

for all $h, g \in H$, where we used Proposition 1.0.2(iv) and easy identities $\varepsilon_t(hg) = \varepsilon_t(h\varepsilon_t(g))$ and $\varepsilon_s(hg) = \varepsilon_t(\varepsilon_s(h)g)$. Dualizing the above arguments we show that S is also a coalgebra anti-homomorphism :

$$\begin{aligned} \varepsilon(S(h)) &= \varepsilon(S(h_{(1)})\varepsilon_t(h_{(2)})) = \varepsilon(S(h_{(1)})h_{(2)}) = \varepsilon(\varepsilon_t(h)) = \varepsilon(h), \\ \Delta(S(h)) &= \Delta(S(h_{(1)})\varepsilon_t(h_{(2)})) \\ &= \Delta(S(h_{(1)}))(\varepsilon_t(h_{(2)}) \otimes 1) \\ &= \Delta(S(h_{(1)}))(h_{(2)}S(h_{(4)}) \otimes \varepsilon_t(h_{(3)})) \\ &= \Delta(\varepsilon_s(h_{(1)}))(S(h_{(3)}) \otimes S(h_{(2)})) \\ &= S(h_{(3)}) \otimes \varepsilon_s(h_{(1)})S(h_{(2)}) = S(h_{(2)}) \otimes S(h_{(1)}). \end{aligned}$$

The proof of the bijectivity of S can be found in ([BNSz], 2.10). □

Next, we investigate the relations between the antipode and counital maps.

Proposition 1.0.6. *We have $S \circ \varepsilon_s = \varepsilon_t \circ S$ and $\varepsilon_s \circ S = S \circ \varepsilon_t$. The restriction of S defines an algebra anti-isomorphism between counital subalgebras H_t and H_s.*

Proof. Using results of Proposition 1.0.5 we compute

$$S(\varepsilon_s(h)) = S(1_{(1)})\varepsilon(h1_{(2)}) = \varepsilon(1_{(1)}S(h))1_{(2)} = \varepsilon_t(S(h)),$$

for all $h \in H$. The second identity is proven similarly. Clearly, S maps H_t to H_s and vice versa. Since S is bijective, and $\dim H_t = \dim H_s$ by Proposition 1.0.3, therefore $S|_{H_t}$ and $S|_{H_s}$ are anti-isomorphisms. □

Proposition 1.0.7. *Any morphism $\alpha : H \to K$ between quantum groupoids preserves counital subalgebras, i.e. $H_t \cong K_t$ and $H_s \cong K_s$. In other words, quantum groupoids with a given target (source) counital subalgebra form a full subcategory.*

Proof. It is clear that $\alpha|_{H_t} : H_t \to K_t$ is a homomorphism. If we write $\Delta(1_H) = \Sigma_{i=1}^n w_i \otimes z_i$ with $\{w_i\}_{i=1}^n$ and $\{z_i\}_{i=1}^n$ linearly independent, then $\Delta(1_K) = \Sigma_{i=1}^n \alpha(w_i) \otimes \alpha(z_i)$. By Proposition 1.0.3, $K_t = \text{span}\{\alpha(z_i)\}$, i.e., $\alpha|_{H_t}$ is surjective. Since $z_j = \varepsilon_t(z_j) = \Sigma_{i=1}^n \varepsilon(w_i z_j)z_i$, then $\varepsilon(w_i z_j) = \delta_{ij}$, therefore, $\dim H_t = n = \Sigma_{i=1}^n \varepsilon_H(w_i z_i) = \Sigma_{i=1}^n \varepsilon_H(w_i S(z_i)) = \varepsilon_H(\varepsilon_t(1_H)) = \varepsilon_H(1_H) = \varepsilon_K(1_K) = \dim K_t$, so $\alpha|_{H_t}$ is bijective. The proof for source subalgebras is similar. □

Integrals and semisimplicity

Definition 1.0.8 ([BNSz], 3.1)**.** A left (right) *integral* in H is an element $l \in H$ ($r \in H$) satisfying, for all $h \in H$, relation

$$hl = \varepsilon_t(h)l, \qquad (rh = r\varepsilon_s(h)). \tag{14}$$

These notions generalize the corresponding notions for Hopf algebras. We denote $\int_H^l$ (respectively, $\int_H^r$) the space of left (right) integrals in H and by $\int_H = \int_H^l \cap \int_H^r$ the space of two-sided integrals.

An integral in H (left or right) is called *non-degenerate* if it defines a non-degenerate functional on $\hat{H}$. A left integral l is called *normalized* if $\varepsilon_t(l) = 1$. Similarly, $r \in \int_H^r$ is normalized if $\varepsilon_s(r) = 1$.

A dual notion to that of left (right) integral is the left (right) invariant measure. Namely, a functional $\phi \in \hat{H}$ is said to be a left (right) *invariant measure* on H if

$$(id \otimes \phi)\Delta = (\varepsilon_t \otimes \phi)\Delta, \qquad (\text{resp.}, (\phi \otimes id)\Delta = (\phi \otimes \varepsilon_s)\Delta).$$

A left (right) invariant measure is said to be normalized if $(\mathrm{id} \otimes \phi)\Delta(1) = 1$ (resp., $(\phi \otimes \mathrm{id})\Delta(1) = 1$).

The next proposition gives a description of the set of left integrals.

Proposition 1.0.9 ([BNSz], 3.2)**.** *The following conditions for $l \in H$ are equivalent :*

(i) $l \in \int_H^l$,

(ii) $(1 \otimes h)\Delta(l) = (S(h) \otimes 1)\Delta(l)$ *for all* $h \in H$,

(iii) $(id \otimes l)\Delta(\hat{H}) = \hat{H}_t$,

(iv) $(Ker\, \varepsilon_t)l = 0$,

(v) $S(l) \in \int_H^r$.

Proof. The proof is a straightforward application of Definitions 1.0.1, 1.0.8, Propositions 1.0.2 and 1.0.6, we leave it as an exercise. □

Definition 1.0.10. A quantum groupoid is said to be *semisimple* if its algebra H is semisimple. A $*$*-quantum groupoid* is a quantum groupoid such that H is a $*$-algebra over the field $\mathbb{C}$ of complex numbers and Δ is a $*$-homomorphism. A C^**-quantum groupoid* is a $*$-quantum groupoid such that H is a finite-dimensional C^*-algebra.

Let us recall that a unital algebra A is said to be *separable* [P] if there is a *separability element* $e \in A \otimes A$ such that $m(e) = 1$ and $(a \otimes 1)e = e(1 \otimes a)$, $(1 \otimes a)e = e(a \otimes 1)$ for all $a \in A$. A semisimple algebra over $\mathbb{C}$ is separable.

Let us note that the algebras H_t and H_s are separable with the separability elements $(S \otimes id)\Delta(1)$ and $(id \otimes S)\Delta(1)$, respectively.

Maschke's Theorem

We have the following generalization of Maschke's Theorem, well-known for Hopf algebras ([M], 2.2.1).

Theorem 1.0.11 ([BNSz], 3.13)**.** *Let H be a finite quantum groupoid, then the following conditions are equivalent :*

(i) *H is semisimple,*

(ii) *There exists a normalized left integral l in H,*

(iii) *H is separable.*

Proof. (i) $\Rightarrow$ (ii) : Suppose that H is semisimple, then since $\mathrm{Ker}\,\varepsilon_t$ is a left ideal in H we have $\mathrm{Ker}\,\varepsilon_t = Hp$ for some idempotent p. Therefore, $\mathrm{Ker}\,\varepsilon_t(1-p) = 0$ and $l = 1 - p$ is a left integral by Lemma 1.0.9(iv). It is normalized since $\varepsilon_t(l) = 1 - \varepsilon_t(p) = 1$. (ii) $\Rightarrow$ (iii) : if l is normalized then $l_{(1)} \otimes S(l_{(2)})$ is a separability element of H by Lemma 1.0.9(ii). (iii) $\Rightarrow$ (i) : this is a standard result [P]. □

Now we will describe the class of quantum groupoids possessing *Haar integrals*, i.e., normalized two-sided integrals (note that if such an integral exists then it is unique and is an S-invariant idempotent).

Theorem 1.0.12 ([BNSz], 3.27)**.** *Let H be a finite quantum groupoid over an algebraically closed field k. Then the following conditions are equivalent :*

(i) *There exists a Haar integral,*

(ii) *H is semisimple and there exists an invertible element $g \in H$ implementing the antipode : $gxg^{-1} = S(x)$ for all $x \in H$, and such that $tr\pi_\alpha(g^{-1}) \neq 0$ for all irreducible representations π_α of H (here tr is a usual trace on a matrix algebra).*

Inclusion matrix, Bratteli diagram and index

Let H be a semisimple quantum groupoid over algebraically closed field k. Then $H = \oplus_{\alpha=1}^{L} M_{m_\alpha}(k),\ H_t = \oplus_{\beta=1}^{K} M_{n_\beta}(k)$. Let us introduce the corresponding *dimension vectors* $\overline{m} := (m_1, ..., m_L)$ and $\overline{n} := (n_1, ..., n_K)$. Clearly $||\overline{n}||^2 = \dim H_t,\ ||\overline{m}||^2 = \dim H$. In what follows we consider the *inclusion matrix* Λ, the *Bratteli diagram* and the *index* $[H : H_t]$ of the inclusion $H_t \subset H$ - for their definitions and properties see, for example, [JS], 3.2, [GHJ], 2.1.

Since the inclusion above is unital, we have : $\overline{m} = \overline{n}\Lambda$, which gives inequality $[H : H_t] \geq \dim H / \dim H_t$. Indeed, for any vector $\overline{f} \in \mathbb{R}^K$ we have :

$$\begin{aligned}[H : H_t] &= ||\Lambda||^2 = (\max_{\overline{f}\neq 0} ||\overline{f}\Lambda||/||\overline{f}||)^2 \geq (||\overline{n}\Lambda||/||\overline{n}||)^2 \\ &= (||\overline{m}||/||\overline{n}||)^2 = \dim H / \dim H_t.\end{aligned}$$

Let us show a more specific property of inclusion matrices of *connected* quantum groupoids, i.e., such that $H_t \cap Z(H) = \mathbb{C}$ (or, equivalently, $H_s \cap Z(H) = \mathbb{C}$) ; any of these conditions implies the irreducibility of the *trivial* left H-module H_t given by $x \cdot \varepsilon_t(y) := \varepsilon_t(xy)$ for all $x, y \in H$ (see ([BNSz], 2.4, [NV1], 2.2).

Indeed, let $\pi_1, \ldots \pi_K$ (resp. $\rho_1, \ldots \rho_L$) be all the classes of irreducible representations of H_t (resp. H), and assume that ρ_1 is the trivial representation of H acting on H_t. Then Λ_{ij}, the ij-th entry of Λ, equals to the multiplicity of π_i in $\rho_j|_{H_t}$.

Since $\rho_1|_{H_t}$ is equivalent to the left regular representation of H_t on itself, we have $\Lambda_{i1} = n_i$ for all $i = 1 \ldots K$, so

$$(\Lambda\Lambda^t)_{ik} = \Sigma_j\, \Lambda_{ij}\Lambda_{kj} \geq \Lambda_{i1}\Lambda_{k1} \geq n_i n_k.$$

Lemma 1.0.13. *The following inequalities hold true for any semisimple connected quantum groupoid over algebraically closed field :*

(i) *$[H : H_t] \geq \dim H_t$;*

(ii) *$[H : H_t]^2 \geq \dim H$.*

Proof. (i) The above mentioned inequality for the entries of $K \times K$ matrices $\Lambda\Lambda^t$ and N, whose entries are $n_i n_j$, gives :

$$[H : H_t] = ||\Lambda\Lambda^t|| = max_{\overline{f}}||\overline{f}\Lambda\Lambda^t||/||\overline{f}|| \geq$$

$$||\overline{n}\Lambda\Lambda^t||/||\overline{n}|| \geq ||\overline{n}N||/||\overline{n}|| = ||\overline{n}||^2 = \dim H_t.$$

(ii) This is a trivial corollary of the two above mentioned inequalities. □

C^*-quantum groupoids

Definition 1.0.10 and the uniqueness of the unit, counit and the antipode (see Proposition 1.0.5) imply that

$$1^* = 1, \quad \varepsilon(x^*) = \overline{\varepsilon(x)}, \quad (S \circ *)^2 = \mathrm{id}$$

for all x in any $*$-quantum groupoid. It is also easy to check the relations

$$\varepsilon_t(x)^* = \varepsilon_t(S(x)^*),\ \varepsilon_t(x)^* = \varepsilon_t(S(x)^*),$$

therefore, H_t and H_s are $*$-subalgebras, and to show that the dual, $\hat{H}$, is also a $*$-quantum groupoid with respect to the $*$-operation

$$<\phi^*, x> = \overline{<\phi, S(x)^*>} \qquad \text{for all } \phi \in \hat{H},\ x \in H.$$

The $*$-operation allows to simplify the axioms of a quantum groupoid (cf. the axioms used in [NV1], [N1]). The second parts of equalities (2) and (3) of Definition 1.0.1 follow from the rest of the axioms, also $S * \mathrm{id} = \varepsilon_s$ is equivalent to $\mathrm{id} * S = \varepsilon_t$. Alternatively, under the condition that the antipode is both algebra and coalgebra anti-homomorphism, the axioms 2 and 3 can be replaced by the identities of Proposition 1.0.2 (ii) and (v) involving the target counital map.

The proof of the following elementary lemma can be found in [BNSz], 4.4 :

Lemma 1.0.14. *Let H be a finite-dimensional C^*-algebra and $S : H \to H$ an algebra anti-isomorphism such that $(S \circ *)^2 = id$. Then there exists an invertible positive element $g \in H$ such that :*

(i) $S^2(x) = gxg^{-1}$, *for all $x \in H$;*

(ii) $tr\pi_\alpha(g^{-1}) = tr\pi_\alpha(g)$ *for all irreducible representations π_α of H (here tr is a usual trace on a matrix algebra);*

(iii) $S(g) = g^{-1}$.

We call this element the *canonical group-like element.*

Theorem 1.0.15. *(*[BNSz]*, 4.5) In a C^*-quantum groupoid Haar integral h exists, $h = h^*$ and*

$$(\phi, \psi) := <\phi^*\psi, h>, \qquad \phi, \psi \in \hat{H}$$

is a scalar product making $\hat{H}$ a Hilbert space where the left regular representation of $\hat{H}$ is faithful. Thus, $\hat{H}$ is a C^*-quantum groupoid, too.

Proof. Clearly, H and the element g from Lemma 1.0.14 verify all the conditions of Theorem 1.0.12, from where the existence of Haar integral follows. Since h is non degenerate, the scalar product $(\cdot, \cdot)$ is also non degenerate. The equality

$$(\phi, \phi) = <\phi^*\phi, h> = \overline{<\phi, S(h_{(1)})^*>} <\phi, h_{(2)}>,$$

implies the positivity of $(\cdot, \cdot)$. □

We will denote by $\hat{h}$ the Haar measure of $\hat{H}$.

Lemma 1.0.16. *ε is a positive functional, i.e., $\varepsilon(x^*x) \geq 0$ for all $x \neq 0$.*

Proof. For all $x \in H$ we have

$$\begin{aligned} \varepsilon(x^*x) &= \varepsilon(x^*1_{(1)})\varepsilon(1_{(2)}1'_{(2)})\varepsilon(1'_{(1)}x) = \varepsilon(\varepsilon_t(x)^*\varepsilon_t(x)) \\ &= <\hat{h}, \varepsilon_t(x)^*\varepsilon_t(x)> \geq 0, \end{aligned}$$

where we used $\hat{h}|_{H_t} = \varepsilon|_{H_t}$, which follows from

$$< \hat{h}, z >=< \hat{\varepsilon}_t(\hat{h}), z >=< \hat{1}, z >$$

for all $z \in H_t$. □

Lemma 1.0.17. *There is a is positive and invertible $g_t \in H_t$ such that the canonical group-like element of H can be written as $g = g_t S(g_t^{-1})$.*

The following property of g is now obvious :

$$\Delta(g) = (g \otimes g)\Delta(1) = \Delta(1)(g \otimes g).$$

Examples : groupoid algebras and their duals

As group algebras and their duals are the easiest examples of Hopf algebras, groupoid algebras and their duals provide "trivial" examples of quantum groupoids.

Let G be a finite *groupoid* (a category with finitely many morphisms, such that each morphism is invertible) then the groupoid algebra kG (generated by morphisms $g \in G$ with the product of two morphisms being equal to their composition if the latter is defined and 0 otherwise) is a quantum groupoid via :

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad g \in G. \tag{15}$$

The counital subalgebras of kG are equal to each other and coincide with the abelian algebra spanned by the identity morphisms : $(kG)_t = (kG)_s = \mathrm{span}\{gg^{-1} \mid g \in G\}$. The target and source counital maps are given by the operations of taking the target (resp. source) object of a morphism :

$$\varepsilon_t(g) = gg^{-1} = \mathrm{id}_{target(g)} \text{ and } \varepsilon_s(g) = g^{-1}g = \mathrm{id}_{source(g)}.$$

The dual quantum groupoid $(kG)^*$ is isomorphic to the algebra of functions on G, i.e., it is generated by idempotents p_g, $g \in G$ such that $p_g p_h = \delta_{g,h} p_g$, with the following structure operations

$$\Delta(p_g) = \sum_{uv=g} p_u \otimes p_v, \quad \varepsilon(p_g) = \delta_{g,gg^{-1}}, \quad S(p_g) = p_{g^{-1}}. \tag{16}$$

The target (resp. source) counital subalgebra is precisely the algebra of functions constant on each set of morphisms of G having the same target (resp. source) object. The target and source maps are

$$\varepsilon_t(p_g) = \Sigma_{vv^{-1}=g}\, p_v \text{ and } \varepsilon_s(p_g) = \Sigma_{v^{-1}v=g}\, p_v.$$

One can show that a C^*-quantum groupoid with a commutative algebra H (resp., co-commutative coproduct Δ) is isomorphic to the object described by the second (resp., first) example above.

Groupoid algebras and their duals give examples of commutative and cocommutative semisimple quantum groupoids, which are C^*-quantum groupoids if the ground field is $\mathbb{C}$ (in the last case $g^* = g^{-1}$ for all $g \in G$).

Example 1.0.18. (i) Let G^0 be the set of units of a finite groupoid G, then the elements $l_e = \sum_{gg^{-1}=e} g\,(e \in G^0)$ span $\int_{kG}^l$ and elements $r_e = \sum_{g^{-1}g=e} g\,(e \in G^0)$ span $\int_{kG}^r$.
(ii) If $H = (kG)^*$ then $\int_H^l = \int_H^r = \mathrm{span}\{p_e,\, e \in G^0\}$.

Temperley-Lieb algebras

Generators and relations :

$$e_i^2 = e_i = e_i^* : \quad e_i e_{i\pm 1} e_i = \lambda e_i, \quad e_i e_j = e_j e_i$$

if $|i-j| \geq 2$, here $\lambda \neq 0$ is real, $i, j = 1, 2, ...$

For fixed $n \geq 2$ and $\lambda^{-1} = 4\cos^2 \frac{\pi}{n+3}$, it is known that
$H = Alg\{1, e_1, ..., e_{2n-1}\}$ is a finite dimensional C^*-algebra,
let us put
$H_t = Alg\{1, e_1, ..., e_{n-1}\}, \quad H_s = Alg\{1, e_{n+1}, ..., e_{2n-1}\}$

In particular, if $n = 2 : H = Alg\{1, e_1, e_2, e_3\} \simeq M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$,
(so $dim(H) = 13$), $H_t = Alg\{1, e_1\} \simeq H_s = Alg\{1, e_3\} \simeq \mathbb{C} \oplus \mathbb{C}$,

$$\lambda^{-1} = 4\cos^2 \tfrac{\pi}{5}.$$

One can write down explicitly Δ, S and ε. One can show that $H \cong \hat{H}$.

2. Lecture 2. Representations of weak Hopf algebras and fusion categories

Abstract.

We construct and study the category of representations of a weak Hopf algebra. Independently, we give a definition of a fusion category, discuss the properties of these categories and give a number of concrete examples. Finally, we explain, using reconstruction theorems, the relation between fusion categories and the categories of representations of semisimple weak Hopf algebras.

Representation categories of quantum groupoids were studied in [NTV], see also [BSz2] for the C^*-case. We do not discuss here quasitriangular, ribbon and modular WHA's as well as the Drinfeld Double construction for WHA's (see [NTV]).

Basic definitions.

Let us recall the definitions of a tensor (monoidal) category and the tensor (monoidal) functor as well as the definition of a fusion category.

A tensor category $\mathcal{C}$ is a category equipped with

– **Tensor product -** a bifunctor $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ with associativity isomorphisms $\alpha(X, Y, Z) : (X \otimes Y) \otimes Z \mapsto X \otimes (Y \otimes Z)$ $(\forall\ X, Y, Z \in Ob(\mathcal{C}))$ satisfying the

Pentagon condition :

$$\begin{array}{ccc}
((W\otimes X)\otimes Y)\otimes Z & \xrightarrow{\alpha(W,X,Y)\otimes id_Z} & (W\otimes(X\otimes Y))\otimes Z \\
\downarrow & & \downarrow{\scriptstyle \alpha(W,X\otimes Y,Z)} \\
\alpha(W\otimes X,Y,Z) & & W\otimes((X\otimes Y)\otimes Z) \\
\downarrow & & \downarrow{\scriptstyle id_W\otimes\alpha(X,Y,Z)} \\
(W\otimes X)\otimes(Y\otimes Z) & \xrightarrow{\alpha(W,X,Y\otimes Z)} & W\otimes(X\otimes(Y\otimes Z))
\end{array}$$

– **Unit object 1** with left and right unit isomorphisms $l_X : \mathbf{1}\otimes X\to X$ and $r_X : X\otimes\mathbf{1}\to X$ satisfying the**Triangle condition**.

A functor between tensor categories

Let $(\mathcal{C},\otimes,\mathbf{1},\alpha)$ and $(\mathcal{D},\otimes',\mathbf{1}',\alpha')$ be two tensor categories. A functor $\mathcal{F}:\mathcal{C}\to\mathcal{D}$ is called a **tensor functor** if it is equipped with a family of isomorphisms $\{\mathcal{F}_{X,Y} : \mathcal{F}(X)\otimes'\mathcal{F}(Y)\to\mathcal{F}'(X\otimes Y)\ |\forall X,Y\in Ob(\mathcal{C})\}$ such that $\mathcal{F}(\mathbf{1})\cong\mathbf{1}'$ and the diagram

$$\begin{array}{ccc}
(\mathcal{F}(X)\otimes'\mathcal{F}(Y))\otimes'\mathcal{F}(Z) & \xrightarrow{\alpha'(\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z))} & \mathcal{F}(X)\otimes'(\mathcal{F}(Y)\otimes'\mathcal{F}(Z)) \\
\downarrow & & \downarrow{\scriptstyle id_{\mathcal{F}(X)}\otimes'\mathcal{F}_{Y,Z}} \\
\mathcal{F}(X\otimes Y)\otimes'\mathcal{F}(Z) & & \mathcal{F}(X)\otimes'\mathcal{F}(Y\otimes Z) \\
\downarrow & & \downarrow{\scriptstyle \mathcal{F}_{X,Y\otimes Z}} \\
\mathcal{F}((X\otimes Y)\otimes Z) & \xrightarrow{\mathcal{F}(\alpha(X,Y,Z))} & \mathcal{F}(X\otimes(Y\otimes Z))
\end{array}$$

is commutative.

A fusion category is a tensor category with

– **Duality (rigidity)** : for any $X\in Ob(\mathcal{C})$, there are left and right dual objects, ${}^*X$ and X^*, with their evaluation and coevaluation morphisms. In particular, for X^* :

$$d_X : X^*\otimes X\mapsto\mathbf{1},\quad b_X:\mathbf{1}\mapsto X\otimes X^*$$

satisfy relations :

$$(id_X\otimes d_X)(b_X\otimes id_x)=id_X,\ (d_X\otimes id_{X^*})(id_{X^*}\otimes b_X)=id_{X^*}.$$

– **Finite semisimplicity** :

The set $Irr(\mathcal{C})$ of (classes of) **simple objects** $(X_i)_{i=1,..,rk(\mathcal{C})}$ (i.e., $End(X_i)=\mathbb{C}$) is finite. Any $X\in Ob(\mathcal{C})$ is isomorphic
to a direct sum of X_i (with multiplicities). In particular :

$$X_i\otimes X_j=\bigoplus_k N_{ij}^k X_k\ \ (\textbf{fusion rule})\quad\text{and}\quad \mathbf{1}=X_{i_0}.$$

– Finally, we suppose that $Hom(X,Y)$ are finite dimensional vector spaces over $\mathbb{C}$, for all $X,Y\in Ob(\mathcal{C})$.

Dimensions of objects.

Frobenius-Perron dimension of X_i - the largest nonnegative eigenvalue of the matrix $(N_{ij}^k)_{j,k}$. We have

$$FPdim(X_i \otimes X_j) = FPdim(X_i)FPdim(X_j),$$

$$FPdim(X_i \oplus X_j) = FPdim(X_i) + FPdim(X_j)$$

which gives a homomorphism of the **fusion ring** of $\mathcal{C}$ to $\mathbb{R}$.
By definition, $FPdim(\mathcal{C}) = \Sigma_i FPdim(X_i)^2$.

Proposition (Etingof, Nikshych, Ostrik, 2005)
If $FPdim(\mathcal{C}) \in \mathbb{N}$, then :
1) $\mathcal{C}$ admits a unique **pivotal structure** (i.e., a family of isomorphisms $a_X : X \mapsto X^{**}$ such that $a_{X\otimes Y} = a_X \otimes a_Y$)
satisfying $Tr(a_X) = FPdim(X)$, where $Tr(a_X) := d_{X^*} \circ \circ (a_X \otimes id_{X^*}) \circ b_X \in End(\mathbf{1}) \cong \mathbb{C}$, $X, Y \in Ob(\mathcal{C})$.
Such categories are called **pseudo-unitary**, they are automatically **spherical**, i.e., $Tr(a_X) = Tr(a_{X^*})$.
2) $Tr(a_{X_i}) = FPdim(X_i) = \sqrt{N}_i$, where $N_i \in \mathbb{N}$.

Examples of fusion categories.

1. The category Vec_f of finite dimensional vector spaces over $\mathbb{C}$ with usual $\otimes$, $\mathbf{1} = \mathbb{C}$, for any $V \in Ob(Vec_f)$, ${}^*V = V^*$ is the usual dual vector space, the only simple object is $\mathbf{1}$, so $rk(Vec_f) = 1$. The associativity isomorphisms are identities.

2. Let G be a finite group, $\omega : G \times G \times G \to \mathbb{C}$ a 3-cocycle. Vec_G^ω is the semisimple category whose simple objects are elements of G,

$$g \otimes h = gh, \quad {}^*g = g^* = g^{-1}, \quad \mathbf{1} = e \in G,$$

$$\alpha_{g,h,k} = \omega(g,h,k)\mathrm{id}_{ghk} \text{ , for all } g,h,k \in G.$$

The pentagon condition follows from the 3-cocycle equality :

$$\omega(gh,k,l)\omega(g,h,kl) = \omega(g,h,k)\omega(g,hk,l)\omega(h,k,l), \ \forall g,h,k,l \in G.$$

3. Representation category $Rep(G)$ with usual $\otimes$, $\mathbf{1}$ is the trivial representation, for any $\pi \in Ob(Vec_f)$, ${}^*\pi = \pi^*$ is the conjugate representation. The associativity isomorphisms are identities.

4. Similarly : representation category of a finite dimensional semisimple Hopf algebra.

5. Representation category of a connected semisimple WHA.

For a quantum groupoid H let Rep(H) be the category of representations of H, whose objects are H-modules of finite rank and whose morphisms are H-linear homomorphisms. We show that, as in the case of Hopf algebras, Rep(H) has a natural structure of a monoidal category with duality.

For objects V, W of $\mathrm{Rep}(H)$ set

$$V \otimes W = \{x \in V \otimes W \mid x = \Delta(1) \cdot x\},$$

with the obvious action of H via the comultiplication Δ. The tensor product of morphisms is the standard tensor product of homomorphisms. The associativity isomorphisms $\Phi_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ are functorial and satisfy the pentagon condition, since Δ is coassociative. We will suppress these isomorphisms and write simply $U \otimes V \otimes W$.

The target counital subalgebra $H_t \subset H$ has an H-module structure given by $a \cdot z = \varepsilon_t(az)$, where $a \in H$, $z \in H_t$.

Lemma 2.0.19. *H_t is the unit object of Rep(H).*

Proof. Define a left unit homomorphism $l_V : H_t \otimes V \to V$ by

$$l_V(1_{(1)} \cdot z \otimes 1_{(2)} \cdot v) = z \cdot v, \qquad z \in H_t,\, v \in V.$$

This map is H-linear, since

$$\begin{aligned} l_V(a \cdot (1_{(1)} \cdot 1 \otimes 1_{(2)} \cdot v)) &= l_V(a_{(1)} \cdot \otimes a_{(2)} \cdot v) \\ &= \varepsilon_t(a_{(1)} z) a_{(2)} \cdot v = az \cdot v \\ &= a \cdot l_V(1_{(1)} \cdot 1 \otimes 1_{(2)} \cdot v), \end{aligned}$$

for all $a \in H$. The inverse map $l_V^{-1} : V \to H_t \otimes V$ is given by

$$l_V^{-1}(v) = S(1_{(1)}) \otimes 1_{(2)} \cdot v.$$

Moreover, the collection $\{l_V\}_V$ gives a natural equivalence between the functor $H_t \otimes (\ )$ and the identity functor. Indeed, for any H-linear homomorphism $f : V \to U$ we have :

$$\begin{aligned} l_U \circ (\mathrm{id} \otimes f)(1_{(1)} \cdot 1 \otimes 1_{(2)} \cdot v) &= l_U(1_{(1)} \cdot 1 \otimes 1_{(2)} \cdot f(v)) \\ &= z \cdot f(v) = f(z \cdot v) \\ &= f \circ l_V(1_{(1)} \cdot 1 \otimes 1_{(2)} \cdot v) \end{aligned}$$

Similarly, one can check that the right unit homomorphism $r_V : V \otimes H_t \to V$ defined by

$$r_V(1_{(1)} \cdot v \otimes 1_{(2)} \cdot z) = S(z) \cdot v, \qquad z \in H_t,\, v \in V,$$

has the inverse $r_V^{-1}(v) = 1_{(1)} \cdot v \otimes 1_{(2)}$ and satisfies the necessary properties. Finally, we verify the triangle axiom. For all objects V, W of $\mathrm{Rep}(H)$ and $v \in V$, $w \in W$ we have

$$\begin{aligned} & (\mathrm{id}_V \otimes l_W)(1_{(1)} \cdot v \otimes 1'_{(1)} 1_{(2)} \cdot z \otimes 1'_{(2)} \cdot w) \\ = \ & 1_{(1)} \cdot v \otimes 1_{(2)} z \cdot w \\ = \ & 1_{(1)} S(z) \cdot v \otimes 1_{(2)} \cdot w \\ = \ & (r_V \otimes id_W)(1'_{(1)} \cdot v \otimes 1'_{(2)} 1_{(1)} \cdot z \otimes 1_{(2)} \cdot w), \end{aligned}$$

therefore $\mathrm{id}_V \otimes l_W = r_V \otimes id_W$ as functors. □

Using the antipode S of H, we can provide $\mathrm{Rep}(H)$ with a duality. For any object V of $\mathrm{Rep}(H)$ define the action of H on $V^* = \mathrm{Hom}_k(V, k)$ by $(a \cdot \phi)(v) = \phi(S(a) \cdot v)$, where $a \in H, v \in V, \phi \in V^*$. For any morphism $f : V \to W$ let $f^* : W^* \to V^*$ be the homomorphism dual to f.

For any V in Rep(H) define the duality homomorphisms

$$d_V : V^* \otimes V \to H_t, \qquad b_V : H_t \to V \otimes V^*$$

as follows. For $\Sigma_j \phi^j \otimes v_j \in V^* \otimes V$ set

$$d_V(\Sigma_j \phi^j \otimes v_j) = \Sigma_j \phi^j(1_{(1)} \cdot v_j)1_{(2)}.$$

Let $\{g_i\}_i$ and $\{\gamma^i\}_i$ be bases of V and V^* respectively, dual to each other. The element $\Sigma_i g_i \otimes \gamma^i$ does not depend on the choice of these bases; moreover, for all $v \in V, \phi \in V^*$ one has $\phi = \Sigma_i \phi(g_i)\gamma^i$ and $v = \Sigma_i g_i \gamma^i(v)$. Set

$$b_V(z) = \Sigma_i z_{(1)} \cdot g_i \otimes z_{(2)} \cdot \gamma^i.$$

Proposition 2.0.20. *The category Rep(H) is a monoidal category with duality.*

Proof. We have already seen that Rep(H) is monoidal, it remains to prove that d_V and b_V are H-linear and satisfy the identities

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \qquad (d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}.$$

Take $\Sigma_j \phi^j \otimes v_j \in V^* \otimes V, z \in H_t, a \in H$. Using the axioms of a quantum groupoid, we have

$$\begin{aligned} a \cdot d_V(\Sigma_j \phi^j \otimes v_j) &= \Sigma_j \phi^j(1_{(1)} \cdot v)\varepsilon_t(a1_{(2)}) \\ &= \Sigma_j \phi^j(\varepsilon_s(1_{(1)}a) \cdot v_j)1_{(2)} \\ &= \Sigma_j \phi^j(S(a_{(1)})1_{(1)}a_{(2)} \cdot v_j)1_{(2)} \\ &= \Sigma_j (a_{(1)} \cdot \phi^j)(1_{(1)} \cdot (a_{(2)} \cdot v_j))1_{(2)} \\ &= \Sigma_j d_V(a_{(1)} \cdot \phi^j \cdot a_{(2)} \cdot v_j) \\ &= d_V(a \cdot \Sigma_j \phi^j \otimes v_j), \end{aligned}$$

therefore, d_V is H-linear. To check the H-linearity of b_V we have to show that $a \cdot b_V(z) = b_V(a \cdot z)$, i.e., that

$$\Sigma_i a_{(1)}z \cdot g_i \otimes a_{(2)} \cdot \gamma^i = \Sigma_i 1_{(1)}\varepsilon_t(az) \cdot g_i \otimes 1_{(2)} \cdot \gamma^i.$$

Since the both sides of the above equality are elements of $V \otimes_k V^*$, evaluating the second factor on $v \in V$, we get the equivalent condition

$$a_{(1)}zS(a_{(2)}) \cdot v = 1_{(1)}\varepsilon_t(az)S(1_{(2)}) \cdot v,$$

which is easy to check. Thus, b_V is H-linear. Using the isomorphisms l_V and r_V identifying $H_t \otimes V$, $V \otimes H_t$ and V, for all $v \in V$ and $\phi \in V^*$ we have :

$$\begin{aligned} (\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V)(v) &= (\mathrm{id}_V \otimes d_V)(b_V(1_{(1)} \cdot 1) \otimes 1_{(2)} \cdot v) \\ &= (\mathrm{id}_V \otimes d_V)(b_V(1_{(2)}) \otimes S(1_{(1)}) \cdot v) \\ &= \Sigma_i (\mathrm{id}_V \otimes d_V)(1_{(2)} \cdot g_i \otimes 1_{(3)} \cdot \gamma^i \otimes S(1_{(1)}) \cdot v) \\ &= \Sigma_i 1_{(2)} \cdot g_i(1_{(3)} \cdot \gamma^i)(1'_{(1)}S(1_{(1)}) \cdot v) \otimes 1'_{(2)} \\ &= 1_{(2)}S(1_{(3)})1'_{(1)}S(1_{(1)}) \cdot v \otimes 1'_{(2)} = v, \\ (d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V)(\phi) &= (d_V \otimes \mathrm{id}_{V^*})(1_{(1)} \cdot \phi \otimes b_V(1_{(2)})) \\ &= \Sigma_i (d_V \otimes \mathrm{id}_{V^*})(1_{(1)} \cdot \phi \otimes 1_{(2)} \cdot g_i \otimes 1_{(3)} \cdot \gamma^i) \\ &= \Sigma_i (1_{(1)} \cdot \phi)(1'_{(1)}1_{(2)} \cdot g_i)1'_{(2)} \otimes 1_{(3)} \cdot \gamma^i \\ &= 1'_{(2)} \otimes 1_{(3)}1_{(1)}S(1'_{(1)}1_{(2)}) \cdot \phi = \phi, \end{aligned}$$

therefore, the proof is complete. □

In order to make $Rep(H)$ a finite semisimple category, we have to require H to be semisimple.

Finally, one can show that H_t is a simple H-module if and only if H is connected (i.e., $H_t \cap Z(H) = \mathbb{C}$).

Thus, the representation category of a connected semisimple WHA is a fusion category.

Classification of fusion categories of rank 2 (V. Ostrik, 2003).

$Irr(\mathcal{C}) = \{\mathbf{1}, X\},\ {}^*\mathbf{1} = \mathbf{1}^* = \mathbf{1},\ {}^*X = X^* = X.$

All possible fusion rules : $\mathbf{1} \otimes X = X \otimes \mathbf{1} = X,\ X \otimes X = \mathbf{1} + nX,\ n = 0, 1, 2, ...$

1) If $n = 0$, then $\mathcal{C} \cong Vec_G^\omega$, where $G = \mathbb{Z}/2\mathbb{Z}$ (2 categories).

2) If $n = 1$, there are two fusion categories with $\dim X = \frac{1 \pm \sqrt{5}}{2}$ (because $(\dim X)^2 = 1 + \dim X$) one of them (with +) is equivalent to the representation category of the **Temperley-Lieb C^*-WHA** with $n = 2$**- Yang-Lee fusion category**.

3) If $n > 1$, there is no fusion category.

From fusion categories to Weak Hopf Algebras.

1. Hayashi's canonical tensor functor.

Given a fusion category $\mathcal{C}$ with $\Omega := Irr(\mathcal{C})$, define a commutative algebra $R := \oplus_{x \in \Omega} \mathbb{C} p_x$, where $p_x p_y = \delta_{x,y} p_x$. The canonical Hayashi's **tensor** functor $\mathcal{F} : \mathcal{C} \to R - Bimod$, where $R - Bimod$ is the tensor category of R-bimodules with $\otimes_R$ and unit object R :

(i) $\mathcal{F}(x) := \oplus_{y,z \in \Omega} Hom(z, y \otimes x), \forall x \in \Omega$ - R-bimodule via :

$$p_y \cdot \mathcal{F}(x) \cdot p_z = Hom(z, y \otimes x).$$

(ii) If $x, y \in \Omega,\ f \in Hom(x, y)$, put $\mathcal{F}(f)(\varphi) := (id_z \otimes f) \circ \varphi$, for any $\varphi \in p_z \cdot \mathcal{F}(x) \cdot p_t = Hom(t, z \otimes x)$, where $z, t \in \Omega$.

The structure of a tensor functor on $\mathcal{F}$:

(iii) $\mathcal{F}_{x,y} : \mathcal{F}(x) \otimes \mathcal{F}(y) \to \mathcal{F}(x \otimes y)$ is given by

$$\mathcal{F}_{x,y}(\varphi \otimes \psi) := \alpha_{z,x,y} \circ (\varphi \otimes id_y) \circ \psi,$$

for all $\varphi \in p_z \cdot \mathcal{F}(x) \cdot p_t,\ \psi \in p_t \cdot \mathcal{F}(y) \cdot p_s,\ z, s, t \in \Omega$.

(iv) $\mathcal{F}_{\mathbf{1}}^{-1} : R \to \mathcal{F}(\mathbf{1})$ is defined by $\mathcal{F}(p_x) := r_x^{-1},\ \forall x \in \Omega$.

2. Szlachányi's reconstruction theorem

Let $\mathcal{C}, \Omega, R$, and $\mathcal{F} : \mathcal{C} \to R - Bimod$ be as above. Define

$$H := \oplus_{x \in \Omega} End_{\mathbb{C}}(\mathcal{F}(x))$$

and inclusions $s : R \to H$ and $t : R \to H$ compatible with left and right actions of R, i.e., $s(r)h = h \cdot r, t(r)h = r \cdot h$, $\forall r \in R, h \in H$. Denote $J := \oplus_{x,y\in\Omega}\mathcal{F}_{x,y}^{-1}$ with $\mathcal{F}_{x,y}$ as above. Then the algebra H has a semisimple connected WHA structure with the coproduct

$$\Delta(h) := \Sigma_{x\in\Omega}(s(p_x) \otimes t(p_x))J \circ h \circ J^{-1}, \quad \forall h \in H.$$

In particular, $\Delta(h) := \Sigma_{x\in\Omega}(s(p_x) \otimes t(p_x))$, $H_t = Vec\{t(p_x))|x \in \Omega\}$, $H_s = Vec\{s(p_x))|x \in \Omega\}$.

$\varepsilon, \varepsilon_t, \varepsilon_s$, and S can be found from relations $(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$, $\varepsilon_t(h) = \varepsilon(1_{(1)}h)1_{(2)}$, $\varepsilon_s(h) = 1_{(1)}\varepsilon(h1_{(2)})$, $m(S \otimes id)\Delta = \varepsilon_s$, $m(id \otimes S)\Delta = \varepsilon_t$.

Finally, $\mathcal{C} \simeq Rep(H)$.

So, we can construct quantum groupoids from fusion categories.

Example : Tambara-Yamagami fusion categories

Let G be an abelian finite group, $\chi : G \times G \to \mathbb{C}$ a non degenerate symmetric bicharacter, and $\beta = \pm(\sqrt{|G|})^{-1}$.

- simple objects are elements of G and an element m ;
- fusion rules and duality :

$$g \otimes h = g + h, \quad {}^*g = g^* = -g, \quad \mathbf{1} = 0 \in G,$$

$$g \otimes m = m \otimes g = m = m^*, \quad m \otimes m = \underset{g\in G}{\oplus} g;$$

- the associativity isomorphisms are identities except for $a_{g,m,h} = \chi(g, h)\mathrm{id}_m$, for all $g, h \in G$,
  $a_{m,g,m} = \bigoplus_{h\in G} \chi(g, h)\mathrm{id}_h$, for all $g \in G$,
  $a_{m,m,m} = \left(\beta\chi(g, h)^{-1}\mathrm{id}_m\right)_{g,h}$.

Weak C^*-Hopf algebras associated with Tambara-Yamagami categories (C. Mével, 2010)

Given a Tambara-Yamagami category, denote $n = |G|$ and $\Omega = G \cup \{m\}$. Then the canonical weak C^*-Hopf algebra is

$$H = \bigoplus_{g\in G} M_{n+1}(\mathbb{C}) \oplus M_{2n}(\mathbb{C}),$$

the coproduct and antipode are written in terms of χ and β.

H is self-dual : $H \simeq \hat{H}$

The target counital subalgebra is of the form $H_t = \bigoplus_{x\in\Omega} \mathbb{C}e^x$ (e^x are minimal idempotents).

The inclusion matrix of $H_t \subset H$:

$$\begin{array}{ccccc} 1 & 1 & ... & 1 & 1 \\ . & . & ... & . & . \\ . & . & ... & . & . \\ . & . & ... & . & . \\ 1 & 1 & ... & 1 & 1 \\ 1 & 1 & ... & 1 & n \end{array}$$

The index of the inclusion $H_t \subset H$ equals to $(n+\sqrt{n})^2$ and its Bratteli diagram is as follows :

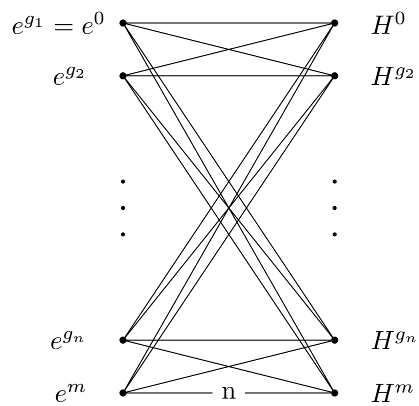


Figure 1

3. Lecture 3. Applications of weak Hopf algebras to von Neumann algebras

Abstract.

We recall basic definitions and facts of the von Neumann algebra theory and, in particular, of the subfactor theory. Then we explain how a finite index and depth 2 subfactor generates a C*-weak Hopf algebra and, more generally, how to describe a structure of a finite index and finite depth subfactor in terms of a C*-weak Hopf algebra associated with it. A number of concrete examples is discussed.

Recall that a II_1-factor is a von Neumann algebra with trivial center admitting a tracial state. An example of such a factor is the group von Neumann algebra of a discrete group whose every non-trivial conjugacy class is infinite, the corresponding trace being generated by its Haar measure. It is also known, that there exists a unique (up to isomorphism) a hyperfinite II_1-factor R (i.e., containing an increasing sequence of finite-dimensional C^*-algebras, whose union is weakly dense in R). There is a notion of index for an inclusion of von Neumann algebras, generalizing a usual notion of index of a subgroup in a group. For the basic results, examples and terminology of the subfactor theory see [GHJ], [JS]).

A II_1-**factor** is (an inifnite-dimensional) von Neumann algebra M such that $Z(M) = \mathbb{C}\mathbf{1}$ equipped with a faithful normal finite **trace**, i.e., a linear form $\tau : M \to \mathbb{C}$ such that $\tau(ab) = \tau(ba), \tau(\mathbf{1}) = 1$ and, for all $a \in M^+$, we have $\tau(a) \geq 0,\ \tau(a) = 0 \Rightarrow a = 0$.

Example of a II_1 factor : $M = \mathcal{L}(H)$ acting on $l^2(H)$, where H a discrete ICC group (in particular, free group with 2 generators), $\tau(m) =< m\dagger\delta_e, \delta_e >$.

A subfactor : $M_0 \subset (M_1, \tau) \longrightarrow e : L^2(M_1, \tau) \to \overline{M_0}$

Main Problem : To study subfactors of a given factor M_1, in particular, when M_1 is a **hyperfinite** II_1-factor.

Index : $[M_1 : M_0] := \tau(e)^{-1}$

V.Jones theorem :

$$[M_1 : M_0] \in \{4\cos^2\frac{\pi}{n} \ \ |n \geq 3\} \cup [4, \infty]$$

Fundamental construction :

$M_2 = \{M_1, e\}$ is a factor and $[M_2 : M_1] = [M_1 : M_0]$.

$M_0 \subset M_1 \subset M_2 \subset ...$ **- V.Jones tower**

Derived tower : $M_0' \cap M_1 \subset M_0' \cap M_2 \subset M_0' \cap M_3 \subset ...$

these C^*-algebras are finite-dimensional if $[M_1 : M_0] < \infty$.

Depth k - fundamental construction from step k in the derived tower. Other invariants : **Principal and dual graphs.**

Example : A finite group action on a II_1-factor

$(M^G)' \cap M = \mathbb{C}\mathbf{1}$.

Then the tower of factors :

$$M^G \subset M \subset M \rtimes G \subset (M \rtimes G) \rtimes \hat{G} \subset ....$$

is of depth 2 and of index $[M : M^G] = [M \rtimes G : M] = ... = |G|$.

Here $\hat{G}$ is the Pontryagin dual of G if G is abelian, and if not we have to view G as **a C*-Hopf (more exactly, G.I. Kac) algebra** : $H = \mathbb{C}G, \Delta : g \mapsto g \otimes g, S : g \mapsto g^{-1}, \varepsilon : g \mapsto 1$, then $\hat{H}$ is the **dual G.I. Kac algebra** (of fuctions on G).

Galois correspondence : $M^G \subset K \subset M$ iff $K = M^L$ for some subgroup L of G.

Quantum groups and subfactors

Theorem (R. Longo, W. Szymanski, M.-C. David, 1994 - 1996)

$M_0 \subset M_1$: type II_1 subfactor of finite index λ and depth 2, **irreducible** : $M_0' \cap M_1 = \mathbb{C}1$, $M_0 \subset M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} M_3$....... Jones tower, then :

- $H = M_0' \cap M_2$, $\hat{H} = M_1' \cap M_3$: **Kac algebras in duality**
- $\hat{H}$ acts on M_2 and $M_1 = M_2^{\hat{H}}$, $M_3 = \theta(M_2 \rtimes \hat{H})$
- $[M_k : M_{k-1}] = \dim H = \dim \hat{H}$ $(k = 1, 2, ..)$

Duality : $< a, b >= \lambda^{-2}\tau(ae_2e_1b)$.

$< \Delta(a), b \otimes c >=< a, bc >$

$< S(a), b >= \overline{< a^*, b^* >}$

$\varepsilon(a) =< a, \mathbf{1} >$

Further motivation : $M_0' \cap M_1 \neq \mathbb{C}1$, non integer index

3.1. Actions of C^*-quantum groupoids on von Neumann algebras.

Definition 3.1.1. An algebra A is a (left) *H-module algebra* if A is a left H-module via $h \otimes a \to h \triangleright a$ and

1) $h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$,
2) $h \triangleright 1 = \varepsilon_t(h) \triangleright 1$.

If A is an H-module algebra we will also say that H acts on A.

Definition 3.1.2. An algebra A is a (right) *H-comodule algebra* if A is a right H-module via $\rho : a \to a^{(0)} \otimes a^{(1)}$ and

1) $\rho(ab) = a^{(0)}b^{(0)} \otimes a^{(1)}b^{(1)}$,
2) $\rho(1) = (\mathrm{id} \otimes \varepsilon_t)\rho(1)$.

It follows immediately that A is a left H-module algebra if and only if A is a right $\hat{H}$-comodule algebra.

Example 3.1.3. (i) The target counital subalgebra H_t is a trivial H-module algebra via $h \triangleright z = \varepsilon_t(hz)$, $h \in H$, $z \in H_t$.

(ii) H is an $\hat{H}$-module algebra via the dual action $\phi \triangleright h = h_{(1)}\langle \phi, h_{(2)} \rangle$, $\phi \in \hat{H}$, $h \in H$.

(iii) Let $A = C_H(H_s) = \{h \in H \mid hy = yh \forall y \in H_s\}$, be the centralizer of H_s in H, then A is an H-module algebra via the adjoint action $h \triangleright a = h_{(1)}aS(h_{(2)})$.

Let A be an H-module algebra, then a *smash product* algebra $A \rtimes H$ is defined on a k-vector space $A \otimes_{H_t} H$ (relative tensor product), where H is a left H_t-module via multiplication and A is a right H_t-module via

$$a \cdot z = S^{-1}(z) \triangleright a = a(z \triangleright 1), \qquad a \in A, z \in H_t.$$

Let $a \rtimes h$ be the class of $a \otimes h$ in $A \otimes_{H_t} H$, then the multiplication in $A \rtimes H$ is given by the familiar formula

$$(a \rtimes h)(b \rtimes g) = a(h_{(1)} \rtimes b) \rtimes h_{(2)}g, \qquad a, b, \in A,\ h.g \in H,$$

and the unit of $A \rtimes H$ is $1 \rtimes 1$.

Example 3.1.4. H is isomorphic to the trivial smash product algebra $H_t \rtimes H$.

Let a von Neumann algebra M be a left H-module algebra in the sense of Definition 3.1.1 via *weakly continuous* action of a C^*-quantum groupoid $H : H \otimes M \ni x \otimes m \mapsto (x \triangleright m) \in M$ such that $(x \triangleright m)^* = S(x)^* \triangleright m^*, x \triangleright 1 = 0$ iff $\varepsilon_t(t) = 0$.

Then it is possible to show that a smash product algebra (now we call it *crossed product algebra*, denote it by $M \rtimes H$ and its elements by $[m \otimes x]$), equipped with an involution $[m \otimes x]^* = [(x^*_{(1)} \triangleright m^*) \otimes x^*_{(2)}]$, is *-isomorphic to a weakly closed algebra of operators on some Hilbert space ([NSzW], 3.4.2), i.e., $M \rtimes H$ is a von Neumann algebra.

The collection $M^H = \{m \in M \mid x \triangleright m = \varepsilon_t(x) \triangleright m,\ \forall x \in H\}$ is a von Neumann subalgebra of M, called *fixed point subalgebra*. The relative commutant $M' \cap M \rtimes H$ always contains a *-subalgebra isomorphic to H_s. Indeed, if $z \in H_s$, then $\Delta(z) = 1_{(1)} \otimes 1_{(2)}z$, therefore

$$[1 \otimes z][m \otimes 1] = [(z_{(1)} \triangleright m) \otimes z_{(2)}] = [(1_{(1)} \triangleright m) \otimes 1_{(2)}z] =$$
$$= [m \otimes z] = [m \otimes 1][1 \otimes z],$$

for any $m \in M$, and $H_s \subset M' \cap M \rtimes H$. An action of H is called *minimal* if $H_s = M' \cap M \rtimes H$.

Like in the case of smash products, one can now define *the dual action*, i.e., the action of a dual C^*-quantum groupoid $\hat{H}$ on the von Neumann algebra $M \rtimes H$, and one can construct the von Neumann algebra $(M \rtimes H) \rtimes \hat{H}$.

If H is a connected C^*-quantum groupoid having a minimal action on a *factor* M (i.e., $Z(M) = \mathbb{C}$), then one can show ([NSzW], 4.2.5, 4.3.5) that $\hat{H}$ is also connected and that

$$N = M^H \subset M \subset M_1 = M \rtimes H \subset M_2 = (M \rtimes H) \rtimes \hat{H} \subset ...$$

is the Jones tower of factors of finite index with *the derived tower*

$$N' \cap N = \mathbb{C} \subset N' \cap M = H_t \subset N' \cap M_1 = H \subset N' \cap M_2 = H \rtimes \hat{H} \subset ...$$

The fact that the last triple of finite-dimensional C^*-algebras is the basic construction, means exactly that the above Jones tower of factors is of depth 2. Moreover, in the case of II_1-factors the finite-dimensional C^*-algebras $H_t, H, \hat{H}, H \rtimes \hat{H}$ form *a canonical commuting square*, which determines completely the equivalence class of the initial subfactor. This implies that any biconnected C^*-quantum groupoid has at most one minimal action on a given II_1-factor and thus corresponds to no more than one (up to equivalence) finite index depth 2 subfactor.

Let us mention the following existence result (D. Nikshych, M.-C. David) : any biconnected involutive C^*-quantum groupoid H has a minimal action on the standard hyperfinite II_1-factor. The idea of the construction is as follows : in the Jones tower $H_t \subset H \subset H \rtimes \hat{H} \subset ...$ all the inclusions are connected, so the union of these finite-dimensional C^*-algebras admits a unique tracial state ; therefore, its von Neumann algebra completion with respect to this trace is a copy of the standard hyperfinite II_1-factor. Then one can extend the actions of H on the above algebras to its actions on this factor and show that it is minimal.

3.2. **A construction of a C^*-quantum groupoid from a depth 2 subfactor.** Let $N \subset M$ be a finite index ($[M : N] = \lambda^{-1}$) depth 2 II_1-subfactor and

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

the corresponding Jones tower, $M_1 = \langle M, e_1\rangle$, $M_2 = \langle M_1, e_2\rangle, \ldots,$ where $e_1 \in N' \cap M_1$, $e_2 \in M' \cap M_2, \cdots$ are the Jones projections. The depth 2 condition means that $N' \cap M_2$ is the basic construction of the inclusion $N' \cap M \subset N' \cap M_1$. Let τ be the trace on M_2 normalized by $\tau(1) = 1$.

Theorem(D. Nikshych-LV, 2000)

$M_0 \subset M_1$: type II_1 subfactors of finite index λ and depth 2

$M_0 \subset M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} M_3$....... Jones tower

- $H = M_0' \cap M_2$, $\hat{H} = M_1' \cap M_3$: C^*-**weak Hopf algebras** in duality.
- $\hat{H}$ acts on M_2 and $M_1 = M_2^{\hat{H}}$, $\quad M_3 = \theta(M_2 \rtimes \hat{H})$.
- The principal graph of the subfactor $M_0 \subset M_1$ is defined by the Bratteli diagram of the inclusion of the finite dimensional algebras $H_t \subset H$ and $[M_k : M_{k-1}] = [H : H_t]$.

Duality : $\boxed{< a, b >= d\lambda^{-2}\tau(ae_2e_1Pb)}$**.**

(there exists a unique $P \in Z(M_1' \cap M_2)$ **s.t.** $\tau(Pz) = Tr(z)$**)**

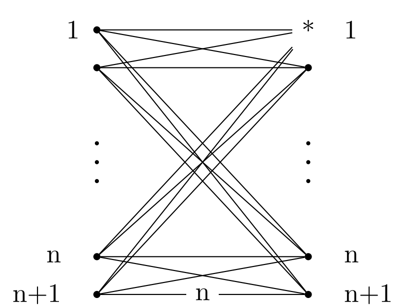
$< \Delta(a), b \otimes c >=< a, bc >$

$< S(a), b >= \overline{< a^*, b^* >}$

$\varepsilon(a) =< a, \mathbf{1} >$.

Example

Applying this theorem to WHA's associated with the Tambara- Yamagami categories (see Lecture 2), we see that there exists
a family of subfactours of the hyperfinite type II_1 factor whose indices are $(n+\sqrt{n})^2$, for all $n \geq 1$. Their principal graphs are given by the Bratteli diagrams of the inclusions $H_t \subset H$:



Galois correspondence :

Left coideal $*$-subalgebra : $I \subset H \quad$ s.t $\Delta(I) \subset H \otimes I$.

Crossed product : $M \rtimes I = span\{[m \otimes b] | m \in M, b \in I\} \subset M \rtimes H$,

Theorem(D. Nikshych-LV, 2000) Two Lattices :
- **Intermediate vN subalgebras** : $M_2 \subset K \subset M_3$ ($K \vee L = (K \cup L)''$, $K \wedge L = K \cap L$)
- **Left coideal $*$-subalgebras** of H
  ($I \vee J = (I \cup J)''$, $I \wedge J = I \cap J$)

are **isomorphic** :

$$K \mapsto M_1' \cap K \subset H, \quad I \mapsto M_2 \rtimes I \subset M_3$$

K is a factor if and only if I is **connected** : $Z(I) \cap H_s = \mathbb{C}\mathbf{1}$.

Characterization of finite index and finite depth subfactors

Observation (S. Popa) :

$$\underbrace{\underbrace{M_0 \subset M_1}_{depth\ n} \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} M_3 .... \subset M_{n-1}}_{depth\ \mathbf{2}},$$

M_1 corresponds to a left coideal $*$-subalgebra I of a quantum groupoid H. Then the previous theorem **characterizes all finite index and finite depth subfactors**.

Theorem (D. Nikshych-LV, 2000) If Depth $(M_0 \subset M_1) = n$,
then the bimodule category $Bimod_{M_0 - M_0}(M_0 \subset M_1)$ with tensor product $\otimes_{M_0}$ is equivalent to $Rep(\hat{H})$, where the C^*-WHA H corresponds to the depth 2 subfactor $M_0 \subset M_{n-1}$.

Example : coideal *-subalgebras I_K et J_K.

Proposition (C. Mével, 2010)

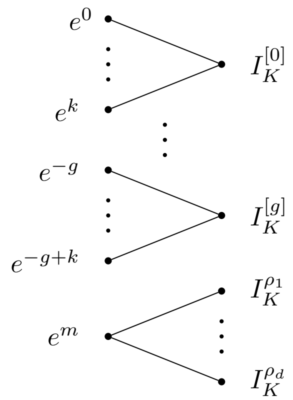
Let H the canonical weak C^*-Hopf algebra associated with the Tambara-Yamagami category $\mathcal{C}(G, \chi, \beta)$. Then with any subgroup
$K < G$ one can associate two connected coideal *-subalgebras,
I_K and J_K, of H whose C^*-algebra structures are, respectively :

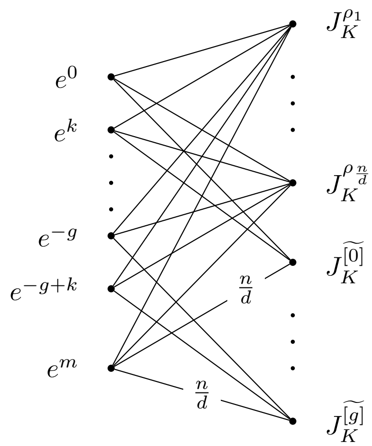
$$I_K = \bigoplus_{M \in G/K} M_d(\mathbb{C}) \oplus \bigoplus_{\rho \in \widehat{K}} \mathbb{C}, \quad \text{and}$$

$$J_K = \bigoplus_{\rho \in \widehat{K^\perp}} M_{n+1}(\mathbb{C}) \oplus \bigoplus_{\tilde{M} \in G/K^\perp} M_{2\frac{n}{d}}(\mathbb{C}),$$

where $n = |G|$, $d = |K|$, and $K^{\perp} = \{g \in G | \chi(g,k) = 1, \text{ for all } k \in K\}$.

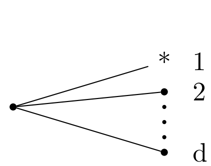
Bratteli diagrams of the inclusions $H_t \subset I_K$ and $H_t \subset J_K$

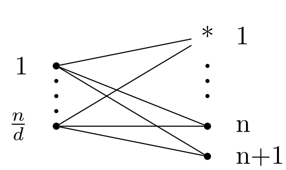




Principal graphs and indices of intermediate subfactors

The above pictures allow to draw all possible principal graphs of intermediate subfactors associated with Tambara-Yamagami categories ($n \in \mathbb{N}, d|n$) :





Their indices are d and $\frac{1}{d}(n+\sqrt{n})^2$, respectively.

Lattices of the coideal subalgebras I_K et J_K

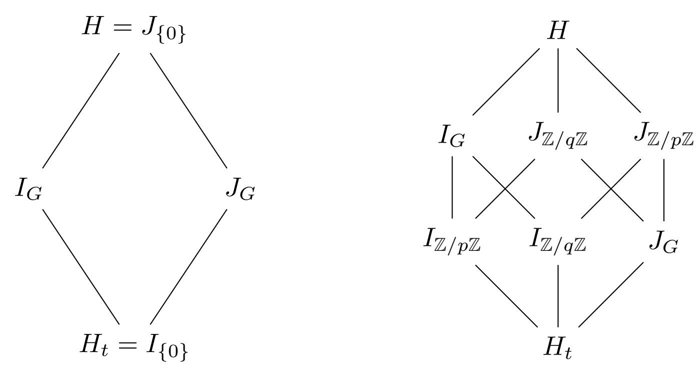
Proposition

The set $\{I_K|K<G\}$ is a sublattice of $l(H)$ isomorphic to the lattice of subgroups of G and the set $\{J_K|K<G\}$ is a sub-
lattice of $l(K)$ anti-isomorphic to the lattice of subgroups of
G. Their union (non-disjoint) is a lattice with the operations $I_K \wedge J_L = I_{K\cap L^\perp}$ and $I_K \vee J_L = J_{K^\perp \cap L}$.

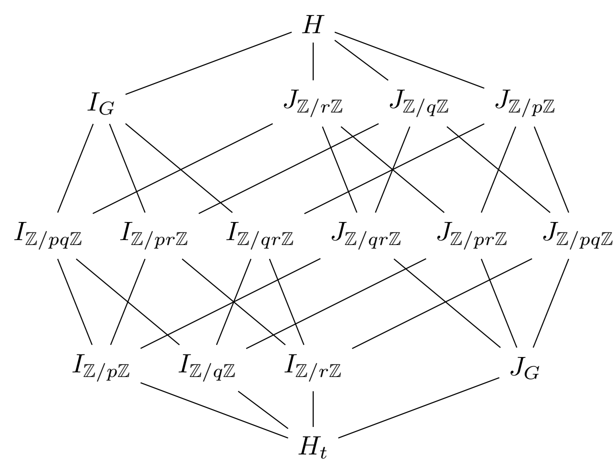
If $n=|G|$ is square free, (i.e., G is of the form $\prod_{i=1}^{k} \mathbb{Z}/p_i\mathbb{Z}$,
where p_i are prime numbers, all different), then the lattice $\{I_K|K<G\}\cup\{J_K|K<G\}$ is isomorphic to the lattice of subsets of a set containing $k+1$ elements.

The figures below illustrate the lattices of the coideal subalgebras and of the intermediate subfactorsthe in the cases $G=\mathbb{Z}/p\mathbb{Z}$, $G=\mathbb{Z}/pq\mathbb{Z}$ and $G=\mathbb{Z}/pqr\mathbb{Z}$.

The cases $G=\mathbb{Z}/p\mathbb{Z}$ and $G=\mathbb{Z}/pq\mathbb{Z}$



Lattices of the coideal subalgebras and of the intermediate subfactors : the case $G = \mathbb{Z}/pqr\mathbb{Z}$



Références

[BNSz] G. Böhm, F. Nill and K. Szlachányi, *Weak Hopf algebras I. Integral theory and C^*-structure*, J. Algebra, **221** (1999), 385–438.

[BSz2] G. Böhm, K. Szlachányi, *Weak Hopf algebras II. Representation Theory, Dimensions and the Markov Trace*, *math.QA*/**9906045** (1999), J. Algebra, **233** (2000), 156212

[D] M.-C. David, *Paragroupe d'Adrian Ocneanu et algebre de Kac*, Pacif. J. of Math., **172**, (1996), 331–363.

[D1] M.-C.David, *C*-groupodes quantiques et inclusions de facteurs : Structure symtrique et autodualit. Action sur le facteur hyperfini de type II_1*, J.operator theory, **54**, n.1 (2005),27–68

[EGNO] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor categories*, Preprint (2009), http ://www-math.mit.edu/ ˜etingof/tenscat1.pdf.

[ENO] P. Etingof, D. Nikshych, V. Ostrik *On fusion categories*, Ann. Math., **162** (2005), 581–642.

[GHJ] F. Goodman, P. de la Harpe, and V.F.R. Jones, *Coxeter Graphs and Towers of Algebras*, M.S.R.I. Publ. **14**, Springer, Heidelberg, 1989.

[JS] V. Jones and V.S. Sunder, *Introduction to Subfactors*, London Math. Soc. Lecture Notes **234**, Cambridge University Press, 1997.

[M] C. Mével, Exemples et applications des groupoides quantiques finis, Ph.D. Thesis, University of Caen, 2010.

[N1] D. Nikshych. *Duality for actions of weak Kac algebras and crossed product inclusions of II_1 factors*, to appear in the Journal of Operator Theory, *math.QA*/**9810049** (1998).

[N2] D. Nikshych. *A duality theorem for quantum groupoids* to appear in Contemporary Mathematics. *math.QA*/**9912226** (1999).

[NV1] D. Nikshych, L. Vainerman. *Algebraic versions of a finite dimensional quantum groupoid*, to appear in Lecture Notes in Pure and Appl. Math. (2000), *math.QA*/**9808054**.

[NV2] D. Nikshych, L. Vainerman. *A characterization of depth 2 subfactors of II_1 factors*, J. Func. Analysis **171** (2000), no. 2, 278-307.

[NV3] D. Nikshych, L. Vainerman. *A Galois correspondence for II_1 factors*, *math.QA*/**0001020** (2000).

[NTV] D. Nikshych, L. Vainerman, and V. Turaev. *Quantum groupoids and invariants of knots and 3-manifolds*, preprint (2000).

[NSzW] F. Nill, K. Szlachányi, H.-W. Wiesbrock. *Weak Hopf algebras and reducible Jones inclusions of depth 2*, I : From crossed products to Jones towers. *math.QA*/**9806130** (1998).

[P] R. Pierce. *Associative algebras*, Graduate Texts in Mathematics, **88**. Springer Verlag, (1982).

[O] V. Ostrik,*Fusion categories of rank 2*, Math. Res. Lett., **10**, no. 2-3, (2003) 177-183.

[Sz] K. Szlachányi, *Finite quantum groupoids and inclusions of finite type*, Fields Inst.Commun., **30**, AMS Providence, RI (2001), 393-407.

[S] W. Szymański, *Finite index subfactors and Hopf algebra crossed products*, Proc. Amer. Math. Soc. **120** (1994), no. 2, 519–528.

[TY] D. Tambara, S. Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, J. Algebra, **209** (2) (1998), 692–707.

[W] Y. Watatani. *Index for C^*-subalgebras*, Memoirs of the AMS **424** (1990).