Lecture I. Basic properties of m-isometries

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Plan of the talk

- Background
- Definition
- Decomposition theorem
- Spectrum and norm
- Example

Some history

Historically, there were first studies of n-symmetric operators (with connection to Sturm-Liouville conjugate point theory) by Helton (1972,74), Agler (1980, 1992), Ball-Helton (1980), Bunce (1983), Rodman and McCullough (1996, 1997, 1998).

The study of m-isometries was started by Agler and Stankus "m-isometric transformations of Hilbert space, I, II, III" (1995).

For two-isometries, there was another approach by Richter, a representation theorem for cyclic analytic two-isometries (1991), and Olofsson, A von Neumann-Wold decomposition of two-isometries (2004). Hellings, Two-isometries on Pontryagin spaces (2008). For 3-isometries, McCullough (1987, 89).

isometry,

unitary,

unilateral shift,

invariant subspace (upper triangular block form),

reducing subspace (direct sum),

von Neumann-Wold decomposition

write on the white board some details

A bounded linear operator T on a Hilbert space H is an m-isometry for a positive integer m if

$$\beta_m(T) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0$$

or equivalently for all $h \in H$,

$$\beta_m(T,h) := \langle \beta_m(T)h, h \rangle = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \left\| T^k h \right\|^2 = 0$$

Note that

$$\beta_{m+1}(T) = T^* \beta_m(T) T - \beta_m(T). \qquad (1)$$

Thus if T is an m-isometry, then T is an n-isometry for all $n \geq m$.

If H_0 is an invariant subspace of T, then

$$\beta_m(T|H_0) = P_{H_0} \beta_m(T)|H_0 \qquad (2)$$

where $T|H_0$ is the restriction of T to H_0.

Write on the board for $m = 1, 2, 3$, proof of recursive formula and about H_0

We say T is a strict m-isometry if T is an m-isometry but not an $(m-1)$-isometry.

We say T is an ∞-isometry if

$$limsup_{m\to\infty} \|\beta_m(T)\|^{1/m} = 0.$$

We say T is a finite-isometry if T is an m-isometry for some $m \geq 1$.

We say T is an ∞-unitary if both T and T^* are ∞-isometries.

Similarly, for $m \geq 1$, T is an m-unitary if both T and T^* are m-isometries.

For an m-isometry T and $l < m$, T is l-pure if T has no nonzero direct summand which is an l-isometry.

For an ∞-isometry, T is a pure ∞-isometry if T has no nonzero direct summand which is a finite-isometry.

Decomposition Theorems

For $m \geq 1$, subspace K_m is defined by

$$K_m(T) = K_m = \bigcap_{i \geq 0} \ker(\beta_m(T)T^i). \qquad (3)$$

It follows from recursive formula that

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots .$$

Proposition 1 *Let $T \in B(H)$. Then K_m is invariant for T and $T|K_m$ is an m-isometry. Furthermore, if $M \subseteq H$ is invariant for T and $T|M$ is an m-isometry, then $M \subseteq K_m$.*

write proof

Proposition 2 *Let $T \in B(H)$. Then for each $m \geq 1$, there exists a unique subspace $M \subseteq H$ that is maximal with respect to the following properties:*
(i) M is reducing for T, and
(ii) $T|M$ is an m-isometry.

Skip proof

Let R_m denote this unique reducing subspace for T.

Theorem 3 *Let $T \in B(H)$ be an ∞-isometry. Let $V_1 = R_1, V_n = R_n \ominus R_{n-1}$ for $n \geq 2$ and*

$$V_\infty = H \ominus \bigvee \{R_i, i \geq 1\} = H \ominus \bigvee \{V_i, i \geq 1\}.$$

Then V_i is reducing for T for each $i = 1, 2, \ldots, \infty$ and with respect to the decomposition

$$H = V_\infty \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots ,$$

T has the following form

$$T = V_\infty \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_n \oplus \cdots$$

where $T|V_\infty$ is an pure ∞-isometry, $T|V_\infty$ is an isometry, and $T|V_n$ is a pure $(m-1)$-isometry.

No need for proof

Nagy-Foias-Langer decomposition theorem for contractions: every contraction is a direct sum of a unitary and a completely nonunitary contraction

Here are two generalizations.

Theorem 4 *Let $T \in B(H)$. Then U_m defined by the formula*

$$
\begin{aligned}
U_m &= K_m(T) \bigcap K_m(T^*) & (4)\\
&= \bigcap_{i \geq 0} \Big[\ker(\beta_m(T)T^i) \bigcap \ker(\beta_m(T^*)T^{i*}) \Big] & (5)
\end{aligned}
$$

is the unique maximal reducing subspace on which T is an m-unitary. Furthermore $T = T_1 \oplus T_2$ with respect to the decomposition $H = U_m \oplus U_m$ where T_1 is an m-unitary and T_2 is an operator which has no direct m-unitary summand.

Theorem 5 *Let $T \in B(H)$. Then R_1 defined by the formula*

$$R_1 = R_1(T) := \bigcap_{i,n \geq 0} \ker(\beta_1(T) T^i T^{*n}) \qquad (6)$$

is the unique maximal reducing subspace on which T is an isometry. Furthermore $T = T_1 \oplus T_2$ with respect to the decomposition $H = R_1 \oplus R_1^{\perp}$ where T_1 is an isometry and T_2 is an operator which has no direct isometry summand.

Maybe write Proof

Spectrum

Proposition 6 *If T is an m-isometry or an ∞-isometry, then $\sigma_{ap}(T) \subseteq \partial D$. Therefore either $\sigma(T) = D^{-}$ or $\sigma(T) \subseteq \partial D$. In particular T is left invertible.*

Proof

Proposition 7 *If T is an m-isometry, then the generalized eigenspaces corresponding to different eigenvalues are orthogonal.*

Skip proof

Reproducing formula for an m-isometry T :

for $n \geq m$

$$T^{*n}T^{n} = \sum_{k=0}^{m-1} \left(\begin{array}{c} n \\ k \end{array} \right) \beta_k(T)$$

write proof

Therefore

$$\beta_{m-1}(T) \geq 0$$

write proof

Norm

$$\begin{aligned} \|T^n h\|^2 &= \langle T^{*n} T^n h, h \rangle = \left\langle \sum_{k=0}^{m-1} \binom{n}{k} \beta_k(T) h, h \right\rangle \\ &= \sum_{k=0}^{m-1} \binom{n}{k} \beta_k(T, h) \end{aligned}$$

Therefore if T is a strict m-isometry, then for constant c, C,

$$Cn^{m-1} \geq \|T^n\|^2 \geq cn^{m-1} \text{ for } n \geq m.$$

A power-bounded m-isometry is an isometry.

write proof

Examples

Example 8 *If H is a finite dimensional Hilbert space, then an ∞-isometry is an m-isometry. An m-isometry is the direct sum of matrices of the form*

$$\lambda I + Q$$

where $Q^\ell = 0$ for some ℓ.

Maybe write proof

Example 9 *Assume T and $Q \in B(H)$ are commuting and T is an m-isometry and Q is a nilpotent operator of order ℓ. Then $T + Q$ is an $(m + 2\ell - 2)$-isometry. If T is an ∞-isometry and Q is a quasinilpotent operator. Then $T + Q$ is an ∞-isometry.*

write Proof

Example 10 *Let l_2 denotes the Hilbert space with basis $\{e_j\}_{j\in\mathbb{N}_0}$. A unilateral weighted shift T on l_2 is defined by $Te_j = w_j e_{j+1}$ for $j \in \mathbb{N}_0$. Without loss of generality, assume all weights are positive. Then T is a strict m-isometry if and only if there exists a polynomial $P(x)$ of degree $m-1$ such that $P(n) > 0$ for $n \in \mathbb{N}_0$ and*

$$(w_n)^2 = \frac{P(n+1)}{P(n)} \text{ for } n \in \mathbb{N}_0. \tag{7}$$

For the bilateral shifts case (m has to be odd), we only need to change both "$n \in \mathbb{N}_0$" in the above to "$n \in \mathbb{Z}$".

will prove in last lecture

Example 11 *An ∞-isometry comes from the limit of a sequence of commuting finite-isometries. Let T_n be $n \times n$ Jordan block*

$$T_n = \begin{bmatrix} \lambda_n & \frac{1}{n} & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{n} \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

where the $\lambda_n \in \partial D$. Then T_n is a strict $(2n-1)$-isometry. Let

$$T = T_1 \oplus T_2 \oplus T_3 \oplus \cdots .$$

Then T is an ∞-isometry but not a finite-isometry. Furthermore $\sigma(T) = \{\lambda_n, n \geq 1\}^-$.

Explain as the limit of a sequence of commuting finite m-isometry.

Lecture II. 2-isometries

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Plan of the talk

- Dilation, Extension and Lifting
- Von Neumann-Wold decomposition for 2-isometries
- Model for Analytic 2-isometries
- Lifting for 2-isometries

Extension and Lifting

Let $T \in L(H)$ and $B \in L(K)$ with $K \supseteq H$. Let B be such that

$$B = \begin{bmatrix} T & * \\ 0 & * \end{bmatrix} \text{ on } K = H \oplus (K \ominus H) = H \oplus H^{\perp}$$

Since H is invariant for B, $T = B|H$, B is called an extension (lifting) of T or T has a lifting (an extension) to B or T is a part of B.

Let $T \in L(H)$ and $B \in L(K)$. We say T is unitarily equivalent to a part of B if there is an isometry $W : H \to K$ such that

$$WT = BW$$

In this case, set $M = WH \subseteq K$, then M is invariant for B and

$$T = W^{-1}(B|M)W.$$

Dilation

Sarason: Let $C \in L(K)$ with $K \supseteq H$. Then $T^n = P_H C^n | H$ if and only if $H^\perp$ is invariant for T^* and

$$C = \begin{bmatrix} T & 0 \\ * & * \end{bmatrix} \text{ on } K = H \oplus (K \ominus H) = H \oplus H^\perp.$$

$$C = \begin{bmatrix} * & * \\ 0 & T \end{bmatrix} \text{on } K = (K \ominus H) \oplus H = H^\perp \oplus H$$

The operator C is called a (power) dilation of T or T has an dilation C. Equivalently C^* is an extension of T^*.

maybe some proof

von Neumann-Wold decomposition for isometries: An isometry T is the direct sum of an unitary and a unilateral shift.

Set $H_\infty := \cap_{n\geq 1} T^n H$, $T|H_\infty$ is unitary

$H_0 := H \ominus TH = \ker(T^*)$ is called the wandering subspace of T

$T|H_0 \oplus TH_0 \oplus T^2 H_0 \oplus \cdots$ is a shift operator

$$H = H_\infty \oplus \left[H_0 \oplus TH_0 \oplus T^2 H_0 \oplus \cdots\right]$$

Corollary: An isometry can be extended to an unitary or An isometry is a part of an unitary.

Nagy isometric dilation theorem: Every contraction has an extension to an coisometry or Every contraction is a part of an coisometry. Let $T \in L(H)$, then $T^* = V^*|H$ (or equivalently $T^n = P_H V^n|H$) where

$$V^* = \begin{bmatrix} T^* & D_T & 0 & \cdots & \cdots \\ 0 & 0 & I & 0 & \ddots \\ 0 & 0 & 0 & I & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \text{ on } K = H \oplus H \oplus \cdots .$$

The isometry V is called an isometric dilation of T and

V^* is called a coisometry extension (lifting) of T^* or T^* has an extension (lifting) to V^*.

write proof

Von Neumann-Wold decomposition for 2-isometries

Let $T \in L(H)$ be an 2-concave or 2-expansive. That is

$$\beta_2(T) = T^{*2}T^2 - 2T^*T + I \leq 0$$

Explain the terminology

$$\begin{aligned} (-1)^m \beta_m(T) &\geq 0 \text{ for contractive, hypercontractive} \\ (-1)^m \beta_m(T) &\leq 0 \text{ for expansive, hyperexpansive} \end{aligned}$$

Recall $H_\infty = \cap_{n \geq 1} T^n H$

Lemma 1 *H_∞ is reducing for T and $T|H_\infty$ is an unitary.*

skip proof

Lemma 2 $\|T^n h\|^2 - \|h\|^2 \leq n(\|Th\|^2 - \|h\|^2) = n\|Dh\|^2$,

$$\beta_1(T) := T^*T - I = D \geq 0.$$

write proof, to start for $n = 1$

$$\left\|T^2 h\right\|^2 - \|Th\|^2 \leq \|Th\|^2 - \|h\|^2$$

Consequence for spectrum and norm
$\sigma_{ap}(T) \subseteq \partial D$. Therefore either $\sigma(T) = D^-$ or $\sigma(T) \subseteq \partial D$. In particular T is left invertible.

$$T^*T - I = D \geq 0 \text{ or } \|Th\|^2 - \|h\|^2 \geq 0$$

T is expansive operator

Late to extend these results to $2m$-expansive operators

Theorem 3 *Let* $T \in L(H)$ *be an 2-concave or 2-expansive. Assume* $H_\infty = \{0\}$ *. Then*

$$H = \bigvee_{i \geq 0} T^n M_0$$

where

$$H_0 = (H \ominus TH) = R(T)^\perp = \ker T^*$$

is called the wondering subspace for S.

PROOF

$$\begin{aligned} L &= (T^*T)^{-1}T^* \text{ is the leftinverse of } T, \\ Q &= TL \text{ is the projection onto } Range(T) \\ P &= I - Q \text{ is the projection on to } R(T)^{\perp} = \ker T^* \end{aligned}$$

Let $x \in H$,

$$\begin{aligned} (I - T^n L^n)h &= \sum_{j=0}^{n-1} (T^j L^j - T^{j+1} L^{j+1})h \\ &= \sum_{j=0}^{n-1} T^j (I - TL) L^j h \\ &= \sum_{j=0}^{n-1} T^j P L^j h \in \bigvee_{i \geq 0} T^n M_0 \end{aligned}$$

Will prove $(I - T^n L^n)h \to h$ weakly to finish the proof.

$$\|h\|^2 = \sum_{j=0}^{n-1} \left\|PL^j h\right\|^2 + \sum_{j=0}^{n} \left\|DL^j h\right\|^2 + \|L^n h\|^2$$

For $n = 1$,

$$\begin{aligned} \|h\|^2 &= \|Ph\|^2 + \|Qh\|^2 \\ &= \|Ph\|^2 + \|Lh\|^2 + \|TLh\|^2 - \|Lh\|^2 \\ &= \|Ph\|^2 + \|Lh\|^2 + \|DLh\|^2 \end{aligned}$$

then by induction (using this formula for $L^n h$).

$$\begin{aligned} &\left(\inf\left\{\|T^n L^n h\|^2 - \|L^n h\|^2 : k \le n \le m\right\}\right) \sum_{n=k}^{m} \frac{1}{n} \\ \le\ & \sum_{n=k}^{m} \frac{1}{n}\left(\|T^n L^n h\|^2 - \|L^n h\|^2\right) \\ \le\ & \sum_{n=k}^{m} \|DL^n h\|^2 \le \|h\|^2 \end{aligned}$$

Since $\|L^n h\|^2$ is decreasing,

$$lim \inf \|T^n L^n h\| = lim \|L^n h\|$$

Thus there exists a weakly convergent subsequence

$$T^{n_j} L^{n_j} h \rightarrow y$$

for some y,

but $T^{n_j} L^{n_j} h \in T^N H$ which is closed hence weakly closed. So $y \in H_\infty = \{0\}$.

THE PROOF IS COMPLETE.

In fact $lim \|L^n h\| \rightarrow 0$ and $\|T^{n_j} L^{n_j} h\| \rightarrow 0$.

$$\|h\|^2 = \sum_{j=0}^{\infty} \left\|P L^j h\right\|^2 + \sum_{j=0}^{\infty} \left\|D L^j h\right\|^2 \qquad (1)$$

Theorem 4 Richter, *Let $T \in L(H)$ be an 2-concave or 2-expansive. Assume $H_\infty = \{0\}$. Then every invariant subspace M of T is of the form*

$$M = \bigvee_{i \geq 0} T^n M_0$$

where

$$M_0 = (M \ominus TM)$$

is called the wondering subspace for S.

There are generalizations of the above result.

By **Olofsson**,

$$\|T^n h\|^2 - c\,\|h\|^2 \;\leq\; c_n(\|Th\|^2 - \|h\|^2),$$
$$\sum_{n\geq 2} \frac{1}{c_n} \;=\; \infty$$

By **Shimorin**, operator related to 2-concave operator including Bergman shift, if $T \in L(H)$ is 2-concave, then $T' = T(T^*T)^{-1}$ satisfying

$$\left\|T'x + y\right\|^2 \leq 2(\|x\|^2 + \left\|T'y\right\|^2)$$

In another connection by **Chavan**, $T' = T(T^*T)^{-1}$ (called by him Cauchy dual to T) is a hyponormal contraction.

Model for Analytic 2-isometries

Richter for $dim\,(H \ominus TH) = 1$

Olofsson for $dim\,(H \ominus TH) > 1$.

Analytic 2-isometries means $H_\infty = \{0\}$.

Let E be a Hilbert space.

Definition 5 *A* **positive** $L(E)$*-valued operator measure on the unit circle* $\mu(e^{i\theta}) = \mu(\theta)$. *Let* Ω *be the* σ*-algebra of Borel sets of the circle.* $\mu(\Omega_0) \geq 0$ *in* $L(E)$, *and for any* $x, y \in E$

$$\mu_{x,y}(\Omega_0) = \langle \mu(\Omega_0)x, y \rangle$$

are all complex regular Borel measures on Ω.

$$T = \int f d\mu \text{ means } \langle Tx, y \rangle = \int f d\mu_{x,y}.$$

The Fourier coefficients of μ are defined by

$$\widehat{\mu}(n) = \int_{\mathbb{T}} e^{-in\theta} d\mu(\theta)$$

$\widehat{\mu}(n)$ are bounded operators in $L(E)$.

The Poisson integral $P[\mu]$ is

$$\begin{aligned} P[\mu](z) &= \int_{\mathbb{T}} P(z, e^{i\alpha}) d\mu(e^{i\alpha}) = \int_{\mathbb{T}} \frac{(1-|z|^2)}{\left|e^{i\alpha}-z\right|^2} d\mu(e^{i\alpha}) \\ &= \sum \widehat{\mu}(n) r^n e^{in\theta}, z = re^{i\theta} \end{aligned}$$

Definition 6 *The Dirichlet space $D(\mu)$. Let $f(z)$ be a E-valued analytic function on $\mathbb{D}$*

$$\|f\|_\mu^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} \left\langle P\left[\mu\right](z) f'(z), f'(z) \right\rangle_E dA(z)$$

If $f = \sum a_j z^j$ is an E-valued analytic polynomial, then

$$\begin{aligned} & \int_{r\mathbb{D}} \left\langle P\left[\mu\right](z) f'(z), f'(z) \right\rangle_E dA(z) \\ = & \sum_{j,k \geq 1} \min\{j,k\} r^{2\max\{j,k\}} \left\langle \widehat{u}(k-j) a_j, a_k \right\rangle \end{aligned}$$

$$\begin{aligned} & \frac{1}{2\pi} \int_{r\mathbb{D}} \left\langle P\left[\mu\right](re^{i\theta}) f(re^{i\theta}), f(re^{i\theta}) \right\rangle_E d\theta \\ = & \sum_{j,k \geq 0} r^{2\max\{j,k\}} \left\langle \widehat{u}(k-j) a_j, a_k \right\rangle \end{aligned}$$

Theorem 7 *M_z on $D(\mu)$ is an analytic 2-isometry.*

Proof

$$\begin{aligned}
&\int_{r\mathbb{D}} \left\langle P[\mu](z)(zf)'(z), (zf)'(z) \right\rangle_E dA(z) \\
= \; & r^2 \int_{r\mathbb{D}} \left\langle P[\mu](z) f'(z), f'(z) \right\rangle_E dA(z) \\
& + \frac{r^2}{2\pi} \int_{r\mathbb{D}} \left\langle P[\mu](re^{i\theta}) f(re^{i\theta}), f(re^{i\theta}) \right\rangle_E d\theta
\end{aligned}$$

$$\begin{aligned}
&\int_{r\mathbb{D}} \left\langle P[\mu](z) \left(z^2 f\right)'(z), (zf)'(z) \right\rangle_E dA(z) \\
& + r^4 \int_{r\mathbb{D}} \left\langle P[\mu](z)(f)'(z), (f)'(z) \right\rangle_E dA(z) \\
= \; & 2r^2 \int_{r\mathbb{D}} \left\langle P[\mu](z)(zf)'(z), (zf)'(z) \right\rangle_E dA(z)
\end{aligned}$$

$$\left\| M_{z^2} f \right\|_\mu^2 + \|f\|_\mu^2 = 2 \|M_z f\|_\mu^2$$

Theorem 8 *Let* $T \in L(H)$ *be an analytic* 2*-isometry. Then* T *is unitarily equivalent to* M_z *on* $D(\mu)$ *for some measure* μ.

Lifting for 2-isometries by **Agler and Stankus**

Recall $T \in L(H)$ is a 2-isometry if

$$\beta_2(T) = T^{*2}T^2 - 2T^*T + I = 0.$$

In this case

$$\Delta = \Delta_T = \beta_1(T) = T^*T - I \geq 0.$$

The simplest 2-isometry is when $rank(\Delta) = 1$.

Difference between isometry and 2-isometry. (isometry of rank$(I - TT^*) = 2$ is the direct sum of rank 1 isometry) but not for 2-isometry.

T on $H^2 \oplus H^2 \oplus C \oplus C$ with $rank(\Delta) = 2$

$$T = \begin{bmatrix} S & 0 & \sqrt{2} \otimes 1 & 0 \\ 0 & S & 0 & \sigma \otimes 1 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Definition 9 *Brownian shift of covariance* σ *(*$\sigma > 0$*) and angle* θ *is the block operator* $B_{\sigma,e^{i\theta}}$ *acting on* $H^2 \oplus C$ *defined by*

$$B_{\sigma,e^{i\theta}} = \left[\begin{array}{cc} S & \sigma(1 \otimes 1) \\ 0 & e^{i\theta} \end{array} \right].$$

Compute $\mathbf{\Delta} = \beta_1(B_{\sigma,e^{i\theta}}) = \sigma^2(1 \otimes 1)$.

Proposition 10 *If* $rank(\Delta) = 1$ *and* T *is pure, then* T *is unitarily equivalent to* $B_{\sigma,e^{i\theta}}$.

skip proof

Definition 11 *Brownian unitary of covariance σ is an operator which is unitarily equivalent to*

$$U \oplus \int_{\oplus} B_{\sigma, e^{i\theta}}^{(n(\theta))} d\mu(\theta) \text{ on } H \oplus \int_{\oplus} (H^2 \oplus C)^{n(\theta)}$$

where μ is a finite positive measure on $[0, 2\pi)$ and $n(\theta)$ is a μ measurable multiplicity function.

Proposition 12 *B is a Brownian unitary of covariance σ if and only if B has the block form*

$$B = \begin{bmatrix} V & \sigma E \\ 0 & U \end{bmatrix} \text{ on } K = K_0 \oplus K_1$$

where V an isometry, U an unitary and E an isometry maps K_1 onto $\ker V^*$.

skip proof

Theorem 13 *Let $T \in L(H)$ be a 2-isometry of covariance σ, then T is unitarily equivalent to a part of a Brownian unitary $B \in L(K)$ of covariance σ.*

PROOF

We need to construct B on K and also an isometry

$$L : H \to K = K_0 \oplus K_1$$

such that

$$LT = BL.$$

Here is the L

$$L = \begin{bmatrix} \sqrt{(I - \frac{1}{\sigma^2}\Delta)} \\ \frac{1}{\sigma}\sqrt{\Delta} \end{bmatrix} = \begin{bmatrix} \delta \\ \frac{1}{\sigma}\sqrt{\Delta} \end{bmatrix}.$$

We will use $LT = BL$ to construct B.

$$
\begin{aligned}
LT &= \begin{bmatrix} \delta T \\ \frac{1}{\sigma}\sqrt{\Delta}T \end{bmatrix} \\
BL &= \begin{bmatrix} V & \sigma E \\ 0 & U \end{bmatrix} \begin{bmatrix} \delta \\ \frac{1}{\sigma}\sqrt{\Delta} \end{bmatrix} \\
&= \begin{bmatrix} V\delta + E\sqrt{\Delta} \\ U\frac{1}{\sigma}\sqrt{\Delta} \end{bmatrix}
\end{aligned}
$$

Equivalently

$$
\begin{aligned}
U\frac{1}{\sigma}\sqrt{\Delta} &= \frac{1}{\sigma}\sqrt{\Delta}T \\
V\delta + E\sqrt{\Delta} &= \delta T
\end{aligned}
$$

Rewrite the second equation as the third equation

$$
V\delta + E\sqrt{\Delta} = VV^*\delta T + (I - VV^*)\delta T
$$

First equation

$$U\frac{1}{\sigma}\sqrt{\Delta} = \frac{1}{\sigma}\sqrt{\Delta}T$$

Now define U_0 on $H_1 = Range(\sqrt{\Delta})^-$ by

$$U_0\sqrt{\Delta}h = \sqrt{\Delta}Th, h \in H$$

and extend U_0 to be an unitary U on K_1.

write proof U_0 is an isometry

Second equation

$$V\delta + E\sqrt{\mathsf{\Delta}} \;=\; \delta T$$

Define V_0 on

$$Range(\delta)^- = R(\delta T)^- \oplus (R(\delta)^- \ominus R(\delta T)^-)$$

by

$$\begin{aligned} V_0 \text{ on } R(\delta T)^- &: \; V_0\delta T h = \delta h, \\ V_0 \text{ on } R(\delta)^- \ominus R(\delta T)^- &: \; V_0 = 0 \end{aligned}$$

and extend V_0 to a coisometry V^* on K_0.

write proof V_0 is a contraction

Third equation

$$V\delta + E\sqrt{\Delta} \;=\; VV^*\delta T + (I - VV^*)\delta T$$

Finally define E from $K_1 = H_1 \oplus (K_1 \ominus H_1)$ onto $(I - VV^*)K_0$ by

$$E \text{ on } H_1 \;=\; R(\sqrt{\Delta})^- : E\sqrt{\Delta}h = (I - VV^*)\delta T h$$

$$E \text{ on } K_1 \ominus H_1 \quad : \quad \text{an arbitrary isometry } F$$

Note E maps H_1 onto $(I - VV^*)Range(\delta T)^-$,
so F has to map $K_1 \ominus H_1$ onto M where

$$M = (I - VV^*)K_0 \ominus (I - VV^*)R(\delta T)^-$$

The existence of F requires that

$$\dim(K_1 \ominus H_1) = \dim(M)$$

which can be achieved by the freedom on K_1.

write proof E on H_1 is an isometry

The original C^*-algbera proof of lifting theorem is based on the following abstract Theorem by **Agler**.

Let $C^{m\times m}[x,y]$ denote the set of the polynomials in x and y with $m \times m$ matrix coefficients. If

$$h = \sum c_{ij} y^j x^i \in C^{m\times m}[x,y]$$

and a is an element of a C^*-algbera with unit, then define $h(a) \in A^{m\times m}$ (the C^*-algbera of $m \times m$ matrices with entries in A) by

$$h(a) = \sum c_{ij} a^{*j} a^i.$$

If $T \in L(H)$, then $h(T)$ is an operator from $H^{(n)} = H \oplus H \oplus \cdots \oplus H$ (m copies) into $H^{(n)}$.

Theorem 14 *Let A to be a C^*-algbera with unit and fix $a \in A$. An operator $T \in L(H)$ has the form*

$$\pi(c)|H$$

*where $\pi : A \to L(K)$ is a unital *-representation, $K \supseteq H$ and H is invariant for $\pi(c)$ if and only if $h(T) \geq 0$ whenever $m \geq 1, h \in C^{m\times m}[x,y]$ and $h(c) \geq 0$.*

Lecture III. M-isometries on Banach spaces and related operators

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Plan of the talk

- Definitions
- Spectrum and norm
- Examples–weighted shifts
- Related operators–Reversing inequality

A bounded linear operator T on a Hilbert space H is an m-isometry if all $h \in H$,

$$\begin{aligned} \beta_m(T,h) \quad : \quad &= \langle \beta_m(T)h, h \rangle \\ &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \left\| T^k h \right\|^2 = 0 \end{aligned}$$

An bounded linear operator T on a Banach space X is called an (m,p)-isometry if

$$\beta_{(m,p)}(T,x) \tag{1}$$
$$: \; = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \left\| T^k x \right\|^p = 0 \text{ for all } x \in X.$$

Then recursive formula

$$\beta_{m+1}(T) = T^* \beta_m(T) T - \beta_m(T) \tag{2}$$

becomes

$$\beta_{(m+1,p)}(T,x) = \beta_{(m,p)}(T,Tx) - \beta_{(m,p)}(T,x).$$

write the proof

If H_0 is an invariant subspace of T, then

$$\beta_m(T|H_0, h_0) = \beta_m(T, h_0) \tag{3}$$

where $T|H_0$ is the restriction of T to H_0.

skip some decomposition theorem

Spectrum

Proposition 1 *If T is an m-isometry or an ∞-isometry, then $\sigma_{ap}(T) \subseteq \partial D$. Therefore either $\sigma(T) = D^{-}$ or $\sigma(T) \subseteq \partial D$. In particular T is left invertible.*

write the proof

Reproducing formula on H for an m-isometry T :
for $n \geq m$

$$T^{*n}T^n = \sum_{k=0}^{m-1} \binom{n}{k} \beta_k(T)$$

Reproducing formula on X

$$\|T^n x\|^p = \sum_{k=0}^{m-1} \binom{n}{k} \beta_{(k,p)}(T, x) \qquad (4)$$

Maybe write the proof.

Therefore for all $x \in X$,

$$\beta_{(m-1,p)}(T, x) \geq 0$$

Write the proof.
Note also the right side of (4) is a polynomial of degree $m-1$ or less.

Norm

Therefore if T is a strict m-isometry, then for constant c, C,

$$Cn^{m-1} \geq \|T^n\|^2 \geq cn^{m-1} \text{ for } n \geq m.$$

A power-bounded m-isometry is an isometry.

Examples

If X is finite dimensional, we do not know.

Example 2 *Assume T and $Q \in B(H)$ are commuting and T is an m-isometry and Q is a nilpotent operator of order ℓ. Then $T + Q$ is an $(m + 2\ell - 2)$-isometry.*

Not true on Banach space

Example 3 *Assume* T *and* $S \in B(X)$ *are commuting. Assume* T *is an* (m, p)*-isometry and* S *is an* (l, p)*-isometry. Then* TS *is an* $(m + l - 1, p)$*-isometry.*

skip the proof

Example 4 *Let l_p denotes the Hilbert space with basis $\{e_j\}_{j \in \mathbb{N}_0}$. A unilateral weighted shift T on l_p is defined by $Te_j = w_j e_{j+1}$ for $j \in \mathbb{N}_0$. Without loss of generality, assume all weights are positive. Then T is a strict m-isometry if and only if there exists a polynomial $P(x)$ of degree $m-1$ such that $P(n) > 0$ for $n \in \mathbb{N}_0$ and*

$$(w_n)^p = \frac{P(n+1)}{P(n)} \text{ for } n \in \mathbb{N}_0. \tag{5}$$

For the bilateral shifts case (m has to be odd), we only need to change both "$n \in \mathbb{N}_0$" in the above to "$n \in \mathbb{Z}$".

In order to prove the above example, we look at the reproducing formula more closely. For $n \geq m$,

$$\|T^n x\|^p = \sum_{k=0}^{m-1} \left(\begin{array}{c} n \\ k \end{array} \right) \beta_{(k,p)}(T,x) \qquad (6)$$

It turns out the above formula is automatically true (T does not have to be an m-isometry) for $0 \leq n \leq m-1$ if one interprets $\binom{n}{k} = 0$ if $n < k$. We state this formally as a lemma.

Lemma 5 *Let $T \in B(X)$. For $x \in X$, Then the unique polynomial $P_x(y)$ interpolating $\left\{(k, \left\|T^k x\right\|^p, 0 \leq k \leq m-1\right\}$ is*

$$P_x(y) = \sum_{k=0}^{m-1} \binom{y}{k} \beta_k(T,h)$$

where for a real number x,

$$\binom{y}{k} = \frac{y(y-1)\cdots(y-k+1)}{k!}.$$

skip the proof.

The following characterization of m-isometry seems to be a slight change of perspective to the reproducing formula, but it is proved to be very powerful. This characterization essentially follows from the following combinatorial (or difference equation) fact. Let Z denote the set of integers and Z_+ denote the set of nonnegative integers.

Lemma 6 *Let $\{a_n\}_{n\in Z_+}$ be a sequence of real number, then*

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}a_{j+k}=0 \text{ for } j\geq 0$$

if and only if there exists a polynomial $P(y)$ of degree less than or equal to $m-1$ such that $a_n = P(n)$. In this case $P(y)$ is the unique polynomial interpolating $\{(k,a_k),0\leq k\leq m-1\}$.

Proposition 7 *Let $T\in B(X)$. For any $x\in X$, set $a_n := \|T^n x\|^p$. Then T is an m-isometry if and only for each x there exists a polynomial $P_x(y)$ of degree less than or equal to $m-1$ such that $a_n = P_x(n)$.*

write the proof.

Theorem 8 *Let T weighted shifts. Then T is an m-isometry if and only if the reproducing formula holds only for $x = e_0$. Equivalently T is a strict m-isometry if and only if there exists a polynomial $P(x)$ of degree equal to $m - 1$ such that $\|T^n e_0\|^p = P(n)$ for $n \in Z_+$ (or $n \in Z$).*

write the proof

Theorem 9 *Let T be a strict (m, q)-isometric weighted shift (bilateral or unilateral) on l_p for $m \geq 2$ and $q \in (0, \infty)$. Then there exist $m_0 \geq 2$ and $k \geq 1$ such that $(m, q) = (k(m_0 - 1) + 1, kp)$ and T is a strict (m_0, p)-isometry on l_p.*

skip the proof

1 Related operators

Recall that for $T \in B(X)$ and $x \in X$,

$$(-1)^m \beta_{(m,p)}(T,x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left\| T^k x \right\|^p .$$

Throughout the paper, in particular in the following definition, $\beta_{(m,p)}(T,x) \geq 0$ really means $\beta_{(m,p)}(T,x) \geq 0$ for all $x \in X$.

We will define several class operators on X by using $\beta_{(m,p)}(T,x)$. These operators have been studied on Hilbert spaces starting by Agler's paper on hypercontractions.

Definition 10 *For* $T \in B(X)$ *and* $m \geq 1$.
(1) T *is* (m,p)*-contractive if* $(-1)^m \beta_{(m,p)}(T,x) \geq 0$;
(2) T *is* (m,p)*-hypercontractive if* $(-1)^k \beta_{(k,p)}(T,x) \geq 0$ *for* $1 \leq k \leq m$;
(3) T *is completely* p*-hypercontractive if* $(-1)^k \beta_{(k,p)}(T,x) \geq 0$ *for* $k \geq 1$;
(4) T *is* (m,p)*-expansive if* $(-1)^m \beta_{(m,p)}(T,x) \leq 0$;
(5) T *is* (m,p)*-hyperexpansive if* $(-1)^k \beta_{(k,p)}(T,x) \leq 0$ *for* $1 \leq k \leq m$;
(6) T *is completely* p*-hyperexpansive if* $(-1)^k \beta_{(k,p)}(T,x) \leq 0$ *for* $k \geq 1$;
(7) T *is* (m,p)*-alternatingly expansive if* $\beta_{(m,p)}(T,x) \geq 0$;
(8) T *is* (m,p)*-alternatingly hyperexpansive if* $\beta_{(k,p)}(T,x) \geq 0$ *for* $1 \leq k \leq m$;
(9) T *is alternatingly* p*-hyperexpansive if* $\beta_{(k,p)}(T,x)$ *for* $k \geq 1$.

write out for $k = 1, 2$.

I will prove an surprising inequality for $\beta_{(m,p)}(T,x)$.

Reversing inequality: $\beta_{(m,p)}(T,x) \leq 0$ for all $x \in X$ implies $\beta_{(m-1,p)}(T,x) \geq 0$ for all $x \in X$.

When $m = 2$ on a Hilbert space, this is due to Richter.

Lemma 11 *Let* $T \in B(X)$, $n \geq m \geq 1$ *and* $x \in X$. *Then*

$$\beta_{(m,p)}(T,x) = \beta_{(m-1,p)}(T,Tx) - \beta_{(m-1,p)}(T,x), \tag{7}$$

$$\beta_{(m,p)}(T,x) = \|T^m x\|^p - \sum_{k=0}^{m-1} \binom{m}{k} \beta_{(k,p)}(T,x), \tag{8}$$

$$\sum_{k=0}^{m-1} \binom{n}{k} \beta_{(k,p)}(T,Tx) = \sum_{k=0}^{m-1} \binom{n+1}{k} \beta_{(k,p)}(T,x) + \binom{n}{m-1} \beta_{(m,p)}(T,x). \tag{9}$$

maybe write the proof of third equality

Theorem 12 *(a) If* $\beta_{(m,p)}(T,x) \leq 0$ *for all* $x \in X$, *then for* $n \geq m$,

$$\|T^n x\|^p \leq \sum_{k=0}^{m-1} \binom{n}{k} \beta_{(k,p)}(T,x), x \in X. \quad (10)$$

(b) If $\beta_{(m,p)}(T,x) \geq 0$ *for all* $x \in X$, *then for* $n \geq m$,

$$\|T^n x\|^p \geq \sum_{k=0}^{m-1} \binom{n}{k} \beta_{(k,p)}(T,x), x \in X. \quad (11)$$

write the proof

write the proof of **Reversing inequality**

Some applications of Reversing inequality

Lemma 13 *Let $T \in B(X)$. If T is invertible, then*

$$\beta_{(m,p)}(T^{-1}, x) = (-1)^m \beta_{(m,p)}(T, T^{-m}x).$$

When $T \in B(H)$, then

$$\beta_m(T^{-1}) = (-1)^m T^{-*m} \beta_m(T) T^{-m}.$$

Corollary 14 *Assume T is invertible.*
If $\beta_{(m,p)}(T, x) \leq 0$ for all $x \in X$ and some even m, then T is an $(m-1, p)$-isometry. In particular if T is an invertible (m, p)-isometry for some even n, then T is also an $(m-1, p)$-isometry.

Conclusion. Invertible strict (m, p)-isometry only for ODD m.

write the proof

Berger-Shaw type result by Agler and Stankus.

We are now back on Hilbert spaces.

Proposition 15 *Let m be even. Let $T \in B(H)$ be an m-isometry. If T is finitely cyclic, then $\beta_{m-1}(T)$ is a compact operator.*

Next we will generalize this result to m-expansive operators by using Reversing inequality in Calkin algebra $B(H)/K(H)$

Let $\mathcal{A}$ denote a C^*-algebra with identity. For $t \in \mathcal{A}$, we write

$$\beta_m(t) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} t^{*k} t^k.$$

We have the following definition similar to Definition 10 but only stated partially.

Definition 16 *Let $t \in \mathcal{A}$. We say t is m-isometric, m-contractive, m-expansive if $\beta_m(t) = 0, (-1)^m \beta_m(t) \geq 0, (-1)^m \beta_m(t) \leq 0$ respectively.*

Theorem 17 *Let $t \in \mathcal{A}$.*
(a) If $\beta_m(t) \leq 0$, then for $n \geq m$

$$t^{*n}t^n \leq \sum_{k=0}^{m-1} \left(\begin{array}{c} n \\ k \end{array} \right) \beta_k(T).$$

If $\beta_m(t) \geq 0$, then the above inequality with $\geq$ holds. If $\beta_m(t) = 0$, then the above inequality becomes an equality.
(b) If $\beta_m(t) \leq 0$, then $\beta_{m-1}(t) \geq 0$.
write the proof of (b).

Theorem 18 *Let $T \in B(H)$ and $\pi(T)$ be its image in the Calkin algebra.*
(a) Assume $\pi(T)$ is invertible. If $\beta_m(\pi(T)) \leq 0$ for some even m, then $\pi(T)$ is an $(m-1)$-isometry. In particular if $\pi(T)$ is an invertible m-isometry for some even n, then $\pi(T)$ is also an $(m-1)$-isometry.

write the proof

Theorem 19 *Let m be even. Let $T \in B(H)$ be an m-expansive operator. If T has a finite-dimensional co-kernel, then $\beta_{m-1}(T)$ is a compact operator.*

write the proof