Various Moment Problems

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Plan of Lectures

Lecture 1. Introduction to multidimensional moment problems

Lecture 2. Solutions to truncated moment problems

Lecture 3. Applications of moment problems and related topics

Lecture 1.
Introduction to multidimensional moment problems

1 Motivation I: Numerical Integration

Definition 1 *A* ***quadrature*** *(or* ***cubature****) rule of size p and precision m is a numerical integration formula which uses p nodes, is exact for all polynomials of degree at most m, and fails to recover the integral some polynomial of degree $m+1$.*

Example 2 [Gaussian Quadrature; size n, precision $2n-1$]

We would like to find nodes t_j ($j = 0, \ldots, n-1$) satisfying

$\int_{-1}^{1} f(t)\, dt = \sum_{j=0}^{n-1} \rho_j f\left(t_j\right)$ *for every polynomial f with $\deg f \leq 2n-1$.*

Consider Interpolating Equations:

$$\sum_{j=0}^{n-1} \rho_j t_j^k = \int_{-1}^{1} t^k \, dt = \begin{cases} 0 & k = 1, 3, \ldots, 2n-1; \\ \frac{2}{k+1} & k = 0, 2, \ldots, 2n-2 \end{cases} \tag{1.1}$$

If $n = 2$*, (1.1) becomes the system of polynomial equations*

$$\begin{cases} \rho_0 + \rho_1 & = & 2 \\ \rho_0 t_0 + \rho_1 t_1 & = & 0 \\ \rho_0 t_0^2 + \rho_1 t_1^2 & = & 2/3 \\ \rho_0 t_0^3 + \rho_1 t_1^3 & = & 0 \end{cases}$$

The solution is $\rho_0 = \rho_1 = 1$*,* $t_0 = -1/\sqrt{3}$*, and* $t_1 = 1/\sqrt{3}$*. Thus we have*

$$\begin{aligned} &\int_{-1}^{1} (a_0 + a_1 t + a_2 t^2 + a_3 t^3) \; dt \\ &\qquad = a_0(\rho_0 + \rho_1) + a_1(\rho_0 t_0 + \rho_1 t_1) + a_2 \left(\rho_0 t_0^2 + \rho_1 t_1^2\right) + a_3 \left(\rho_0 t_0^3 + \rho_1 t_1^3\right) \\ &\qquad = \int_{-1}^{1} \left(a_0 + a_1 t + a_2 t^2 + a_3 t^3\right) \; d\mu, \end{aligned}$$

where $\mu := \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1}$ *(*δ *stands for the point mass).*

Numerical analysis textbooks prove this by using Legendre polynomials. We can do this as follows: Let $\beta_0 := 2, \beta_1 := 0, \beta_2 := 2/3, \beta_3 := 0$ *and let*

$$H := \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \\ \beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_3 & \alpha \end{pmatrix} = \begin{array}{c} \begin{array}{ccc} \mathit{1} & T & T^2 \end{array} \\ \begin{pmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & \alpha \end{pmatrix} \end{array}$$

For the sake of a minimal number of nodes, we want $\operatorname{rank}\ H = 2$*; thus,* $\alpha = 2/9$ *and the column relation in* H *can be written as* $T^2 = (1/3)$***1****. It is known the roots of the equation* $t^2 = 1/3$ *(that is,* $t_0 = -1/\sqrt{3}$ *and* $t_1 = 1/\sqrt{3}$*) are the nodes. We may compute the densities by solving the Vandermonde system:*

$$\begin{pmatrix} 1 & 1 \\ t_0 & t_1 \\ t_0^2 & t_1^2 \\ t_0^3 & t_1^3 \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \qquad \Longrightarrow \qquad \rho_0 = \rho_1 = 1.$$

2 Motivation II

We begin with a simple question:

How do we find a formula to represent the Fibonacci sequence?

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

For now, let us take the first six terms and write:

$$\beta \equiv \{\beta_j\}_{j=0}^{5} = \{1, 1, 2, 3, 5, 8\} \text{ or } \beta \equiv \{\beta_{ij}\} = \{1, 1, 2, 3, 5, 8\} \ \ (0 \le i + j \le 2)$$

In a way, we can find a formula for β

$$\beta_j = \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^j + \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^j$$

for $j = 0, \ldots, 6$, that is, the representing measure μ for β supported in the real line is

$$\mu = \left(\frac{5-\sqrt{5}}{10}\right)\delta_{\frac{1-\sqrt{5}}{2}} + \left(\frac{5+\sqrt{5}}{10}\right)\delta_{\frac{1+\sqrt{5}}{2}};$$

or

$$\beta_{ij} = (1)\left(\frac{1}{2}\right)^i (1)^j + \left(\frac{1}{7}\right)\left(\frac{9}{2}\right)^i (7)^j + \left(-\frac{1}{7}\right) \cdot (1)^i (0)^j,$$

that is, the representing measure ν for β supported in the plane is

$$\nu = 1 \cdot \delta_{\left(\frac{1}{2},1\right)} + \frac{1}{7} \cdot \delta_{\left(\frac{9}{2},7\right)} - \frac{1}{7} \cdot \delta_{(1,0)}.$$

The coefficients in the formulas are called **densities** and the points are **atoms** of the representing measure. This is an elementary example of the **truncated moment problem** on the real line or on the plane. The moment problem is to find some conditions for the existence of such a measure for a given sequence, and if possible, we also would like to discover a concrete formula of the measure.

3 What is the Moment Problem?

Inverse problems naturally occur in many branches of science and mathematics. An inverse problem entails finding the values of one or more parameters using the values obtained from observed data.

This problem is intimately connected with image reconstruction for X-ray computerized tomography.

Moment problems are a special class of inverse problems. While the classical theory of moments dates back to the beginning of the 20th century, the systematic study of truncated moment problems began only about 20 years ago.

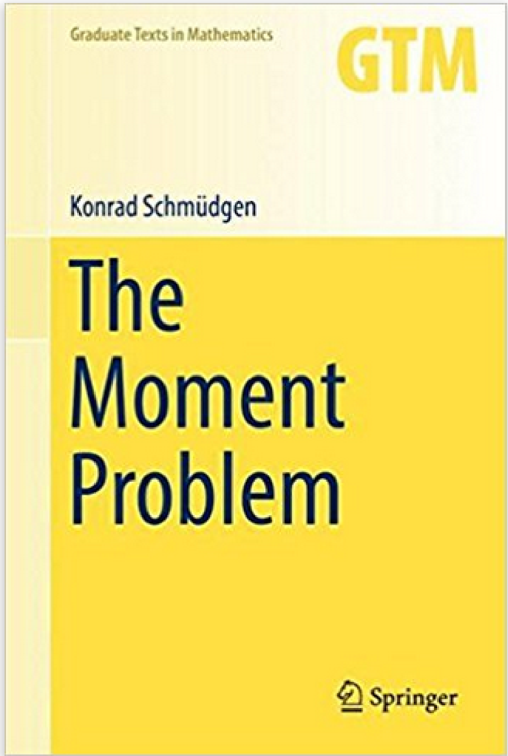
H. J. Landau said in the article “Moments in Mathematics”:

> “The moment problems is a classical question in analysis, remarkable not only for its own elegance, but also for its extraordinary range of subjects theoretical and applied, which it has illuminated. From it flow developments in function theory, in functional analysis, in spectral representation of operators, in probability and statics, in Fourier analysis and the prediction of stochastic process, in approximation and numerical methods, in inverse problems and the design of algorithms for simulating physical systems.”

By the recent developments of the multivariable moment problem, we may add the following to Landau’s list:

real algebraic geometry, optimization,
convex analysis, matrix analysis, and so on.

Good References



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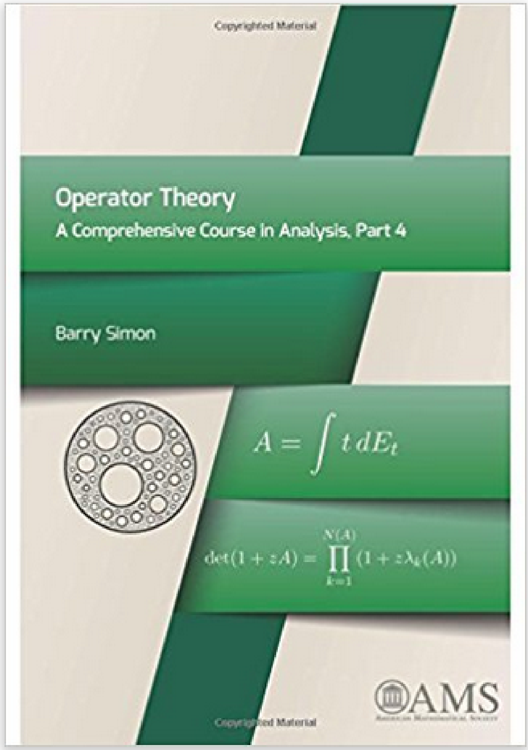

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Barry Simon

4 Finitely Atomic Measures

Definition 3 *Let $(X, \mathfrak{M}, \mu)$ be a measure space. A set $E \in \mathfrak{M}$ is called an* ***atom*** *if $\mu(E) > 0$ and for any measurable set $F \subset E$, $\mu(F) = 0$ or $\mu(E \setminus F) = 0$.*

If E is an atom, and if E contains a singleton atom $\{t\}$, then $\mu(E) = \mu(\{t\})$. In this case, a singleton atom is described as a "point mass".

A finitely atomic Borel measure in the Euclidean space is defined as

$$\mu := \sum_{k=1}^{\ell} \rho_k \delta_{w_k},$$

where ρ_k is called a density and δ_{w_k} is the point mass measure at the atom w_k for $k = 1, \ldots, \ell$.

For example, if $\{(x_i, y_i)\}_{k=1}^{\ell}$ is the set of atoms in $\mathbb{R}^2$, then

$$\int x^i y^j d\mu = \sum_{k=1}^{\ell} \rho_k x_k^i y_k^j.$$

5 Univariate Full Moment Problem

For an **infinite** real moment sequence $\beta := \{\beta_n\}_{n=0}^{\infty}$, the **full moment problem** (FMP) entails finding necessary and sufficient conditions for the existence of a positive Borel measure μ such that

$$\beta_n = \int t^n \, d\mu, \qquad n \geq 0.$$

According to the location of the support of the measure μ, the problem is classified as:

$\mathrm{supp}\ \mu \subseteq [0, \infty)$	(Stieltjes MP)
$\mathrm{supp}\ \mu \subseteq \mathbb{R}$	(Hamburger MP)
$\mathrm{supp}\ \mu \subseteq [a, b]$	(Hausdorff MP)
$\mathrm{supp}\ \mu \subseteq \mathbb{T}$	(Toeplitz MP)

5.1 Linear Functional

By a **semigroup** $(S, \circ)$ we mean a nonempty set S with an associative composition $\circ$, that is, for $s_1, s_2, s_3 \in S$

$$(s_1, s_2) \mapsto s_1 \circ s_2 \in S \text{ such that } s_1 \circ (s_2 \circ s_3) = (s_1 \circ s_2) \circ s_3,$$

and a neutral element $e \in S$, that is, $e \circ s = s \circ e = s$ for $s \in S$.

Definition 4 *A* $*$-***semigroup*** $(S, \circ, *)$ *is a semigroup* $(S, \circ)$ *endowed with a mapping* $* : S \to S$*, called an* ***involution****, such that*

$$(s \circ t)^* = t^* \circ s^*, \qquad (s^*)^* = s, \qquad s, t \in S.$$

In the sequel, let S be a $*$-semigroup. If there is no confusion to arise, we write simply S instead of $(S, \circ, *)$.

Definition 5 *A function $\varphi : S \to \mathbb{K}$ on a $*$-semigroup S is* ***positive semidefinite*** *(psd) if for arbitrary elements $s_1, \ldots, s_n \in S$, numbers $\xi_1, \ldots, \xi_n \in \mathbb{K}$ and $n \in \mathbb{N}$,*

$$\sum_{i,\,j=0}^{n} \varphi(s_i^* \circ s_j)\bar{\xi}_i \xi_j \geq 0.$$

For a $*$-semigroup S, we can define the semigroup $*$-algebra

$$\mathbb{K}[S] := \left\{ \sum_{s \in S} \alpha_s s : \alpha_s \in \mathbb{K}, \text{ only finitely many } \alpha_s \text{ are nonzero} \right\}.$$

The vector space $\mathbb{K}[S]$ becomes a unital $*$-algebra over $\mathbb{K}$ with product and involution defined by

$$\left(\sum_{s \in S} \alpha_s s \right) \left(\sum_{t \in S} \beta_t t \right) := \left(\sum_{s,\ t \in S} \alpha_s \beta_t (s \circ t) \right), \qquad \left(\sum_{s \in S} \alpha_s s \right)^* := \sum_{s \in S} \bar{\alpha}_s s^*.$$

Since the elements of S form a basis of $\mathbb{K}[S]$, there is a one-to-one correspondence between functions $\varphi : S \to \mathbb{K}$ and linear functionals $L_\varphi : \mathbb{K}[S] \to \mathbb{K}$ given by

$$L_\varphi(s) := \varphi(s), \qquad s \in S.$$

The unitial $*$-algebra $\mathbb{K}[S]$ over $\mathbb{K}$ is the **semigroup $*$-algebra** of S and the functional L_φ is often called the **Riesz functional associated with the function φ.**

Proposition 6 *For a function $\varphi : S \to \mathbb{K}$, TFAE:*

(i) φ is a psd function;

(ii) L_φ is a positive linear functional on $\mathbb{K}[S]$;

(iii) $H(\varphi) = \big(\varphi(s^* \circ t)\big)_{s,\, t \in S}$ is a psd Hermitian matrix.

5.2 Solutions to Univariate FMP

Let $\beta \equiv \{\beta_n\}_{n=0}^{\infty}$ be a real psd sequence, that is, for all $\xi_0, \xi_1, \ldots, \xi_n \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\sum_{k,\,\ell=0}^{n} \beta_{k+\ell}\, \xi_k\, \xi_\ell \geq 0.$$

When the sum is strictly positive, β is said to be **positive definite**.

Let Λ_β be the **Riesz functional** on $\mathbb{R}[x]$ defined by $\Lambda_\beta(x^n) := \beta_n$, $n \in \mathbb{N}_0$. Let $E\beta$ denote the shifted sequence given by

$$(E\beta)_n := \beta_{n+1}, \qquad n \in \mathbb{N}_0.$$

Clearly, $\Lambda_{E\beta}(p(x)) = \Lambda_\beta(x\,p(x))$ for $p(x) \in \mathbb{R}[x]$. Also, we define the Hankel matrices:

$$A(k) := \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_k \\ \beta_1 & \beta_2 & \beta_3 & \iddots & \beta_{k+1} \\ \beta_2 & \beta_3 & \iddots & \iddots & \beta_{k+2} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \beta_k & \beta_{k+1} & \beta_{k+2} & \cdots & \beta_{2k} \end{pmatrix} \quad \text{and} \quad B(k) := \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{k+1} \\ \beta_2 & \beta_3 & \beta_4 & \iddots & \beta_{k+2} \\ \beta_3 & \beta_4 & \iddots & \iddots & \beta_{k+3} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \beta_{k+1} & \beta_{k+2} & \beta_{k+3} & \cdots & \beta_{2k+1} \end{pmatrix}.$$

Theorem 7 [Hamburger, 1921] *For a real sequence* $\beta = \{\beta_n\}_{n=0}^{\infty}$, *TFAE:*

(i) There is a positive Borel measure μ *such that* $\beta_n = \int_{\mathbb{R}} x^n d\mu(x)$ *for* $n \in \mathbb{N}_0$*;*

(ii) The sequence β *is psd;*

(iii) All Hankel matrices $A(k)$ *are psd for all* $k \geq 0$*;*

(iv) Λ_β *is a positive linear functional on* $\mathbb{R}[x]$, *that is,* $\Lambda_\beta(p^2) \geq 0$ *for* $p \in \mathbb{R}[x]$.

Proof. Based on Proposition 6. ■

Proposition 8 *For a Hamburger moment sequence* $\beta = \{\beta_n\}_{n=0}^{\infty}$, *TFAE:*

(i) Each representing measure μ *of* β *has infinite support;*

(ii) The sequence β *is positive definite;*

(iii) All Hankel matrices $A(k)$ *are positive definite for all* $k \geq 0$.

Theorem 9 [Stieltjes FMP, 1894] *For a real sequence for $\beta = \{\beta_n\}_{n=0}^{\infty}$, TFAE:*

(i) There is a positive Borel measure μ with $\operatorname{supp}\ \mu \subseteq [0, \infty)$ such that $\beta_n = \int_0^{\infty} x^n d\mu(x)$ for $n \in \mathbb{N}_0$;

(ii) The sequences β and $E\beta$ are psd;

(iii) $\Lambda_\beta(p^2) \geq 0$ and $\Lambda_\beta(x\, q^2) \geq 0$ for all $p,\ q \in \mathbb{R}[x]$;

(iv) All Hankel matrices $A(k)$ and $B(k)$ are psd for all $k \geq 0$;

(v) All Hankel matrices $A(k)$ are strongly totally positive for all $k \geq 0$.

Proof. (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv): similar to that of Hamburger's Theorem. (iv) $\Longleftrightarrow$ (v): based on matrix analysis. ∎

Question 1 *Can we use the total positivity to solve multidimensional moment problems?*

6 Univariate Truncated Moment Problem

(Based on Curto-Fialkow, 1991)

Given a finite real sequence $\{\beta_n\}_{n=0}^{m}$, the **truncated moment problem** (TMP) entails finding necessary and sufficient conditions for the existence of a positive Borel measure μ satisfying $\beta_n = \int t^n d\mu \ (0 \le n \le m)$.

The solution is described based on the Hankel matrices consisting of moments:

$$A(k) := \begin{array}{c} \begin{matrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \end{matrix} \\ \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_k \\ \beta_1 & \beta_2 & \beta_3 & \iddots & \beta_{k+1} \\ \beta_2 & \beta_3 & \iddots & \iddots & \beta_{k+2} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \beta_k & \beta_{k+1} & \beta_{k+2} & \cdots & \beta_{2k} \end{pmatrix} \end{array} \quad \text{and} \quad B(k) := \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{k+1} \\ \beta_2 & \beta_3 & \beta_4 & \iddots & \beta_{k+2} \\ \beta_3 & \beta_4 & \iddots & \iddots & \beta_{k+3} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \beta_{k+1} & \beta_{k+2} & \beta_{k+3} & \cdots & \beta_{2k+1} \end{pmatrix}.$$

The j-th column of $A(k)$ will be denoted by $\mathbf{v}_j := (\beta_{j+\ell})_{\ell=0}^{k}$, $0 \le j \le k$, so that we may write $A(k) = \begin{pmatrix} \mathbf{v}_0 & \cdots & \mathbf{v}_k \end{pmatrix}$.

The **(Hankel) rank** of β, denoted $\operatorname{rank}\ \beta$, is now defined as follows: If $A(k)$ is nonsingular, $\operatorname{rank}\ \beta := k+1$; if $A(k)$ is singular, $\operatorname{rank}\ \beta$ is the smallest integer i, $1 \le i \le k$, such that $\mathbf{v}_i \in \operatorname{span}(\mathbf{v}_0 \ \cdots \ \mathbf{v}_{i-1})$. Thus, if $A(k)$ is singular, there exists a unique $(\phi_0, \ldots, \phi_{i-1})$ such that $\mathbf{v}_i = \phi_{i-1}\mathbf{v}_0 + \phi_{i-2}\mathbf{v}_1 + \cdots + \phi_0\mathbf{v}_{i-1}$. The polynomial

$$g_\beta(t) := t^i - \phi_0 t^{i-1} - \cdots - \phi_{i-1} t - \phi_{i-1} \tag{6.1}$$

is called the **generating function** of β.

A key to prove the coming results is:

Proposition 10 *Let $\tilde{\beta} = (\beta_0, \ldots, \beta_{2k+1})$, $\beta_0 > 0$. Assume $A(k)$ is positive definite. Then the generating function $g_{\tilde{\beta}}$ has $k+1$ distinct real roots, $x_0, \ldots, x_k$. Thus the Vandermonde matrix V of the points $x_0, \ldots, x_k$ is invertible, and if $\rho = \begin{pmatrix} \rho_0 & \cdots & \rho_k \end{pmatrix} := V^{-1}\mathbf{v}_0$, then $\rho_j > 0$ for $0 \le j \le k$. Moreover, if $\mu := \sum_{i=0}^{k} \rho_i \delta_{x_i}$, then $\beta_j = \int t^j \, d\mu$, $0 \le j \le 2k+1$.*

Theorem 11 [Hamburger TMP, Existence of Odd Cases]
Let $\tilde{\beta} = (\beta_0, \dots, \beta_{2k+1})$, $\beta_0 > 0$, *and let* $r := \text{rank}\ \beta$. *TFAE:*

(i) There exists a (*r-atomic) positive Borel measure* μ *satisfying* $\beta_j = \int t^j d\mu$ ($j = 0, \dots, 2k+1$) *and* $\text{supp}\ \mu \subseteq \mathbb{R}$*;*

(ii) $A(k) \geq 0$, $\mathbf{v}_{k+1} \in \text{Ran}\ A(k)$*;*

(iii) $A(k+1) \geq 0$ *for some choice of* $\beta_{2k+2} \in \mathbb{R}$, *that is,* $A(k)$ *has a positive Hankel extension.*

Proof. Based on Proposition 10. ■

Theorem 12 [Hamburger TMP, Existence of Even Cases]
Let $\tilde{\beta} = (\beta_0, \dots, \beta_{2k})$, $\beta_0 > 0$, *and let* $r := \text{rank}\ \beta$. *TFAE:*

(i) There exists a (*r-atomic) positive Borel measure* μ *satisfying* $\beta_j = \int t^j d\mu$ ($j = 0, \dots, 2k$) *and* $\text{supp}\ \mu \subseteq \mathbb{R}$*;*

(ii) $A(k) \geq 0$, $\text{rank}\ A(k) = \text{rank}\ \beta$*;*

(iii) $A(k)$ *has a positive Hankel extension.*

Theorem 13 [Stieltjes TMP, Existence of Odd Cases]
Let $\tilde{\beta} = (\beta_0, \ldots, \beta_{2k+1})$, $\beta_0 > 0$, *and let* $r := \operatorname{rank}\ \beta$. *TFAE:*

(i) There exists a positive Borel measure μ *satisfying* $\beta_j = \int t^j d\mu$ $(j = 0, \ldots, 2k+1)$ *and* $\operatorname{supp}\ \mu \subseteq [0, \infty)$*;*

(ii) There exists a r*-atomic representing measure* μ *for* $\tilde{\beta}$ *satisfying* $\operatorname{supp}\ \mu \subseteq [0, \infty)$*;*

(iii) $A(k) \geq 0$, $B(k) \geq 0$, *and* $\mathbf{v}(k+1, k) := (\beta_{k+1} \cdots \beta_{2k+1})^T \in \operatorname{Ran}\ A(k)$.

Theorem 14 [Stieltjes TMP, Existence of Even Cases]
Let $\tilde{\beta} = (\beta_0, \ldots, \beta_{2k})$, $\beta_0 > 0$, *and let* $r := \operatorname{rank}\ \beta$. *TFAE:*

(i) There exists a positive Borel measure μ *satisfying* $\beta_j = \int t^j d\mu$ $(j = 0, \ldots, 2k)$ *and* $\operatorname{supp}\ \mu \subseteq [0, \infty)$*;*

(ii) There exists a r*-atomic representing measure* μ *for* $\tilde{\beta}$ *satisfying* $\operatorname{supp}\ \mu \subseteq [0, \infty)$*;*

(iii) $A(k) \geq 0$, $B(k-1) \geq 0$, *and* $\mathbf{v}(k+1, k-1) := (\beta_{k+1} \cdots \beta_{2k})^T \in \operatorname{Ran}\ B(k-1)$.

Theorem 15 [Hausdorff TMP, Existence of Odd Cases]

Let $\tilde{\beta} = (\beta_0, \ldots, \beta_{2k+1})$, $\beta_0 > 0$, *and let* $r := \text{rank } \beta$, *and let* $g_{\tilde{\beta}}$ *as in (6.1). There exists a positive Borel measure* μ *satisfying* $\beta_j = \int t^j d\mu$ *(*$j = 0, \ldots, 2k+1$*) and* $\text{supp } \mu \subseteq [a, b]$ *if and only if* $A(k) \geq 0$, $bA(k) \geq B(k) \geq aA(k)$, *and* $\mathbf{v}(k+1, k) := (\beta_{k+1} \cdots \beta_{2k+1})^T \in \text{Ran } A(k)$.

Theorem 16 [Hausdorff TMP, Existence of Even Cases]

Let $\tilde{\beta} = (\beta_0, \ldots, \beta_{2k})$, $\beta_0 > 0$, *and let* $r := \text{rank } \beta$. *There exists a positive Borel measure* μ *satisfying* $\beta_j = \int t^j d\mu$ *if and only if* $A(k) \geq 0$, $bA(k) \geq B(k) \geq aA(k)$, *and there exists* β_{2k+1} *such that* $\mathbf{v}(k+1, k) := (\beta_{k+1} \cdots \beta_{2k+1})^T \in \text{Ran } A(k)$.

Example 17 *Consider an example:* $\beta \equiv \{\beta_{j=0}^{5}\} = \{1, 1, 2, 3, 5, 8\}$. *This is an odd case with* $k = 2$*; thus,*

$$A(2) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{pmatrix} \quad \textit{and} \quad B(2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 8 \end{pmatrix}$$

Note that $A(2) \geq 0$ *but* $B(2) \not\geq 0$*; thus, in the view of Stieltjes MP, this sequence has no solution. However, in the view of Hausdorff MP, it may have a measure on some* $[a, b]$*, where* $bA(2) \geq B(2) \geq aA(2)$*. There are infinitely many desired* a *and* b*. Indeed,* $A(2)$ *has a unique column relation* $T^2 = 1 + T$*, where* T^i *stands for the* ($i+1$)*-th column in* $A(2)$)*. Thus the generating function* $g_{\tilde{\beta}}(t) = t^2 - 1 - t$ *has the two roots* $(1 \pm \sqrt{5})/2$ *that are the atoms of the unique representing measure* μ*. Solving the Vandermonde system*

$$\begin{pmatrix} 1 & 1 \\ (1-\sqrt{5})/2 & (1+\sqrt{5})/2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we find the densities $\rho_1 = (5 - \sqrt{5})/10$ *and* $\rho_2 = (5 + \sqrt{5})/10$.

7 Representation of General Fibonacci Sequences

Let $c_0, c_1, \ldots, c_{r-1}$ with $c_{r-1} \neq 0$ ($r \geq 2$) be fixed real numbers. For any sequence of real numbers $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})$, we define the **$r$-generalized Fibonacci sequence** $\{Y_\alpha(n)\}_{n\geq 0}$ as follows: $Y_\alpha(n) = \alpha_n$ for $n = 0, 1, \ldots, r-1$ and

$$Y_\alpha(n+1) = c_0 Y_\alpha(n) + c_1 Y_\alpha(n-1) + \cdots + c_{r-1} Y_\alpha(n-r+1) \qquad (7.1)$$

for all $n \geq r-1$.

Theorem 18 [Rachidi-Wahbi, 2001] *Let $Y_\alpha = \{Y_\alpha(n)\}_{n\geq 0}$ be given by sequence (7.1), where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})$ with $\alpha_0 > 0$ and let $r = \operatorname{rank}(Y_\alpha)$. TFAE:*

(i) There exists a positive a Borel measure μ with $\operatorname{supp} \mu \subseteq [a, b]$ such that $Y_\alpha(n) = \int_a^b t^n d\mu(t)$ for all $n \geq 0$;

(ii) There exists a r-atomic representing measure μ for Y_α such that $\operatorname{supp} \mu \subseteq [a, b]$;

(iii) $A(r) \geq 0$ and $bA(r) \geq B(r) \geq aA(r)$.

8 Multidimensional Full Moment Problems

(Based on B. Fuglede, 1984)

Notations

(i) $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

(ii) $s = (s_\alpha)_{\alpha \in \mathbb{N}_0^n}$: a multisequence ($n$-sequence); that is, $s : \mathbb{N}_0^n \to \mathbb{R}$

(iii) $\mathcal{P}_0$: complex vector space of all polynomials, $\mathcal{P}_d := \{p \in \mathcal{P}_0 : \deg p \leq d\}$

(iv) E_j: the shift operator on the vector space of all real n-sequences; that is, $(E_j s)_\alpha = s_{\alpha + e^{(j)}}$, where $e^{(j)} = (0, \ldots, 1, \ldots, 0)$ with 1 as the j-th entry.

If $p = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha \in \mathcal{P}_0$ with real coefficients, then

$$p(E) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha E^\alpha \qquad \text{and} \qquad (p(E)s)_\alpha = \sum_{\beta \in \mathbb{N}_0^n} a_\beta s_{\alpha+\beta}.$$

where $E^\alpha = E_1^{\alpha_1} \cdots E_n^{\alpha_n}$.

(v) $\mathcal{M}^*$: the convex cone of all positive Borel measures μ on $\mathbb{R}^n$ such that the integrals, $s_\alpha = \int_{\mathbb{R}^n} x^\alpha d\mu(x)$ ($\alpha \in \mathbb{N}_0^n$), converge absolutely. A measure μ is referred to as a **representing measure**.

(vi) V_s (or V_μ): the convex (and weak* compact) set of all $\mu \in \mathcal{M}^*$ having the moments s_α

(vii) K: a closed subset of $\mathbb{R}^n$

If an n-sequence s has a representing measure μ satisfying $\operatorname{supp}\ \mu \subseteq K$, then μ is said to be a K**-representing measure**.

Definition 19 *The* ***Riesz functional*** $\Lambda \equiv \Lambda_s$ *on* $\mathcal{P}_0$ *defined by*

$$\Lambda_s(p) := \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha s_\alpha, \qquad p = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha \in \mathcal{P}_0.$$

In particular, $\Lambda_s(x^\alpha) = s_\alpha$.

Definition 20 *A real n-sequence $s = (s_\alpha)_{\alpha \in \mathbb{N}_0^n}$ is said to be* ***positive semidefinite*** *(psd) if*

$$\Lambda_s(p\bar{p}) \geq 0 \qquad \textit{for every } p \in \mathcal{P}_0.$$

Explicitly, this amounts to the condition that

$$\sum s_{\alpha+\beta} a_\alpha \bar{a}_\alpha \geq 0$$

for every complex (or, equivalently, for every real) n-sequence $a = (a_\alpha)_{\alpha \in \mathbb{N}_0^n}$ such that $a_\alpha = 0$ for all but finitely many $\alpha \in \mathbb{N}_0^n$. In particular, if $\Lambda_s(p\bar{p}) > 0$ for every nonzero $p \in \mathcal{P}_0$, then s is said to be ***positive definite****.*

Note that one can easily see that TFAE:

(i) s is psd;

(ii) the **symmetric kernel** $(\alpha, \beta) \mapsto s_{\alpha+\beta}$ is positive semidefinite;

(iii) $\det \left(s_{\alpha^{(i)}+\alpha^{(j)}}\right) \geq 0$ for every r-tuple $(\alpha^{(1)}, \ldots, \alpha^{(r)})$ of distinct n-indices $\alpha^{(j)} = (\alpha_1^{(j)}, \ldots, \alpha_n^{(j)})$, $j = 1, \ldots, r$.

In the presence of a representing measure μ for s, we may write for $p \in \mathcal{P}_0$,

$$\Lambda_s(p) = \int p \, d\mu.$$

It follows that if every moment n-sequence has a representing measure, then it must be psd by the fact that for $p \in \mathcal{P}_0$,

$$\Lambda_s(p\bar{p}) = \int |p|^2 \, d\mu \geq 0.$$

Note If s has a K-representing measure μ, then Λ_s is K**-positive**; that is, $p \in \mathcal{P}_0$ and $p|_K \geq 0 \Longrightarrow \Lambda_s(p) \geq 0$.

Definition 21 *If a moment n-sequence s has a unique representing measure, then s is said to be **determinate**.*

Note When K is compact, every K-moment sequence is determinate on account of Weierstrass' approximation theorem.

8.1 Summary of Well-known Results

(a) A sequence s has a representing measure supported on $K = [0, 1]$;

$\iff$ s and $E(I - E)s$ are both psd [Hausdorff];

$\iff$ s, Es, and $(I - E)s$ are psd [Devinatz].

(b) [Hamburger, 1920-21]
Every psd real sequence has $\mathbb{R}$-representing measure ($n = 1$).

(c) [M. Riesz, Haviland (provided an alternative proof)]
A sequence s has a representing measure supported on $K = [0, \infty)$
$\iff$ s and Es are both psd.

(d) [Filstinskii]
For the case when $\mathbb{R} \backslash K$ is the disjoint union of finitely many bounded open intervals (α_j, β_j), $j = 1, \ldots, k$:
A sequence s has a representing measure supported on K
$\iff \prod_{j \in J}(E - \alpha_j I)(E - \beta_j I)s$ is psd for each subset J of $\{1, 2, \ldots, k\}$.

(e) [Riesz ($n = 1$ and $K = \mathbb{R}$), Haviland (general cases)]

Theorem 22 [Riesz-Haviland]

A sequence s has a K-representing measure $\iff$ s is K-positive;

that is, $\Lambda_s(p) \geq 0$ for every $p \in \mathcal{P}_0$ such that $p|_K \geq 0$.

Proof. ($\Longrightarrow$) Clear!

($\Longleftarrow$) Key 1: The positive linear form Λ_s on the vector space $\mathcal{P}_0(K)$ considered as functions on K extends to a Radon measure on K because $\mathcal{P}_0(K)$ is an adapted space in the sense defined by Choquet (1962).

Key 2: F. Riesz representation theorem (1909). ■

This Riesz-Haviland Theorem can be used in some cases (for example, Hamburger MP, Stieltjes MP, Schmüdgen's results) to obtain "concrete" solutions; that is, solutions expressed in terms of positivity of matrices closely associated with an n-sequence s.

Hilbert's 17th Question

We refer to $p \in \mathcal{P}_{2d}$ as a **sum of squares** (sos) if there exist $p_1, \ldots, p_k \in \mathcal{P}_d$ such that $p = \sum_{i=1}^{k} p_i^2$.

To apply the Riesz-Haviland Theorem, one must establish the positivity of Λ_s on K. In particular, the positivity of Λ_s is easily established if each psd polynomial is sos, since then Λ_s is positive if and only if the associated **moment matrix** (see the definition in Lecture 2) is psd.

Hilbert showed that every nonnegative (psd) polynomial is a sum of squares of polynomials only in the following 3 cases: univariate polynomials, quadratic polynomials, and bivariate polynomials of degree 4.

Thus, verifying the positivity of Λ_s is highly nontrivial beyond the scope of Hilbert's result. This fact is the main reason why multidimensional moment problems are much more difficult than the classical one variable problems. For some K, checking K-positivity was successful and we have the solutions as follows:

(f) [Berg-Maserick]

Let $p_1, \ldots, p_m$ be real polynomials such that

$$K = \bigcap_{j=1}^{m} \{x \in \mathbb{R} : p_j(x) \geq 0\}$$

is compact. A psd sequence s is a K-moment sequence if and only if each sequence $p_j(E)s$, $j = 1, \ldots, m$, is psd.

(g) [Berg-Maserick]

If $K = \{x \in \mathbb{R} : p(x) \geq 0\}$, $K \neq \mathbb{R}$ (non-compact), then a psd sequence s has a K-representing measure if and only if $p(E)s$ is psd and $\deg p \leq 2$.

(h) [Hildebrandt-Schoenberg, 1933]

A sequence s is a moment sequence supported on $K = [0, 1]^n$

$\iff$ $(I - E)^{\alpha} s \geq 0$ for all $\alpha \in \mathbb{N}_0^n$.

(i) [Berg-Christensen-Jensen, Schmüdgen, 1979]

There is a psd multisequence which has no representing measure.

(j) As to further necessary and sufficient conditions for solubility of the multidimensional K-moment problems see:

(1) [Herglotz, F. Riesz] K: the unit circle in $\mathbb{R}^2 = \mathbb{C}$ (**Trigonometric MP**)

(2) [Akhiezer-Krein] K: any circular arc

(3) [Devinatz] K: a product of compact intervals and/or circular arcs

(4) [Devinatz] K: a circular cylinder

(5) [Atzmon] K: the unit disc in $\mathbb{R}^2 = \mathbb{C}$

(6) [Maserick] K: a convex body in $\mathbb{R}^n$

(7) [McGregor] K: a ball or a sphere in $\mathbb{R}^n$ or a solid torus in $\mathbb{R}^3$

(8) [Schmüdgen] K: a compact semialgebraic subset in $\mathbb{R}^n$

(k) [Berg-Maserick, 1982]

If $K = \{x \in \mathbb{R}^n : p(x) \geq 0\}$ is compact for any real polynomial p on $\mathbb{R}^n$, then an n-sequence s has a K-representing measure if and only if both s and $p(E)s$ have a representing measure.

(l) [Lindahl-Maserick, 1971]

Every “bounded” psd n-sequence is a determinate moment n-sequence if and only if the representing measure is supported by $[-1, 1]^n$.

(m) [Berg-Maserick, 1984]

A psd n-sequence of at most exponential growth is the same as the n-sequence of moments of a measure of compact support (hence determinate).

(n) [Berg-Christensen,1979]

If $(s_\alpha)_{\alpha\in\mathbb{N}_0^n}$ and $(t_\alpha)_{\alpha\in\mathbb{N}_0^n}$ admit representing measures, then so does $(s_\alpha t_\alpha)_{\alpha\in\mathbb{N}_0^n}$.

More results are obtained by means of the operator theoretic approach by Devinatz, Kostyučenko-Mityagin, Èskin, and Nussbaum.

8.2 Spectral Theory for Commuting Families of Self-adjoint Operators

Define a sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{P}_0$ by

$$\langle p, q \rangle = \Lambda_s(p\bar{q}) = \sum_{\alpha,\, \beta \in \mathbb{N}_0^n} s_{\alpha+\beta}\, a_\alpha \bar{b}_\beta$$

for any two polynomials $p = \sum a_\alpha x^\alpha$, $q = \sum b_\alpha x^\beta \in \mathcal{P}_0$.

Consider a subspace $\mathcal{N}$ of $\mathcal{P}_0$ defined by $\mathcal{N} := \{p \in \mathcal{P} : \langle p, p \rangle = 0\}$.

We denote $\mathcal{P}$ the Hilbert space completion of the prehilbert space $\mathcal{P}_0/\mathcal{N}$.

In the view of the Cauchy-Schwartz inequality, $\mathcal{N}$ is therefore an “ideal” in the commutative ring $\mathcal{P}_0$.

Denoting by $X_j : \mathcal{P}_0 \to \mathcal{P}_0$ ($j = 1, \ldots, n$) the operator of multiplication by the j-th coordinate x_j, we thus have

$$X_j \mathcal{N} \subset \mathcal{N}, \quad j = 1, \ldots, n.$$

This inclusion is closely related to the (RG)-property of TMP (see the definition in Lecture 2).

If s is the moment n-sequence of some measure $\mu \in \mathcal{M}^*$, then

$$\langle p, q \rangle = \int p\bar{q} d\mu, \qquad \text{for all } p,\ q \in \mathcal{P}_0.$$

The ideal $\mathcal{N}$ consists of all polynomials vanishing on $\operatorname{supp}\ \mu$. In particular, s is positive definite (that is, $\mathcal{N} = \{0\}$) if and only if $\operatorname{supp}\ \mu$ is not contained in the union of finitely many real algebraic varieties $(\neq \mathbb{R}^n)$.

Anyhow it follows from that $\mathcal{P}_0/\mathcal{N}$ is embedded linearly and isometrically as a subspace of the Hilbert space $L^2(\mu)$. The completion $\mathcal{P}$ of $\mathcal{P}_0/\mathcal{N}$ may therefore be identified with the "closure" $\mathcal{P}^\mu$ of $\mathcal{P}_0/\mathcal{N}$ in $L^2(\mu)$. In the sequel we do not distinguish notationally between $\mathcal{P}_0$ and $\mathcal{P}_0/\mathcal{N}$.

Theorem 23 [Nussbaum, 1965; Fuglede, 1983]
A psd n-sequence $s = (s_\alpha)_{\alpha \in \mathbb{N}_0^n}$ has a representing measure if and only if there exist a Hilbert space $\mathcal{H}$ in which $\mathcal{P}_0$ is linearly and isometrically embedded, and a family $H = (H_1, \ldots, H_n)$ of commuting self-adjoint operators in $\mathcal{H}$ such that H_j extends the multiplication operator X_j on $\mathcal{P}_0$:

$$H_j p(x) = x_j p(x), \qquad p \in \mathcal{P}_0, \qquad j = 1, \ldots, n. \tag{8.1}$$

In the affirmative case, the mapping $E \mapsto \langle E\mathit{1}, \mathit{1}\rangle$ carries the set $\mathcal{E}$ of all spectral measures E on $\mathbb{R}^n$ of such families H (in such spaces $\mathcal{H}$) onto the equivalence class V_s of all measures $\mu \in \mathcal{M}^*$ having the moments s_α.

Proof. ($\Longleftarrow$) Suppose that commuting families as stated exist, and let $E \in \mathcal{E}$ denote the spectral measure of such a family H. Then $\mu := \langle E\mathit{1}, \mathit{1}\rangle$ is a (positive) measure on $\mathbb{R}^n$, defined by

$$\mu(\sigma) := \langle E(\sigma)\mathit{1}, \mathit{1}\rangle, \qquad \sigma \in \text{Borel}\,\mathbb{R}^n.$$

For any $\alpha, \beta \in \mathbb{N}_0^n$ we then have from (8.1):

$$\langle E(\sigma)x^\alpha, x^\beta\rangle = \langle E(\sigma)H^\alpha \mathit{1}, H^\beta \mathit{1}\rangle = \int_\sigma \lambda^{\alpha+\beta} d\mu(\lambda). \tag{8.2}$$

This shows that $\mu \in \mathcal{M}^*$, and also that the restriction of $PE(\sigma)$ to $\mathcal{P}$ is uniquely determined from μ (the monomials $x \mapsto x^\alpha$ being total in $\mathcal{P}$). Taking $\sigma = \mathbb{R}^n$ and $\beta = 0$ in (8.2), we obtain

$$\int \lambda^\alpha d\mu(\lambda) = \langle x^\alpha, \mathit{1} \rangle = \Lambda_s(x^\alpha) = s_\alpha.$$

Thus, $\mu = \langle E\mathit{1}, \mathit{1} \rangle$ has indeed the moments s_α.
($\Longrightarrow$) Let $\mu \in V_s$ denote any measure from $\mathcal{M}^*$ representing s. Consider the "cannonical extension"

$$\mathcal{H} = L^2(\mu), \qquad \mathcal{P} = \mathcal{P}^\mu, \qquad H = x\cdot, \tag{8.3}$$

that is, H_j is the self-adjoint operator in $L^2(\mu)$ defined by $H_j f(x) = x_j f(x)$ for all $f \in L^2(\mu)$ such that $x \mapsto x_j f(x)$ is in $L^2(\mu)$. Then $\mathcal{P}$ is isometrically embedded as the closure $\mathcal{P}^\mu$ of $\mathcal{P}_0$ in $L^2(\mu)$, and (8.1) is fulfilled. Moreover, $H_1, \ldots, H_n$ commute, and the spectral measure E of H is given by $E(\sigma) = \mathit{1}_\sigma\cdot$, that is

$$E(\sigma)f = \mathit{1}_\sigma f, \qquad f \in L^2(\mu),$$

where $\mathit{1}_\sigma$ denotes the indicator function of σ. Clearly $\langle E(\sigma)\mathit{1}, \mathit{1} \rangle = \langle E\mathit{1}_\sigma, \mathit{1} \rangle = \mu(\sigma)$. ■

9 Moment Problems via Dimensional Extension

(Based on Putinar and Vasilescu, 1999)

Theorem 24 *Let $s = (s_\alpha)_{\alpha\in\mathbb{N}_0^n}$ ($s_0 > 0$) be a n-moment sequence of real numbers. Let $p = (p_1, \ldots, p_m) \in \mathcal{P}_n$ and let $p_k(t) = \sum_{\xi\in I_k} a_k t^\xi$, $k = 1, \ldots, m$ with an index set $I_k \subseteq \mathbb{N}_0^n$ finite for all k. Then there is a representing measure for s supported on $\cap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$ if and only if there exists a psd $(n+1)$-sequence $t = \left(t_{(\alpha,\beta)}\right)_{(\alpha,\beta)\in\mathbb{N}_0^n\times\mathbb{N}_0}$ with the following properties: For all $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0$, and $k = 1, \ldots, m$,*

(i) $s_\alpha = t_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{N}_0^n$;

(ii) $t_{(\alpha,\beta)} = t_{(\alpha,\beta+1)} + \sum_{j=1}^n t_{(\alpha+2e_j,\beta+1)} + \sum_{k=1}^m \sum_{\xi,\eta\in I_k} a_{k\xi} a_{k\eta} t_{(\alpha+\xi+\eta,\beta+1)}$;

(iii) The $(n+1)$*-sequences* $\left(\sum_{\xi\in I_k} a_{k\xi} a_{k\eta} t_{(\alpha+\xi,\beta)}\right)_{(\alpha,\beta)\in\mathbb{N}_0^n\times\mathbb{N}_0}$ *are psd.*

Moreover, s has a uniquely determined representing measure on $\cap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$ if and only if the $(n+1)$-sequence t is unique.

10 Recurrence Formulas and Favard's Theorem

Recall that for a real psd sequence $s = (s_n)_{n\in\mathbb{N}_0}$, the equation

$$\langle p, q\rangle_s := \Lambda_s(p\bar{q}), \qquad p,\ q \in \mathbb{C}[x],$$

defines a scalar product (sesquilinear form) on the vector space $\mathbb{C}[x]$.

Proposition 25 *There exists an orthonormal basis $(p_n)_{n\in\mathbb{N}_0}$ of the unitary space $(\mathbb{C}[x], \langle\cdot,\cdot\rangle_s)$ such that each polynomial p_n has degree j and a positive leading coefficient. The basis $(p_n)_{n\in\mathbb{N}_0}$ is uniquely determined by these properties. Moreover, $p_n \in \mathbb{R}[x]$.*

Here, the sequence $(p_n)_{n\in\mathbb{N}_0}$ is **orthonormal** means that

$$\langle p_k, p_n\rangle = \delta_{k,n} \qquad k,\ n \in \mathbb{N}_0.$$

Definition 26 *The polynomials p_n, $n \in \mathbb{N}_0$, are called **orthonormal polynomials** associated with the positive definite sequence s.*

Proposition 27 *Set* $D_{-1} = 1$*. Then* $p_0(x) = 1/\sqrt{s_0}$ *and for* $n \in \mathbb{N}$ *and* $k \in \mathbb{N}_0$*,*

$$p_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_n \\ s_1 & s_2 & s_3 & \cdots & s_{n+1} \\ s_2 & s_3 & s_4 & \cdots & s_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix},$$

$$\langle x^n, p_n \rangle_s = \sqrt{d_n/d_{n-1}}, \quad \textit{and} \quad \langle x^k, p_n \rangle_s = 0 \quad \textit{if} \quad k < n.$$

The leading coefficient of p_n *is* $\sqrt{\frac{d_n}{d_{n-1}}}$*. In particular,* $p_1(x) = \frac{s_0 x - s_1}{\sqrt{s_0(s_0 s_2 - s_1^2)}}$*.*

Definition 28 *A sequence* $(R_n)_{n \in \mathbb{N}_0}$ *is called a sequence of* ***orthogonal polynomials*** *(OPS) with respect to* s *if* $R_n(x) \in \mathbb{R}[x]$, $\deg R_n = n$*, and*

$$\langle R_k, R_n \rangle = 0 \qquad \textit{for} \quad k \neq n, k, n \in \mathbb{N}_0.$$

While there are many OPS for a given sequence s, there is a unique OPS consisting of monic polynomials. Since p_n has the leading term $\sqrt{D_{n-1}/D_n}$, the polynomial

$$P_n(x) := \sqrt{D_n/D_{n-1}}\, p_n(x), \qquad n \in \mathbb{N}_0,$$

is monic. Set $P_0(x) = 1$. Then $(P_n)_{n\in\mathbb{N}_0}$ is the unique monic OPS for s.

Orthogonal polynomials can be characterized and studied by means of three term recurrence relations. In particular, for a monic OPS $P_n(x) = \sqrt{D_n/D_{n-1}}\, p_n(x)$ for a sequence s:

Proposition 29 *Set* $a_n = \sqrt{D_{n-1}D_{n+1}}D_n^{-1}$ *and* $b_n = \Lambda_s(x\, p_n^2)$ *for* $n \in \mathbb{N}_0$*. Then we have* $a_n > 0$ *and* $b_n \in \mathbb{R}$ *for* $n \in \mathbb{N}_0$*, and*

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_{n-1}^2 P_{n-1}(x), \qquad n \in \mathbb{N}_0,$$

where $a_{-1} := 1$ *and* $P_{-1} := 0$*. In particular,*

$$P_0(x) = 1, \qquad P_1(x) = x - b_0, \qquad P_2(x) = (x - b_0)(x - b_1) - a_0^2.$$

The converse to Proposition 29 is known as **Favard's theorem**:

Theorem 30 *Let* $(\alpha_n)_{n\in\mathbb{N}_0}$ *and* $(\beta_n)_{n\in\mathbb{N}_0}$ *be complex sequences and set* $\alpha_{-1} := 1$*. Let* $(R_n)_{n\in\mathbb{N}_0}$ *denote the sequence of monic polynomials* R_n *which is uniquely determined by the relations, for* $n \in \mathbb{N}_0$*,*

$$R_{n+1}(x) = (x - \beta_n)R_n(x) - \alpha_{n-1}R_{n-1}(x), \qquad R_{-1} = 0, \qquad R_0(x) = 1.$$

There exists a positive definite real sequence s *such that* $(R_n)_{n\in\mathbb{N}_0}$ *is the monic OPS for* s *if and only if* $\alpha_n > 0$ *and* $\beta_n \in \mathbb{R}$ *for all* $n \in \mathbb{N}_0$*. If* s_0 *is a given positive number, then this sequence* $s = (s_n)n \in \mathbb{N}_0$ *is uniquely determined.*

Further, if $\alpha_n > 0$ *and* $\beta \in \mathbb{R}$ *for all* $n \in \mathbb{N}_0$ *and* $s_0 > 0$ *are given, then there exists a measure* $\mu \in \mathcal{M}_s$ *such that* $\mu(\mathbb{R}) = s_0$ *and for* $j, k \in \mathbb{N}_0$*,* $j \neq k$*, and* $n \in \mathbb{N}$*,*

$$\int_{\mathbb{R}} R_j(x)R_k(x)\, d\mu(x) = 0, \qquad \int_{\mathbb{R}} R_n^2(x)\, d\mu(x) = \alpha_{n-1}\alpha_{n-2}\cdots\alpha_0 s_0.$$

Question 2 *Can we have a multidimensional version of the Favard Theorem?*

11 Symmetric Moment Problems

(Based on T. S. Chihara, 1982)

We consider a moment sequence $\beta \equiv \{\beta_j\}_{j=0}^{\infty}$ satisfying

$$\beta_{2j} > 0, \qquad \beta_{2j+1} = 0. \tag{11.1}$$

We assume β satisfies Hamburger's criterion

$$A(n) = \left(\beta_{i+j}\right)_{i,\,j=0}^{n} > 0, \qquad n \geq 0,$$

so that there is a measure μ with an infinite spectrum $\mathcal{S}(\mu)$ such that

$$\beta_j = \int_{-\infty}^{\infty} t^j \, d\mu(t). \tag{11.2}$$

A moment sequence and its corresponding Hamburger moment problem will be called **symmetric** if (11.1) is satisfied. A measure μ will be called symmetric if there is a constant C such that

$$\mu(t) + \mu(-t) = C \tag{11.3}$$

at all points of continuity.

Note A soluble symmetric Hamburger MP always has a symmetric solution since μ_1 is any solution of (11.1) and (11.2), then

$$\mu(t) = \frac{1}{2}\left[\mu_1(t) - \mu_1(-t)\right]$$

yields a symmetric solution.

Let $\{\beta_j\}_{j=0}^{\infty}$ be a symmetric moment sequence satisfying Hamburger's criterion and let μ be a symmetric solution of the resulting Hamburger MP. Then there is a uniquely determined sequence $S = \{S_j(t)\}$ of monic polynomials orthogonal with respect to μ. The sequence S is a symmetric orthogonal polynomial sequence:

$$S_j(-t) = (-1)^j S_j(t).$$

The related sequence $P := \{P_j(t)\}$ defined by

$$P_j(t^2) = S_{2j}(t)$$

is an orthogonal polynomial sequence (OPS) with respect to the measure

$$d\psi(t) = 2d\mu(\sqrt{t}), \qquad t \geq 0. \tag{11.4}$$

The sequence $K := \{K_j(t)\}$ defined by

$$tK_j(t^2) = S_{2j+1}(t)$$

is then the corresponding sequence of monic **kernel polynomials** which are orthogonal with respect to

$$d\omega(t) = td\psi(t), \qquad t \geq 0. \tag{11.5}$$

A measure have the form (11.5) will be said to be of **kernel type**.

We will assume all measures have been nomalized by

$$\mu(x) = \frac{1}{2}\left[\mu(t^+) - \mu(t^-)\right], \qquad \mu(0) = 0.$$

We also always take $\beta_0 = 1$ so the correspondence between $\{\beta\}$ and S is one-to-one. The measure defined by (11.4) is a solution of Stieltjes moment problem corresponding to the moment sequence $\{\beta_{2j}\}_{n=0}^{\infty}$, while (11.5) yields a solution of the Stieltjes moment problem for $\{\beta_{2j+2}\}_{n=0}^{\infty}$. The Stieltjes moment problem for $\{\beta_{2j}\}_{n=0}^{\infty}$ will be called the Stieltjes moment problem **associated** with the symmetric Hamburger moment problem $\{\beta_j\}_{n=0}^{\infty}$ (and conversely).

Theorem 31 [T. S. Chihara, 1982]

A symmetric Hamburger moment sequence $\beta_H = \{\beta_j\}_{n=0}^{\infty}$ *is determinate if and only if the associated Stieltjes moment sequence* $\beta_S = \{\beta_{2j+2}\}_{n=0}^{\infty}$ *is determinate.*

Proof. Suppose ν is the representing measure of β_S. Let

$$d\mu(t) = \frac{1}{2}\left[\chi_{[0,\infty)}(t)\, d\nu(t^2) + \chi_{(-\infty,0]}(t)\, d\nu(t^2)\right].$$

Then the moments of μ are β_H. Thus the uniqueness for β_H on $(-\infty,\infty)$ implies the uniqueness for β_S on $[0,\infty)$. ■

A related topic will be discussed with a different approach in Lecture 3.

12 Strong Moment Problems

(Based on Jones-Thorn-Njåstad, 1984)

Given a moment sequence $c \equiv \{c_j\}_{j\in\mathbb{Z}}$ of real numbers, The **strong Hamburger moment problem** is find necessary and sufficient conditions for the existence of a nonnegative measure ψ defined on the Borel sets of the real line and with infinite support, such that

$$c_j = \int_{-\infty}^{\infty} (-t)^j \, d\psi(t), \qquad \text{for all } j \in \mathbb{Z}.$$

The solution is given in terms of positivity of certain Hankel determinants associated with the double sequence $\{c_n\}$. The main tool used is the theory of orthogonal (and quasiorthogonal) Laurent polynomials.

For each $n \in \mathbb{Z}$, the **Hankel determinants** $H_n^{(n)}$ associated with the double sequence c are given by

$$H_0^{(n)} = 1, \qquad H_k^{(n)} = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+k-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n+k-1} & c_{n+k} & \cdots & c_{n+2k-2} \end{vmatrix}, \qquad k \geq 1.$$

Theorem 32 *The strong Hamburger moment problem for a moment sequence $c \equiv \{c_j\}_{j \in \mathbb{Z}}$ has a solution if and only if the following determinant criteria are satisfied:*

$$H_{2m}^{(-2m)} > 0, \qquad H_{2m+1}^{(-2m)} > 0, \qquad m \geq 0.$$

13 Multidimensional Truncated Moment Problems

Let $\beta \equiv \beta^{(m)} = \left\{\beta_{\mathbf{i}} \in \mathbb{R} : \mathbf{i} \in \mathbb{Z}_+^d,\ |\mathbf{i}| \le m\right\}$, with $\beta_{\mathbf{0}} > 0$, be a d-dimensional multisequence of degree m. It is called a **truncated moment sequence**. For a closed set $K \subseteq \mathbb{R}^d$, the **truncated K-moment problem** (TKMP) entails finding necessary and sufficient conditions for the existence of a positive Borel measure μ on $\mathbb{R}^d$ with $\operatorname{supp}\ \mu \subseteq K$ such that

$$\beta_{\mathbf{i}} = \int \mathbf{x}^{\mathbf{i}}\, d\mu(\mathbf{x}) \ \ (\mathbf{i} \in \mathbb{Z}_+^d,\ |\mathbf{i}| \le m),$$

where $\mathbf{x} \equiv (x_1, \ldots, x_d)$, $\mathbf{i} \equiv (i_1, \ldots, i_d) \in \mathbb{Z}_+^d$, and $\mathbf{x}^{\mathbf{i}} := x_1^{i_1} \cdots x_d^{i_d}$.

The measure μ is said to be a K-**representing measure** for β. For the typical case $K = \mathbb{R}^d$, the problem is referred to as the **truncated real moment problem** (TRMP) and μ is a **representing measure**.

14 Another Scope of TMP

We may consider the following question:

> If there is a K-representing measure, is there a "finitely" K-representing measure?

A theorem of Tchakalloff in 1957 provides an affirmative answer for K compact. The complete answer found 50 years later:

Theorem 33 [Bayer-Teichmann, 2006] *If a d-dimensional multisequence $\beta^{(m)}$ has a K-representing measure on $\mathbb{R}^m$, then $\beta^{(m)}$ has a finitely-atomic K-representing measure μ, with $\operatorname{card}\ \operatorname{supp}\ \le \dim \mathbb{R}[x_1, \ldots, x_d]$.*

Thus, we may regard solving TRMP (in particular, $d = 2,\ m = 2n$) as solving the system of the following polynomial equations: For some $\ell \le \dim \mathcal{P}_{2n}$,

$$\beta_{ij} = \sum_{k=1}^{\ell} \rho_k x_k^i y_k^j, \qquad i,\ j \in \mathbb{Z}_+,\ 0 \le i + j \le 2n.$$

Various Moment Problems

Lecture 2. Solutions to truncated moment problems

Operator Theory and Operator Algebras Winter School
(Mungyeong; December 20-23, 2017)

1 Truncated Real Moment Problems (Two-variable)

Given a **truncated real moment sequence** (of degree m), $\beta \equiv \beta^{(m)} = \{\beta_{00}, \beta_{10}, \beta_{01}, \ldots, \beta_{m,0}, \beta_{m-1,1}, \ldots, \beta_{1,m-1}, \beta_{0,m}\}$ with $\beta_{00} > 0$, the truncated real moment problem (**TRMP**) entails seeking necessary and sufficient conditions for the existence of a positive Borel measure μ supported in $\mathbb{R}^2$ such that

$$\beta_{ij} = \int x^i y^j \, d\mu \quad (i, j \in \mathbb{Z}_+, \ 0 \le i + j \le m).$$

Even Order Moment Problems. When $m = 2n$, R. Curto and L. Fialkow have made a great contribution to various moment problems in a series of papers (complete solutions were found for $m = 2, 4$).

Odd Order Moment Problems. When $m = 2n + 1$, a general solution is given by D. Kimsey in 2016. A complete solution to the cubic complex moment problem (when $m = 3$) was also given by D. Kimsey in 2014 and some cases of quintic moment problems (when $m = 5$) were solved by L. A. Fialkow in 2014.

FMP vs TMP

Theorem 1 [J. Stochel, 2001]
$\beta^{(\infty)}$ *has a representing measure on a closed set* $K \subseteq \mathbb{R}^d$
$\Longleftrightarrow$ *for each* m*,* $\beta^{(m)}$ *has a representing measure supported in* K*.*

This result says that in a sense TMP is more general than FMP.

2 Complex One-variable Moment Problems

- Given $\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0},$ with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij},$ the **truncated complex moment problem (TCMP)** entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \qquad (0 \leq i + j \leq 2n);$$

  μ is called a **representing measure** for γ.

- The **full** complex moment problem considers finding a representing measure for an **infinite** moment sequence $\gamma := \{\gamma_{ij}\}_{i,j \geq 0}$ with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$.

3 Moment Matrix

When $m = 2n$, we define the **(real) moment matrix** $M_d(n)$ of $\beta \equiv \beta^{(2n)}$ as

$$M_d(n) \equiv M_d(n)(\beta) := \left(\beta_{\,\mathbf{i}+\mathbf{j}}\right)_{\mathbf{i},\mathbf{j}\in\mathbb{Z}_+^d:\ |\mathbf{i}|,|\mathbf{j}|\leq n}.$$

R. Curto and L. Fialkow have used the “functional calculus” in the column space of $M_d(n)$; to introduce the functional calculus, we label the columns and rows of $M_d(n)$ with monomials $X^{\mathbf{i}} := X_1^{i_1}\cdots X_d^{i_d}$ in the degree-lexicographic order. Note that each block with the moments of the same order in $M_d(n)$ is Hankel and that $M_d(n)$ is symmetric. In addition, one can define a sesquilinear form: for $\mathbf{i}, j \in \mathbb{Z}_+^d$,

$$\langle X^{\mathbf{i}}, X^{\mathbf{j}}\rangle_{M_d(n)} := \langle M_d(n)\widehat{X^{\mathbf{i}}}, \widehat{X^{\mathbf{j}}}\rangle = \beta_{\mathbf{i}+\mathbf{j}},$$

where $\widehat{X^{\mathbf{i}}}$ is the column vector associated to the monomial $X^{\mathbf{i}}$.

We will put all the moments of the given sequence into the moment matrix and then label the columns with the lexicographical order; in particular, for $d = 2$:

$$\text{TCMP: } \mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, Z^3, \bar{Z}Z^2, \bar{Z}^2Z, \bar{Z}^3, \ldots$$
$$\text{TRMP: } \mathit{1}, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3, \ldots$$

Complex Moment Matrix (Block Toeplitz):

$$M(3)(\gamma) = \begin{array}{c} \begin{array}{c|cc|ccc|cccc} 1 & Z & \bar{Z} & Z^2 & \bar{Z}Z & \bar{Z}^2 & Z^3 & \bar{Z}Z^2 & \bar{Z}^2Z & \bar{Z}^3 \end{array} \\ \left[\begin{array}{c|cc|ccc|cccc} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \hline \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \hline \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \hline \gamma_{30} & \gamma_{31} & \gamma_{40} & \gamma_{32} & \gamma_{41} & \gamma_{50} & \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{60} \\ \gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{51} \\ \gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{42} \\ \gamma_{03} & \gamma_{04} & \gamma_{13} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{06} & \gamma_{15} & \gamma_{24} & \gamma_{33} \end{array}\right] \end{array}$$

Real Moment Matrix (Block Hankel):

$M_2(3)(\beta^{(6)})$ is written by

$$
\begin{array}{c}
\\ \mathit{1} \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^3 \\ X^2Y \\ XY^2 \\ Y^3
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
\mathit{1} & X & Y & X^2 & XY & Y^2 & X^3 & X^2Y & XY^2 & Y^3
\end{array} \\
\left(\begin{array}{c|cc|ccc|cccc}
\beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{03} \\
\hline
\beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{13} \\
\beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{04} \\
\hline
\beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{23} \\
\beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{32} \\
\beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} & \beta_{32} & \beta_{23} & \beta_{32} & \beta_{23} \\
\hline
\beta_{30} & \beta_{40} & \beta_{31} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{60} & \beta_{51} & \beta_{42} & \beta_{33} \\
\beta_{21} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{51} & \beta_{42} & \beta_{33} & \beta_{24} \\
\beta_{12} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{13} & \beta_{42} & \beta_{33} & \beta_{24} & \beta_{15} \\
\beta_{03} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{13} & \beta_{05} & \beta_{33} & \beta_{24} & \beta_{15} & \beta_{06}
\end{array}\right)
\end{array}.
$$

4 TCMP $\cong$ TRMP

Define:

- For $z = x + iy$, $\psi(x, y) := z \equiv x + iy$ and $\Psi(x, y) := (z, \bar{z})$.
- A map $L := \bigoplus_{k=0}^{n} L_k$, where

$$L_0 = (1), L_1 = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \text{and } L_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2i & 0 & -2i \\ -1 & 1 & -1 \end{pmatrix}.$$

We can easily check that L is invertible.

Note The columns in L_k correspond to monomials

$$z^k = (x + iy)^k, \qquad \bar{z}z^{k-1} = (x - iy)(x + iy)^{k-1}, \ldots,$$
$$\bar{z}^{k-1}z = (x - iy)^{k-1}(x + iy), \qquad \bar{z}^k = (x - iy)^k.$$

Proposition 2 [Curto-Fialkow, 2002]

Let $\begin{cases} M(n)(\gamma) \equiv \textit{moment matrix associated with a complex sequence } \gamma \\ M(n)(\beta) \equiv \textit{moment matrix associated with a real sequence } \beta \end{cases}$

and $M(n)(\gamma) = L^* M(n)(\beta) L$. *Then the following hold:*

(i) $M(n)(\gamma) \geq 0 \iff M(n)(\beta) \geq 0$

(ii) $\mathrm{rank}\ M(n)(\gamma) = \mathrm{rank}\ M(n)(\beta)$

(iii) $\mu_\beta = \mu_\gamma \circ \Psi$

(iv) $\psi($*supp* $\mu_\beta) =$ *supp* μ_γ.

(v) $M(n)(\gamma)$ *admits a* $\mathrm{rank}\ M(n)(\gamma)$*-atomic representing measure*
$\iff M(n)(\gamma)$ *admits a* $\mathrm{rank}\ M(n)(\beta)$*-atomic representing measure.*

(vi) For $p \in \mathcal{P}_n$, $p(Z, \bar{Z}) = L^*((p \circ \Psi)(X, Y))$.

5 Main Approaches to Solve TMP

- Positive Extension of Riesz Functional;

  (Rieze-Haviland Theorem; need to define proper higher order moments)

- Rank-preserving Positive Moment Matrix Extension;

  (Flat Extension Theorem; need to define proper higher order moments)

- Consistency for Extremal Cases;

  (Need to find a representation theorem for certain polynomials)

- Rank-one Decomposition.

All the upcoming solutions to TMPs in Lecture 2 were derived through one of these approaches. We will take a look at them in detail.

6 Necessary Conditions

Although most the following arguments are valid for $M_d(n)$, $d \geq 3$, we focus on bivariate real moment problems ($d = 2$) in the sequel; so let $M(n) \equiv M(n)(\beta) \equiv M_2(n)(\beta^{(2n)})$.

6.1 Basic Positivity Condition

For $p(x, y) = \sum_{i,j} x^i y^j \in \mathcal{P}_n$,

$$M(n) \geq 0 \iff 0 \leq \int |p|^2 \, d\mu = \sum_{i,j,k,l} a_{ij} a_{kl} \int x^{i+l} y^{j+k} \, d\mu$$
$$= \sum_{i,j,k,l} a_{ij} a_{kl} \beta_{i+l} \beta_{j+k}$$

How to check: Use the nested determinant test or see if all eigenvalues of the moment matrix are nonnegative.

6.2 Recursively Generated

Real case If $M(n)$ satisfies the condition that

$$p, q, pq \in \mathcal{P}_n,\ p(X,Y) = \mathbf{0} \Longrightarrow (pq)(X,Y) = \mathbf{0},$$

then $\beta^{(2n)}$ or $M(n)$ is said to be **recursively generated.** It gives rigidity to $M(n)$ and its extensions; for, once a column relation arises, then all its multiples in the polynomial level must appear as column relations in $M(n)$ and its extensions.

Complex case For $p \in \mathcal{P}_n$, $p(z,\bar{z}) \equiv \sum_{0 \le i+j \le n} a_{ij} \bar{z}^i z^j$ define $p(Z,\bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j$. A complex moment sequence $\gamma^{(2n)}$ is said to be **recursively generated**:

If $p, q, pq \in \mathcal{P}_n,$ and $p(Z,\bar{Z}) = \mathbf{0}$, then $(pq)(Z,\bar{Z}) = \mathbf{0}$.

Importance of a column relation

Proposition 3 *If there exists a representing measure μ for $\beta^{(2n)}$, then*

$$p(X,Y) = \boldsymbol{0} \iff \text{supp } \mu \subseteq \mathcal{Z}(p).$$

Example 4 *Consider a recursively generated $M(2)$ with a column relation $X =$ 1 (that is, $\beta_{10} = \beta_{00}$, $\beta_{20} = \beta_{10}$, $\beta_{11} = \beta_{01}$, $\beta_{30} = \beta_{20}$, $\beta_{21} = \beta_{11}$, and $\beta_{12} := \beta_{02}$):*

$$
\begin{array}{cccccc}
\mathit{1} & X & Y & X^2 & XY & Y^2 \\
\end{array}
$$
$$
\begin{pmatrix}
\boldsymbol{\beta}_{00} & \boldsymbol{\beta}_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\
\boldsymbol{\beta}_{10} & \boldsymbol{\beta}_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\
\boldsymbol{\beta}_{01} & \boldsymbol{\beta}_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\
\boldsymbol{\beta}_{20} & \boldsymbol{\beta}_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\
\boldsymbol{\beta}_{11} & \boldsymbol{\beta}_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\
\boldsymbol{\beta}_{02} & \boldsymbol{\beta}_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04}
\end{pmatrix}
$$

To have a representing measure, $M(2)$ must have additional column relations:

$$X^2 = \mathit{1} \ (\textit{from } x \cdot x = 1 \cdot x = 1) \quad \textit{and} \quad XY = Y \ (\textit{from } x \cdot y = 1 \cdot y = y);$$

that is, the higher-order moments must be fixed as follows:

$$
\begin{gathered}
\beta_{20} := \beta_{00},\ \beta_{30} := \beta_{10},\ \beta_{21} := \beta_{01},\ \beta_{40} := \beta_{20},\ \beta_{31} := \beta_{11},\ \beta_{22} := \beta_{02}, \\
\beta_{11} := \beta_{01},\ \beta_{21} := \beta_{11},\ \beta_{12} := \beta_{02},\ \beta_{31} := \beta_{21},\ \beta_{22} := \beta_{12},\ \beta_{13} := \beta_{03}.
\end{gathered}
$$

6.3 Variety Condition

The **algebraic variety** of $\beta \equiv \beta^{(2n)}$ or $M(n)(\beta)$ is defined by

$$\mathcal{V} \equiv \mathcal{V}(\beta) \equiv \mathcal{V}(M(n)) = \bigcap_{p(X,Y)=\mathbf{0},\ \deg p \le n} \mathcal{Z}(p),$$

where $\mathcal{Z}(p) = \{(x, y) : p(x, y) = 0\}$.

If μ is a representing measure for $M(n)$, then the inequality

$$\text{rank}\, M(n) \le \text{card supp } \mu \le \text{card}\, \mathcal{V}(M(n))$$

is said to be the **variety condition.**

Example 5 *Consider $M(2)(\beta^{(4)})$ generated by a 4-atomic representing measure $\mu = \delta_{(-2,4)} + \delta_{(-1,1)} + \delta_{(1,1)} + \delta_{(2,4)}$. Thus,*

$$\beta_{ij} = \int x^i y^j \, d\mu = (-2)^i (4)^j + (-1)^i (1)^j + (1)^i (1)^j + (2)^i (4)^j.$$

We see:

$$M(2) = \begin{pmatrix} 4 & 0 & 10 & 10 & 0 & 34 \\ 0 & 10 & 0 & 0 & 34 & 0 \\ 10 & 0 & 34 & 34 & 0 & 130 \\ 10 & 0 & 34 & 34 & 0 & 130 \\ 0 & 34 & 0 & 0 & 130 & 0 \\ 34 & 0 & 130 & 130 & 0 & 514 \end{pmatrix} \xrightarrow{\textit{row red.}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$M(2)$ *has two column relations $X^2 = Y$ and $Y^2 = -4 \cdot \mathit{1} + 5Y$. Now solve the system of polynomial equations $x^2 = y$ and $y^2 = -4 + 5y$; we find 4 zeros; that is, $\mathcal{V}(M(2)) = \{(-2,4), (-1,1), (1,1), (2,4)\}$. In this case,* $\operatorname{rank} M(n) = \operatorname{card}\ \operatorname{supp}\ \mu = \operatorname{card} \mathcal{V}(M(n)) = 4$ *and the variety condition clearly holds.*

Example 6 [Complex Case] *Consider a psd* $M(3)(\gamma^{(6)})$ *with 3 column relations*

$$p(Z,\bar{Z}) := Z^3 - 2iZ - \frac{5}{4}\bar{Z} = \mathbf{0},\ \overline{p(Z,\bar{Z})} = \mathbf{0},\ \bar{Z}^2 Z + i\bar{Z}Z^2 - i\frac{5}{4}Z - \frac{5}{4}\bar{Z} = \mathbf{0}.$$

$$M(3)(\gamma^{(6)}) = \begin{pmatrix} 224 & 0 & 0 & 176i & 208 & -176i & 0 & 0 & 0 & 0 \\ 0 & 208 & -176i & 0 & 0 & 0 & 196i & 236 & -196i & -92 \\ 0 & 176i & 208 & 0 & 0 & 0 & -92 & 196i & 236 & -196i \\ -176i & 0 & 0 & 236 & -196i & -92 & 0 & 0 & 0 & 0 \\ 208 & 0 & 0 & 196i & 236 & -196i & 0 & 0 & 0 & 0 \\ 176i & 0 & 0 & -92 & 196i & 236 & 0 & 0 & 0 & 0 \\ 0 & -196i & -92 & 0 & 0 & 0 & 277 & -227i & -97 & -61i \\ 0 & 236 & -196i & 0 & 0 & 0 & 227i & 277 & -227i & -97 \\ 0 & 196i & 236 & 0 & 0 & 0 & -97 & 227i & 277 & -227i \\ 0 & -92 & 196i & 0 & 0 & 0 & 61i & -97 & 227i & 277 \end{pmatrix}$$

The algebraic variety is

$$\mathcal{V} = \Big\{ w_0 := 0,\ w_1 := \sqrt{6}/4 + \sqrt{6}/4i,\ w_2 := -\sqrt{6}/4 - \sqrt{6}/4i,$$
$$w_3 := 1 + (1/2)i,\ w_4 := 1/2 + i,\ w_5 := -1 - (1/2)i,\ w_6 := -1/2 - i\big\}.$$

Thus, we know $\mathrm{rank}\ M(3) = \mathrm{card}\ \mathcal{V} = 7$.

6.4 Weak Consistency and Consistency

- **(Riesz Functional)** $\Lambda_\gamma(\bar{z}^i z^j) := \gamma_{ij}$ or $\Lambda_\beta(x^i y^j) := \beta_{ij}$
  (Riesz functional is linear and preserves conjugate.)

  Two more necessary conditions:

- **(Weak Consistency)** $p \in \mathcal{P}_n$, $p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0$.

- **(Consistency)** $p \in \mathcal{P}_{2n}$, $p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0$.

Note $\beta^{(2n)}$ is consistent $\Longrightarrow$ $\beta^{(2n)}$ is recursively generated.

Lemma 7 [Curto-Fialkow-Möller, 2008] *A sequence has an interpolating measure if and only if the sequence is consistent on a certain subset of* $\mathbb{R}^d$.

List of Necessary Conditions (Review)

- Positivity: $M(n) \geq 0$
- Recursively Generated: $p, q, pq \in \mathcal{P}_n$, $p(X,Y) = 0 \Longrightarrow (pq)(X,Y) = 0.$
- Variety Condition: $\operatorname{rank} M(n) \leq \operatorname{card} \mathcal{V}(M(n))$
- Weak Consistency: $p \in \mathcal{P}_n$, $p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0$.
- Consistency: $p \in \mathcal{P}_{2n}$, $p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0$.

All these conditions are not sufficient for $n \geq 3$!

Thus, we need to discover another necessary conditions to study higher degree moment problems.

7 The First Solution: Flat Extension Theorem

The moment sequence $\beta^{(2n)}$ or $M(n)$ is said to be **flat** if

$$\mathrm{rank}\ M(n) = \mathrm{rank}\ M(n-1).$$

(This case subsumes all previous results for the Hamburger, Stieltjes, Hausdorff, and Toeplitz TMP's.)

Theorem 8 [Curto-Fialkow, Mem. AMS, 1996]
If a positive $M(n)$ is flat, then it has a unique $\mathrm{rank}\ M(n)$-atomic representing measure.

Theorem 9 [Flat Extension Theorem; Curto-Fialkow,1996]
If $\beta^{(2n)}$ has a $\mathrm{rank}\ M(n)$-atomic representing measure if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.

7.1 Key Facts in the Flat Extension Theorem

We need to define:

(i) $\varphi : \mathbb{C}[z, \bar{z}] \to \mathcal{C}_{M(n)(\gamma)}$ defined by $\varphi(\bar{z}^i z^j) := \bar{Z}^i Z^j$

(ii) $\mathcal{N} := \{p : \langle M\hat{p}, \hat{p}\rangle = 0\}\,,\ \ker\varphi = \{p : \varphi(p) = \mathbf{0}\} \Longrightarrow \ker\varphi \subseteq \mathcal{N}$

(iii) $\langle\cdot,\cdot\rangle_M : \mathbb{C}[z, \bar{z}] \to \mathbb{C}$ defined by $\langle p, q\rangle_M(n) := \langle M\hat{p}, \hat{q}\rangle$

(iv) $\langle\cdot,\cdot\rangle : \mathbb{C}[z, \bar{z}]/\mathcal{N} \to \mathbb{C}$ defined by $\langle f + \mathcal{N}, g + \mathcal{N}\rangle := \langle f, g\rangle_M = \langle M\hat{f}, \hat{g}\rangle$

(This sesquilinear form is well-defined and positive semi-definite.)

($\langle f + \mathcal{N}, f + \mathcal{N}\rangle = 0 \implies f \in \mathcal{N}$)

Lemma 10 $M(\infty)$*: infinite positive moment matrix* $\implies \ker\varphi = \mathcal{N}$.

Lemma 11 $M(\infty)$*: infinite positive moment matrix* $\implies \ker\varphi$ *is an ideal.*

By the preceding two lemmas, $\mathcal{N}$ is an ideal of $\mathbb{C}[z,\bar{z}]$, and hence we can define a multiplication operator

$$M_z : \mathbb{C}[z,\bar{z}]/\mathcal{N} \to \mathbb{C}[z,\bar{z}]/\mathcal{N} \qquad \text{by } M_z(f) := zf.$$

Lemma 12 *Let* $M(\infty)$ *be a positive infinite moment matrix of finite-rank. Then the following hold:*

(i) $\mathbb{C}[z,\bar{z}]/\mathcal{N}$ *is a finite dimensional Hilbert space.*

(ii) $\dim \mathbb{C}[z,\bar{z}]/\mathcal{N} = \operatorname{rank}\ M$*.*

(iii) M_z *is a normal operator.*

Proposition 13 $M(\infty)$*: infinite moment matrix with a representing measure* μ $\Longrightarrow \text{card supp } \mu = \text{rank } M(\infty)$*.*

Theorem 14 $M(\infty)$*: infinite moment matrix of finite-rank* $\Longrightarrow M(\infty)$ *has a unique representing measure, which is* $\text{rank } M$*-atomic.*

Proof. Since M_z is normal, it follows from the Spectral Theorem that $C^*(M_z) \cong C(\sigma(M_z))$. The operator $M_z : \mathbb{C}[z, \bar{z}] \to \mathbb{C}[z, \bar{z}]$ leaves $\mathcal{N}$ invariant, and $M_z : \mathbb{C}[z, \bar{z}]/\mathcal{N} \to \mathbb{C}[z, \bar{z}]/\mathcal{N}$ is normal, that is, $M_z^* M_z = M_z M_z^*$. The equivalence of $C^*(M_z)$ and $C(\sigma(M_z))$ preserves the order, so that if a continuous function is nonnegative on $\sigma(M_z)$, then its associated element of the C^*-algebra will be positive. This element is called $f(M_z)$. When it acts on cosets $1 + \mathcal{N}$ it still remains positive, this time as an operator on the quotient space. Thus, whenever the continuous function f is nonnegative, $f(M_z)$ is actually a positive, self-adjoint operator, and therefore the inner product $\langle f(M_z)(1 + \mathcal{N}), 1 + \mathcal{N} \rangle$ is nonnegative.

This means that the linear functional

$$\eta(f) := \langle f(M_z)(1+\mathcal{N}), 1+\mathcal{N}\rangle, \qquad f \in C(\sigma(M_z))$$

is positive.

Thus, the Riesz Representation Theorem implies that there exists a positive Borel measure μ, with $\text{supp}\ \mu \subseteq \sigma(M_z)$, such that $\eta(f) = \int f d\mu$. Then

$$\begin{aligned}\int \bar{z}^i \bar{z}^j d\mu &= \eta(\bar{z}^i \bar{z}^j) = \langle M_z^{*i} M_z^j (1+\mathcal{N}), 1+\mathcal{N}\rangle \\ &= \langle z^j + \mathcal{N}, z^i + \mathcal{N}\rangle = \langle z^j, z^i\rangle_M = \gamma_{ij}.\end{aligned}$$

The desired representing measure is the scalar spectral measure μ of M_z, and $\text{supp}\ \mu = \sigma(M_z)$. In particular, $\text{card}\ \text{supp}\ \mu = \text{rank}\ M(n) = r$. ■

A proof of the Flat Extension Theorem is easily derived from Theorem 14.

The Flat Extension Theorem is valid for any dimension.

General Flat Extension Theorem

Moreover, an extended version of the Flat Extension Theorem says if $M(n)$ admits a positive extension $M(n+k)$ for some $k \in \mathbb{Z}_+$ that has a flat extension $M(n+k+1)$, then β has a $\operatorname{rank}\ M(n+k)$-atomic measure μ.

Let $\tau := \operatorname{rank}\ M(n+k)$. According to this flat extension theorem, the algebraic variety $\mathcal{V}(M(n+k+1))$ consists of exactly τ points, and hence we may write $\mathcal{V}(M(n+k)) = \{(x_1, y_1), \ldots, (x_\tau, y_\tau)\}$, which serves as the support of a measure. Solving a Vandermonde equation, we can find the densities of the measure and establish a concrete formula of the measure.

7.2 How to Build a Positive Extension

To build a flat extension moment matrix

$$M(n+1) = \begin{pmatrix} M(n) & B(n+1) \\ B(n+1)^* & C(n+1) \end{pmatrix},$$

we need to allow new moments (parmeters) $\beta_{n,0}, \beta_{n-1,1}, \ldots, \beta_{0,n}$ with keeping recursiveness and then check if $C(n+1)$ is Hankel.

A useful tool to build an extension is:

Theorem 15 [Smul'jan, 1959]

$$\tilde{A} := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \iff \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases}.$$

Moreover, $\mathrm{rank}\ \tilde{A} = \mathrm{rank}\ A \iff C = W^*AW.$

Example 16 *Consider a quadratic moment sequence:*

$$\beta^{(4)} : \{\beta_{ij}\} = \{5, 5, 14, 5, 14, 50\} \Longrightarrow M(1) = \begin{pmatrix} 5 & 5 & 14 \\ 5 & 5 & 14 \\ 14 & 14 & 50 \end{pmatrix}$$

Note that $M(1)$ has a column relation $X = 1$.

To build a flat $M(2)$, we impose on $M(2)$ to have $X^2 = 1$ and $XY = 1$:

$$\begin{pmatrix} M(1) & B(2) \end{pmatrix} = \begin{pmatrix} 5 & 5 & 14 & 5 & 14 & 50 \\ 5 & 5 & 14 & 5 & 14 & 50 \\ 14 & 14 & 50 & 14 & 50 & \beta_{03} \end{pmatrix}$$

We now find W such that $M(1)W = B(2)$ and get, for $k_1, k_2, k_3 \in \mathbb{R}$,

$$W = \begin{pmatrix} 1 - k_1 & -k_2 & (-7\,\beta_{03} - 27\,k_3 + 1250)/27 \\ k_1 & k_2 & k_3 \\ 0 & 1 & (5\beta_{03} - 700)/54 \end{pmatrix}.$$

We then evaluate $C(2) = W^* M(1) W$ *and get*

$$M(2) = \begin{pmatrix} 5 & 5 & 14 & 5 & 14 & 50 \\ 5 & 5 & 14 & 5 & 14 & 50 \\ 14 & 14 & 50 & 14 & 50 & \beta_{03} \\ 5 & 5 & 14 & 5 & 14 & 50 \\ 14 & 14 & 50 & 14 & 50 & \beta_{03} \\ 50 & 50 & \beta_{03} & 50 & \beta_{03} & 5(\beta_{03}^2 - 280\beta_{03} + 25000)/54 \end{pmatrix}$$

The column relations in $M(2)$ *are* $X = \mathit{1}, X^2 = \mathit{1}, XY = Y$, *and*

$$Y^2 = \frac{-7\beta_{03} - 27k + 1250}{27}\mathit{1} + kX + \frac{(5\beta_{03} - 700)}{54}Y$$

for some $k \in \mathbb{R}$*, that is,* $M(2)$ *is a flat extension of* $M(1)$ *for any* β_{03}*. In particular, if we take* $\beta_{03} = 194$*, then the algebraic variety* $\mathcal{V} = \{(1,1),(1,4)\}$*. To find the densities, solve the Vandermonde system:*

$$\begin{pmatrix} 1 & 1 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \beta_{00} \\ \beta_{01} \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \end{pmatrix}.$$

Thus, we get $\rho_1 = 2$*,* $\rho_2 = 3$ *and we may write a representing measure as* $\mu = 2\,\delta_{(1,1)} + 3\,\delta_{(1,4)}$*.*

8 Summary of Solutions of TMP

We now list the cases of soluble TMP as follows (remember that TCMP$\cong$TRMP):

Complex Cases

(i) TCMP is of **flat data** type, i.e., $\text{rank} \;\; M(n) = \text{rank} \;\; M(n-1)$ (this case subsumes all previous results for the Hamburger, Stieltjes, Hausdorff, and Toeplitz truncated moment problems [Curto-Fialkow, 1991]);

(ii) The column $\bar{Z}$ is a linear combination of the columns 1 and Z [Curto-Fialkow, 1996];

(iii) For some $k \leq [n/2] + 1,$ the analytic column Z^k is a linear combination of columns corresponding to monomials of lower degree [Curto-Fialkow, 1996];

(iv) The analytic columns of $M(n)$ are linearly dependent and span $\mathcal{C}_{M(n)}$, the column space of $M(n)$ [Curto-Fialkow, 1996];

Real Cases

(v) $M(n)$ is singular and subordinate to conics [Curto-Fialkow, 2000, 2002, 2004, 2005];

(vi) $M(n)$ with a finite algebraic variety [Fialkow, 2008];

(vii) $M(n)$ is *extremal*, that is, $\text{rank}\ M(n) = \text{card}\ \mathcal{V}$ [Curto-Fialkow-Möller, 2008];

(viii) $M(n)$ is *recursively determinate*, that is, if $M(n)$ has only column dependence relations of the form

$$X^n = p(X,Y)\ (p \in \mathcal{P}_{n-1});$$
$$Y^m = q(X,Y)\ (q \in \mathcal{P}_m,\ q \text{ has no } y^m \text{ term}, m \le n),$$

where $\mathcal{P}_k$ denotes the subspace of polynomials in $\mathbb{R}[x,y]$ whose degree is less than or equal to k. [Curto-Fialkow, 2013].

In the sequel, we will talk about solutions of lower degree TMP's.

9 Quadratic Moment Problems: $M_2(1)$

Theorem 17 [Curto-Fialkow, 1996]

For $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$ *and* $r :=$ *rank* $M(1)$*, TFAE:*

- γ *has a rep. meas.;*
- γ *has an* r*-atomic rep. meas.;*
- $M(1) \geq 0.$

In this case,

(i) $r = 1 \Longrightarrow \exists$ a unique representing measure;

(ii) $r = 2 \Longrightarrow \exists$ 2-atomic representing measures parameterized by a line;

(iii) $r = 3 \Longrightarrow \exists$ 3-atomic representing measures.

10 Multivariable Quadratic Moment Problems: $M_d(1)$ for any $d \geq 2$

Theorem 18 [Fialkow-Nie, 2010] *If $M_d(1)(\beta)$ is psd for any $d \in \mathbb{Z}_+$, then β has a $\operatorname{rank} M_d(1)$-atomic representing measure.*

Proof. Through the following lemma. ■

Proposition 19 [Sturm-Zhang, 2003]

Q, X: $n \times n$ symmetric matrices, $X \geq 0$, and rank $X = r$

$\Longrightarrow$ There are nonzero vectors $u_1, \ldots, u_r \in \mathbb{R}^m$ such that

$$X = u_1 u_1^T + \cdots + u_r u_r^T, \qquad u_1^T Q u_1 = \cdots = u_r^T Q u_r = \frac{Q \bullet X}{r},$$

where $\bullet$ is the Frobenius inner product, that is, $Q \bullet X = \operatorname{Tr}(QX^*)$.

11 Bivariate Quartic Moment Problems: $M_2(2)(\beta^{(4)})$

11.1 Invariance under Degree-one Transformations

For $a, b, c, d, e, f \in \mathbb{R}$ with $bf \neq ce$, let $\Psi(x, y) \equiv (\Psi_1(x, y), \Psi_2(x, y)) := (a + bx + cy, d + ex + fy)$ for $x, y \in \mathbb{R}$. A new moment sequence $\tilde{\beta}^{(2n)} : \left\{\tilde{\beta}_{ij}\right\}$ is constructed with the definition $\tilde{\beta}_{ij} := \Lambda_\beta(\Psi_1^i \Psi_2^j)$ $(0 \leq i + j \leq 2n)$. We can immediately check that $\Lambda_{\tilde{\beta}}(p) = \Lambda_\beta(p \circ \Psi)$ for every $p \in \mathcal{P}_n$.

Proposition 20 [Curto-Fialkow, 2002] *Let $M(n)$ and $\tilde{M}(n)$ be the moment matrices associated with β and $\tilde{\beta}$, and let $J\hat{p} := \widehat{p \circ \Psi}$ ($p \in \mathcal{P}_n$). Then the following are true:*

(i) $\tilde{M}(n) = J^* M(n) J$, $\tilde{M}(n) \geq 0 \Leftrightarrow M(n) \geq 0$ and $\operatorname{rank}\ \tilde{M}(n) = \operatorname{rank}\ M(n)$;

(ii) The formula $\mu = \tilde{\mu} \circ \Psi$ establishes a one-to-one correspondence between the sets of representing measures for β and $\tilde{\beta}$, which preserves measure class and cardinality of the support; moreover, $\varphi(\operatorname{supp}\ \mu) = \operatorname{supp}\ \tilde{\mu}$;

(iii) $M\,(n)$ admits a flat extension if and only if $\tilde{M}\,(n)$ admits a flat extension.

11.2 Singular Quartic Moment Problem

Five Canonical Conics

It is well known that the existence of representing measures for TMP is invariant under an invertible degree-one transformation.
Thus, a generic conic column relation in $M(n)$, under such an affine mapping, can be transformed into one of 5 canonical types:

$$X^2 + Y^2 = \mathit{1}, \;\; Y = X^2, \;\; XY = \mathit{1}, \;\; XY = \mathbf{0}, \;\; \text{and} \;\; X^2 = X.$$

Each case requires an independent result as listed below.

TMP on the Unit Circle

The moment problem for the unit circle $\mathbb{T}$ corresponds to the case in which $M(n)(\gamma^{(2n)})$the column relation $\bar{Z}Z = \mathit{1}$. This is the case in which there exists a sequence $\alpha : \alpha_{-2n}, \ldots, \alpha_0, \ldots, \alpha_{2n}$ such that $\gamma_{ij} = \alpha_{j-i}$ $(i,\ j \in \mathbb{Z}_+;\ 0 \leq i+j \leq 2n)$ (so that $\alpha_{-k} = \bar{\alpha}_k$). Let $T(2n)(\alpha)$ denote the Toeplitz matrix $(\alpha_{j-i})_{i,\,j=0,\ldots,2n}$. For example,

$$M(2) = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_{-1} & \alpha_2 & \alpha_0 & \alpha_{-2} \\ \alpha_{-1} & \alpha_0 & \alpha_{-2} & \alpha_1 & \alpha_{-1} & \alpha_{-3} \\ \alpha_1 & \alpha_2 & \alpha_0 & \alpha_3 & \alpha_1 & \alpha_{-1} \\ \alpha_{-2} & \alpha_{-1} & \alpha_{-3} & \alpha_2 & \alpha_{-2} & \alpha_{-4} \\ \alpha_0 & \alpha_1 & \alpha_{-1} & \alpha_0 & \alpha_0 & \alpha_{-2} \\ \alpha_2 & \alpha_3 & \alpha_1 & \alpha_4 & \alpha_2 & \alpha_0 \end{pmatrix} \longleftrightarrow TM(4) = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_{-1} & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_{-2} & \alpha_{-1} & \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_{-3} & \alpha_{-2} & \alpha_{-1} & \alpha_0 & \alpha_1 \\ \alpha_{-4} & \alpha_{-3} & \alpha_{-3} & \alpha_{-1} & \alpha_0 \end{pmatrix}$$

Thus, the result reduces the truncated $\mathbb{T}$-moment problem for γ to the **truncated trigonometric moment problem**:

$$\alpha_j = \int z^k \, d\mu \ (0 \leq k \leq 2n);\ \ \mu \geq 0;\ \operatorname{supp}\ \mu \subseteq \mathbb{T}.$$

Proposition 21 [Fialkow, 1995] *For $n \geq 0$, $M(n)(\gamma^{(2n)}) \geq 0 \iff T(2n) \geq 0$.*

Proposition 22 [Fialkow, 1995] *Suppose $M(n)(\gamma^{(2n)})$ has the column relation $\bar{Z}Z = 1$, that is, $\gamma_{ij} = \alpha_{j-i}$ ($i, j \in \mathbb{Z}_+; 0 \leq i+j \leq 2n$). Then TFAE:*

(i) $\gamma^{(2n)}$ *has a representing measure;*

(ii) $M(n)(\gamma^{(2n)}) \geq 0$*;*

(iii) $T(2n) \geq 0$*;*

(iv) α *has a* $\operatorname{rank} T(2n)$*-atomic representing measure;*

(v) $\gamma^{(2n)}$ *has a* $\operatorname{rank} M(n)$*-atomic representing measure.*

Parabolic TMP

Proposition 23 [Curto-Fialkow, 2004] *$M(n)(\beta)$ admits a representing measure supported in $y = x^2$ if and only if $M(n)$ is positive, recursively generated, satisfies $X^2 = Y$, and $\operatorname{rank}\ M(n) \le \operatorname{card}\ \mathcal{V}(M(n))$. In this case, $M(n)$ admits a flat extension $M(n+1)$ and β admits a $\operatorname{rank}\ M(n)$-atomic (minimal) representing measure.*

Hyperbolic TMP

Proposition 24 [Curto-Fialkow, 2005] *Let $Q(x, y) = 0$ be an hyperbola in the plane. A sequence $\beta \equiv \beta^{(2n)}$ has a representing measure supported in $Q(x, y) = 0$ if and only if $M(n)$ is positive, recursively generated, $Q(X, Y) = \mathbf{0}$ in $\mathcal{C}_{M(n)}$, and $\operatorname{rank}\ M(n) \le \operatorname{card}\ \mathcal{V}(M(n))$. In this case, $\operatorname{rank}\ M(n) \le 2n + 1$; if $\operatorname{rank}\ M(n) \le 2n$, then there is a $\operatorname{rank}\ M(n)$-atomic representing measure, while if $\operatorname{rank}\ M(n) = 2n + 1$, then there is a representing measure μ for which $2n + 1 \le \operatorname{card}\ \operatorname{supp}\ \mu \le 2n + 2$.*

Two Intersecting Lines

Proposition 25 [Curto-Fialkow, 2005] *Assume that $M(n)$ is positive, recursively generated, and satisfies $XY = \mathbf{0}$ and $\mathrm{rank}\ M(n) \leq \mathrm{card}\ \mathcal{V}(M(n))$. Then $\mathrm{rank}\ M(n) \leq 2n+1$; if $\mathrm{rank}\ M(n) \leq 2n$, then $M(n)$ admits a flat extension $M(n+1)$. If $\mathrm{rank}\ M(n) = 2n+1$, then $M(n)$ admits a positive, recursively generated extension $M(n+1)$, satisfying $2n+1 \leq \mathrm{rank}\ M(n+1) \leq 2n+2$, and $M(n+1)$ admits a flat extension $M(n+2)$ (so β admits a representing measure supported in $xy = 0$, with $2n+1 \leq \mathrm{card\ supp}\ \mu \leq 2n+2$).*

Two Parallel Lines

Theorem 26 [Fialkow, 2014] *Let $n \geq 2$. Suppose $\deg p(x,y) = 2$ and $\mathcal{Z}(p)$ consists of 2 parallel lines. Then $\beta \equiv \beta(2n)$ has a representing measure supported in $\mathcal{Z}(p)$ if and only if $M(n)$ is positive, recursively generated, satisfies the variety condition, and $p(X,Y) = \mathbf{0}$ in the column space of $M(n)$.*

We may also conclude that such $M(n)$ admits a flat extension, and hence that $\beta(2n)$ has a $\mathrm{rank}\ M(n)$-atomic representing measure.

11.3 Nonsingular Quartic Moment Problems: $M(2) > 0$

Theorem 27 [Fialkow-Nie, 2009] *If $M(2)(\beta) > 0$, then β has a representing measure.*

The proof is based on convex analysis and SOS.

Theorem 28 [Curto-Yoo, 2016] *If $M(2)(\beta) > 0$, then β has a 6-atomic representing measure.*

The proof is based on a rank-one decomposition.

Proof of Theorem 28. Normalize $M(2)$: Using a degree-one transformation, any positive definite $M(2)$ can be translated to

$$\widetilde{M(2)} := \begin{pmatrix} 1 & 0 & 0 & \tilde{\beta}_{20} & \tilde{\beta}_{11} & \tilde{\beta}_{02} \\ 0 & 1 & 0 & \tilde{\beta}_{30} & \tilde{\beta}_{21} & \tilde{\beta}_{12} \\ 0 & 0 & 1 & \tilde{\beta}_{21} & \tilde{\beta}_{12} & \tilde{\beta}_{03} \\ \tilde{\beta}_{20} & \tilde{\beta}_{30} & \tilde{\beta}_{21} & \tilde{\beta}_{40} & \tilde{\beta}_{31} & \tilde{\beta}_{22} \\ \tilde{\beta}_{11} & \tilde{\beta}_{21} & \tilde{\beta}_{12} & \tilde{\beta}_{31} & \tilde{\beta}_{22} & \tilde{\beta}_{13} \\ \tilde{\beta}_{02} & \tilde{\beta}_{12} & \tilde{\beta}_{03} & \tilde{\beta}_{22} & \tilde{\beta}_{13} & \tilde{\beta}_{04} \end{pmatrix}.$$

Rank-one decomposition: Let m_{11} be the $(1,1)$ entry in the positive definite matrix $\widetilde{M(2)}^{-1}$ and let $u := \frac{\det \widetilde{M(2)}}{m_{11}}$. Then $u > 0$ and we may write

$$\widetilde{M(2)} = \begin{pmatrix} 1-u & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & \beta_{30} & \beta_{21} & \beta_{12} \\ 0 & 0 & 1 & \beta_{21} & \beta_{12} & \beta_{03} \\ 1 & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ 0 & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ 1 & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix} + \begin{pmatrix} u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \equiv M_c + M_o$$

M_c has a representing measure by the solution of singular quartic moment problems and M_o also has the representing measure $\delta_{(0,0)}$. ■

12 Classification of Cubics for Singular Sextic Moment Problems: $M_2(3)(\beta^{(6)})$

Cubics are classified into 78 species under affine transformations.
(Too many for the study of TMP!)

Classification of Irreducible Cubics: according to singularity and the existence of a flex (i.e., a generalized inflection point).

Any irreducible cubic can be transformed into one of the form:

$$y^2 = x^3 + fx^2 + gx + h \tag{12.1}$$

for some $f, g, h \in \mathbb{R}$.

However, this classification is not valid for the purpose of TMP.

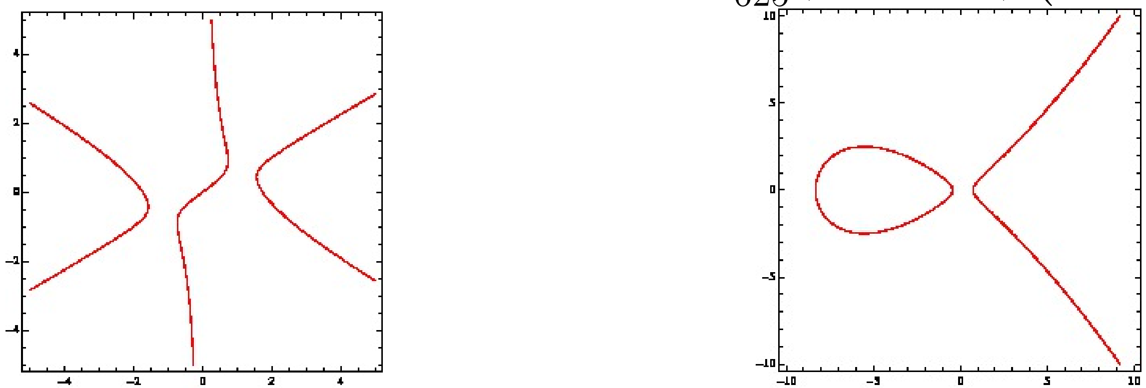
Theorem 29 *A nonsingular, irreducible cubic has a flex if and only if it can be transformed into*

$$y^2 = x(x-1)(x-w) \qquad \textit{and} \qquad y^2 = x(x^2+kx+1) \tag{12.2}$$

for $w > 1$ *and* $-2 < k < 2$*.*

For example, we will study TCMP associated to a specific complex cubic $z^3 = itz + u\bar{z}$, whose real part $-ux + x^3 + ty - 3xy^2 = 0$. For $t = 4$ and $u = 3$,

Figure 1: $-3x + x^3 + 4y - 3xy^2 = 0 \to y^2 = \frac{1}{625}(-3+4x)\left(39+96x+11x^2\right)$.



The two cubics are equivalent but not in the scope of TMP.

13 Extremal Moment Problems

We say that $\beta^{(2n)}$ or $M(n)(\beta^{(2n)})$ are **extremal** if $\mathrm{rank}\ M(n) = \mathrm{card}\ \mathcal{V}(M(n))$.

Theorem 30 [Curto-Fialkow-Möller, 2008]

For an extremal $\beta \equiv \beta^{(2n)}$, TFAE:

(i) β has a representing measure;

(ii) β has a unique representing measure, which is rank $M(n)$-atomic;

(iii) $M(n) \geq 0$ and β is consistent.

Question *Suppose the following:*

- $M(n)(\beta)$ *is singular;*
- $M(n)$ *is positive;*
- β *is consistent;*
- $r \leq v$*.*

Then does β *admit a representing measure?*

Fialkow[2010] found a psd $M(3)$ such that

$\operatorname{rank} M(3) = 9$ with a column relation $Y = X^3$,
CONSISTENT but $M(3)$ does not have a measure.

Thus, we know Consistency is NOT sufficient!

14 Sextic Moment Problems

14.1 Cubic Harmonic Column Relations

We first want to study $M(3) \equiv M(3)(\gamma^{(6)})$ with a column relation $Z^3 = p(Z, \bar{Z})$, $\deg p \leq 2$. By the structure of the moment matrix, $M(3)$ must have another column relation $\bar{Z}^3 = \bar{p}(Z, \bar{Z})$. Thus, $\text{rank } M(3) \leq 8$.

However, it was shown that cubic harmonic polynomials $q(z, \bar{z}) := f(z) - \overline{z}$, with $\deg\ q = 3$ have seven or fewer zeros. To satisfy the variety condition, $M(3)$ must have more column relations.

Theorem 31 [Khavinson-Swiatek, Wilmhurst, Sarason]
If $\deg\ p = n \geq 1$*, then*

$$\text{card } \mathcal{Z}(f(z) - \bar{z}) \leq 3n - 2.$$

In the case when $\deg f(z) = 3$, we have $\text{card } \mathcal{Z}(f(z) - \overline{z}) \leq 7$.

How to Choose a Good Cubic Harmonic Cubic Column Relations

Using a degree-one transformation $w = z + b/3$, we may ignore the quadratic term:

$$z^3 + bz^2 + cz + d \Longrightarrow w^3 + \tilde{c}w + \tilde{d}$$

WLOG, we can always assume that the quadratic term in the analytic piece is absent. That is, it suffices to study column relations like $Z^3 = A1 + BZ + C\bar{Z}$, where $A, B, C \in \mathbb{C}$.

Use the symmetry of the cubic: Let K be the zero set of $z^3 = A + Bz + C\bar{z}$. To have more points in K, we impose symmetry on K; we want to satisfy the conditions for symmetry:

(i) About the line $x = 0$ (equivalently, $z \in K \Longrightarrow -z \in K$): $A = 0 \Longrightarrow 0 \in K$.
(ii) About the line $y = x$ (equivalently, $z = i\bar{z}$): $B \in i\mathbb{R} \Longrightarrow C \in \mathbb{R}$.

Finally, we have a cubic column relation of the form $Z^3 = itZ + u\bar{Z}$, with $t, u \in \mathbb{R}$. Under these symmetries, if we can find only two nonzero points, one on the lines $y = x$ or $y = -x$ and the other outside that line, then we will have seven points in the zero set.

Variety Condition

Having the maximum number of zeros in $\mathcal{Z}(q_7)$ is crucial for us to solve a sextic truncated moment problem with a column relation of the form $q_7(Z, \bar{Z}) = \mathbf{0}$.

Since a singular moment problems is well understood, we are only interested in the cases when $M(2) > 0$.

Let $r_2 :=$ rank $M(2)$, $r_3 :=$ rank $M(3)$ and $v_3 :=$ card $\mathcal{V}(M(3))$. Then the possible cases when $r_2 = 6$ are:

r_2	r_3	v_3	
6	6	6 or 7	$M(3)$ is a flat extension of $M(2)$
6	7	6 or less than	No representing measure
6	7	7	Extremal

The case $(r_2, r_3, v_3) = (6, 6, 7)$ does not seem to be interesting but we should mention that it cannot happen.

The Solution to Sextic MP with the Cubic Harmonic Column Relation

Theorem 32 [Curto-Yoo, 2014] *Let $M(3) \geq 0$, with $M(2) > 0$ and $q_7(Z, \bar{Z}) := Z^3 - itZ - u\bar{Z} = 0$ ($0 < u < |t| < 2u$). Then TFAE:*

(i) There exists a 7-atomic representing measure for $M(3)$;

(ii) $q_{LC}(Z, \bar{Z}) := \bar{Z}^2 Z + i\bar{Z}Z^2 - iuZ - u\bar{Z} = 0$;

(iii) $\Lambda(q_{LC}) = 0$ *and* $\Lambda(zq_{LC}) = 0$;

(iv) $\mathrm{Re}\ \gamma_{12} - \mathrm{Im}\ \gamma_{12} = u(\mathrm{Re}\ \gamma_{01} - \mathrm{Im}\ \gamma_{01})$ *and* $\gamma_{22} = (t+u)\gamma_{11} - 2u\ \mathrm{Im}\ \gamma_{02}$*.*

Proof. Note that $|\mathcal{Z}(q_7)| = 7$ on the union of cones, $0 < u < |t| < 2u$.
($\Longrightarrow$) Easy.
($\Longleftarrow$) Checking consistency via the following Proposition:

Proposition 33 [Representation of Polynomials]
Let $\mathcal{Q}_6 := \{p \in \mathcal{P}_6 : p|_{\mathcal{Z}(q_7)} \equiv 0\}$ and let $\mathcal{I} := \{p \in \mathcal{P}_6 : p = fq_7 + g\bar{q}_7 + hq_{LC}$ for some $f, g, h \in \mathcal{P}_3\}$. Then $\mathcal{Q}_6 = \mathcal{I}$.

Another Cubic Harmonic Column Relation

Theorem 34 [Curto-Yoo, 2014]
Let $M(3) \geq 0$*, with* $M(2) > 0$*.* $M(3)$ *satisfies the column relation* $W^3 = 2\alpha W - \beta \overline{W}$ *for* $0 < \alpha < |\beta| < 2\alpha$*. Then TFAE:*

(i) There exists a representing measure for $M(3)$*;*

(ii) $\Lambda(\hat{q}_{LC}) = 0$ *and* $\Lambda(w\hat{q}_{LC}) = 0$*, where* $\hat{q}_{LC}(w, \bar{w}) = \overline{w}^2 w - \overline{w} w^2 + \beta w - \beta \overline{w}$*.*

Proof. Using a degree one transformation $\varphi(z, \bar{z}) = (1+i)\bar{z}$, we have $p(W, \overline{W}) = J^*(p \circ \Phi)(Z, \bar{Z})$. ∎

Example 35 *Consider again* $M(3)$ *in Example 6:*

$$M(3) = \begin{pmatrix} 224 & 0 & 0 & 176i & 208 & -176i & 0 & 0 & 0 & 0 \\ 0 & 208 & -176i & 0 & 0 & 0 & 196i & 236 & -196i & -92 \\ 0 & 176i & 208 & 0 & 0 & 0 & -92 & 196i & 236 & -196i \\ -176i & 0 & 0 & 236 & -196i & -92 & 0 & 0 & 0 & 0 \\ 208 & 0 & 0 & 196i & 236 & -196i & 0 & 0 & 0 & 0 \\ 176i & 0 & 0 & -92 & 196i & 236 & 0 & 0 & 0 & 0 \\ 0 & -196i & -92 & 0 & 0 & 0 & 277 & -227i & -97 & -61i \\ 0 & 236 & -196i & 0 & 0 & 0 & 227i & 277 & -227i & -97 \\ 0 & 196i & 236 & 0 & 0 & 0 & -97 & 227i & 277 & -227i \\ 0 & -92 & 196i & 0 & 0 & 0 & 61i & -97 & 227i & 277 \end{pmatrix}$$

$M(3)$ is psd and has three column relations

$$q_7(z,\bar{z}) = z^3 - 2iz - \frac{5}{4}\bar{z} = 0, \qquad q_{LC}(z,\bar{z}) := (\bar{z}+iz)\left(\bar{z}z - \frac{5}{4}\right) = 0,$$

and $\bar{q}_7(z,\bar{z}) = 0$. As seen earlier, rank $M(3) =$ card $\mathcal{V} = 7$. Since $M(3)$ satisfies the conditions in Theorem 34, it admits a 7-atomic representing measure.

The following $\tilde{M}(3)$ is obtained by the degree-one transformation $\varphi(z,\bar{z}) = (1+i)\bar{z}$ from $M(3)$ in the above:

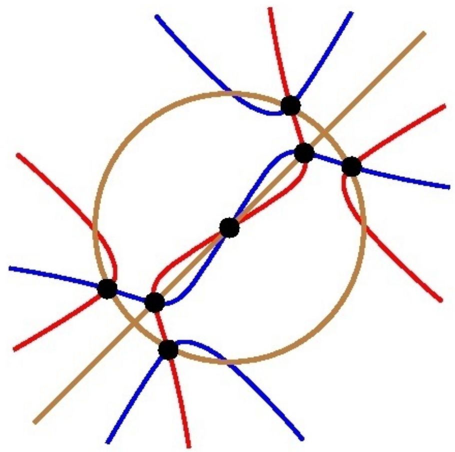
$$\tilde{M}(3) = \begin{pmatrix} 28 & 0 & 0 & 44 & 52 & 44 & 0 & 0 & 0 & 0 \\ 0 & 52 & 44 & 0 & 0 & 0 & 98 & 118 & 98 & 46 \\ 0 & 44 & 52 & 0 & 0 & 0 & 46 & 98 & 118 & 98 \\ 44 & 0 & 0 & 118 & 98 & 46 & 0 & 0 & 0 & 0 \\ 52 & 0 & 0 & 98 & 118 & 98 & 0 & 0 & 0 & 0 \\ 44 & 0 & 0 & 46 & 98 & 118 & 0 & 0 & 0 & 0 \\ 0 & 98 & 46 & 0 & 0 & 0 & 277 & 227 & 97 & -61 \\ 0 & 118 & 98 & 0 & 0 & 0 & 227 & 277 & 227 & 97 \\ 0 & 98 & 118 & 0 & 0 & 0 & 97 & 227 & 277 & 227 \\ 0 & 46 & 98 & 0 & 0 & 0 & -61 & 97 & 227 & 277 \end{pmatrix}$$

$\tilde{M}(3)$ *has three column relations*

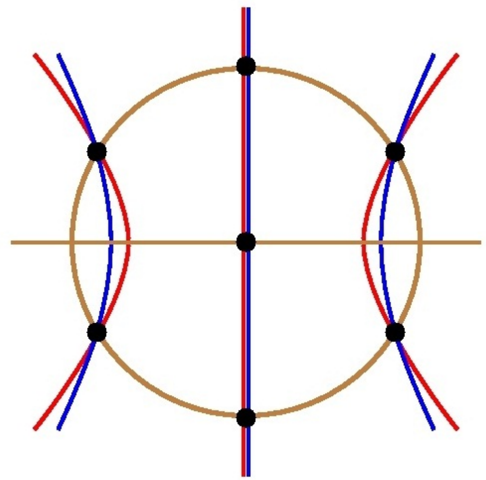
$$\tilde{q}_7(z,\bar{z}) \;=\; z^3 - 4z + \frac{5}{2}\bar{z} = 0,$$
$$\tilde{q}_{LC}(z,\bar{z}) \;=\; \frac{5}{2}z - \frac{5}{2}\bar{z} - \bar{z}z^2 + \bar{z}^2 z = 0, \quad \textit{and} \;\; \bar{\tilde{q}}_7(z,\bar{z}) = 0.$$

It also has a 7-atomic representing measure.

In the complex plane, we show the varieties of $M(3)$ *and* $\tilde{M}(3)$*:*



$\mathcal{V}(M(3))$



$\mathcal{V}(\tilde{M}(3))$

14.2 Classification of (Real) Sextic Moment Problems

We are mostly interested in $M(3)$ with an invertible $M(2)$-block since a complete solution to quartic moment problems was archived recently. So we assume $M(3) \geq 0$ and $M(2) > 0$. Let $r_3 := \mathrm{rank}\ M(3)$ and $v_3 := \mathrm{card}\ \mathcal{V}(M(3))$.

r_3	v_3	Eg. with/without a measure	Solutions by
7	7	known/impossible	Curto-Yoo, Fialkow
7	8	unknown/unknown	Curto-Yoo, Fialkow
7	9	unknown/unknown	Curto-Yoo, Fialkow
7	∞	known/unknown	Curto-Yoo
8	8	known/unknown	Curto-Yoo, Fialkow
8	9	known/known	Curto-Yoo, Fialkow
8	∞	known/unknown	Curto-Yoo
9	∞	known/known	Fialkow, Yoo (particular cases)
10	∞	known/known	N/A

15 Non-extremal Sextic Moment Problems

15.1 Rank-one Decompositions.

Any rank-one positive matrix must be of the form $\mathbf{x}\mathbf{x}^*$ for some nonzero vector $\mathbf{x} \in \mathbb{C}^n$. In addition, a positive matrix A can be written as a sum:

$$A = \sum_{i=1}^{k} \mathbf{x}_i \mathbf{x}_i^*,$$

for some nonzero vectors $\mathbf{x}_i \in \mathbb{C}^n$ for $i = 1, \ldots, k$.

This representation is not unique and the minimal k is known to be the rank of A.

Notations

(i) $\mathbf{v}(a,b) := \begin{pmatrix} 1 & a & b & a^2 & ab & b^2 & \cdots & a^n & a^{n-1}b & \cdots & ab^{n-1} & b^n \end{pmatrix}$

(ii) $P(a,b) := \mathbf{v}(a,b)^T \mathbf{v}(a,b)$

For example, if $n = 2$, then

$$P(a,b) = \begin{pmatrix} 1 & a & b & a^2 & ab & b^2 \\ a & a^2 & ab & a^3 & a^2b & ab^2 \\ b & ab & b^2 & a^2b & ab^2 & b^3 \\ a^2 & a^3 & a^2b & a^4 & a^3b & a^2b^2 \\ ab & a^2b & ab^2 & a^3b & a^2b^2 & ab^3 \\ b^2 & ab^2 & b^3 & a^2b^2 & ab^3 & b^4 \end{pmatrix} \longleftrightarrow \text{moment matrix generated by } \delta_{(a,b)}$$

If $M(n)(\beta)$ has a ℓ-atomic representing measure, then we may write

$$M(n) = \sum_{k=1}^{\ell} \rho_k P(x_k, y_k),$$

where $\rho_k > 0$ and $(x_k, y_k) \in \mathcal{V}(M(n))$ for $k = 1, \ldots, \ell \leq \dim \mathcal{P}_{2n}$.

15.2 Solution to Non-extremal Sextic Moment Problems

The Case: $\text{rank } M(3) = 7$**,** $\text{card } \mathcal{V}(M(3)) > 7$

Theorem 36 [Curto-Yoo, 2105] *Let $\mathcal{V} \equiv \mathcal{V}(M(3))$ be the algebraic variety of $M(3)\left(\beta^{(6)}\right)$ and let $v := \text{card } \mathcal{V}$. If $M(3)$ is consistent, psd with a nonsingular $M(2)$, $\text{rank } M(3) = 7$, and $v \geq 8$ ($v = +\infty$ possible), then $\beta^{(6)}$ has a 7-atomic measure.*

Proof. Key ideas:

- There is a point $(a, b) \in \mathcal{V}$ such that no conic can contain all the points in $\mathcal{V} - \{(a, b)\}$.

- Set (a, b) as the point (x_j, y_j) and then consider a vector **v** $= \left(1, a, b, a^2, ab, b^2,\right.$ $\left. a^3, a^2b, ab^2, b^3\right)^T$. Notice that $\mathbf{v}\mathbf{v}^T$ is a rank one moment matrix with the measure $\delta_{(a,b)}$ and we may write

$$\widetilde{M(3)} = M(3) - \rho\, \mathbf{v}\mathbf{v}^T,$$

for some $\rho := \det\left(M(3)_{\mathcal{B}}\right) / \left(\lambda \det\left([M(3)_{\mathcal{B}}]_{\{2,3,\ldots,7\}}\right)\right) > 0$.

- $\widetilde{M(3)}$ has exactly 6 positive eigenvalues along with zero whose multiplicity is 4. In other words, $\widetilde{M(3)}$ is positive semidefinite.

  Finally, we observe that $\widetilde{M(2)}$ is positive definite, and so $\widetilde{M(3)}$ is flat, which means that $\widetilde{M(3)}$ admits a 6-atomic representing measure. Therefore, we conclude that $M(3)$ has a 7-atomic representing measure. ∎

In fact, we can generalize the previous result as follows:

Corollary 37 *Let $\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(n))$ be the algebraic variety of $\mathcal{M}(n)\left(\beta^{(2n)}\right)$ and let v be the cardinality of $\mathcal{V}$. Suppose $\mathcal{M}(n-1) > 0$ and its associated moment sequence has an r-atomic representing measure. If $\mathcal{M}(n)(\beta^{(2n)}) \geq 0$, $\operatorname{rank} \mathcal{M}(n) = \frac{n(n+1)}{2}+1$, and $v \geq \frac{n(n+1)}{2}+2$, then $\beta^{(2n)}$ has a (r+1)-atomic measure.*

The Case: $\operatorname{rank}\ M(3) = 8$**,** $\operatorname{card}\ \mathcal{V}(M(3)) = +\infty$

Problem 38 *Let* $\mathcal{V}$ *be the algebraic variety of* $M(3)$*. Assume* $M(3) \geq 0$ *of rank 8 with* $M(2) > 0$*, and* $\operatorname{card}\ \mathcal{V} = \infty$*. Under what conditions, does the moment sequence admits a representing measure?*

Algorithm to Solve Problem 38.

Step 1. Using the generalize Vandermonde matirx of $\mathcal{V}$, we can show there is a new cubic polynomial $r(x, y)$ vanishing on $\mathcal{V}' := \mathcal{V} - \{(a, b), (c, d)\}$ for some $(a, b), (c, d) \in \mathcal{V}$ besides the two polynomials $p(x, y)$ and $q(x, y)$ from the column relations.

Step 2. We may write

$$M(3) = \widetilde{M(3)} + m_1 \mathbf{v}(a, b)\mathbf{v}(a, b)^T + m_2 \mathbf{v}(c, d)\mathbf{v}(c, d)^T,$$

where m_1 and m_2 are nonnegative (not simultaneously zero) for $(a, b), (c, d) \in \mathcal{V}$.

Note that $\widetilde{M(3)}$ must have the column relation $r(X, Y) = \mathbf{0}$. ■

15.3 TMP with a Single Cubic Column Relation

TMP with the column relation $Y = X^3$

For $p \in \mathcal{P}_n$, $M(n)$ is p-**pure** if the only dependence relation in $\mathcal{C}_{M(n)}$ are those of the form $(pq)(X,Y) = \mathbf{0}$ ($q \in \mathcal{P}_{n-\deg p}$). Let $p(x,y) = y - x^3$ and let Γ denote the curve $y = x^3$.

Theorem 39 [Fialkow, 2011] *Suppose $n \geq 3$, $p(x,y) = y - x^3$, and $M(n)$ is p-pure. TFAE for $\beta \equiv \beta^{(2n)}$:*

(i) β has a $\operatorname{rank}\ M(n)$-atomic Γ-representing measure;

(ii) $M(n)$ admits a psd, recursively generated moment matrix extension $M(n+1)$;

(iii) $M(n) \geq 0$ and $\beta_{1,2n-1} > \psi(\beta)$, where $\psi(\beta)$ is the rational expression in the moment data.

Note The numerical condition $\beta_{1,2n-1} > \psi(\beta)$ in (iii) is a new type of condition discovered for the first time in truncated moment theory. No such condition was necessary for $n = 1$ or 2!

$\beta^{(6)}$ **with a Reducible Cubic Column Relation (Based on Yoo, 2017)**

For $M \equiv M(3) \equiv M(3)(\beta)$ with a reducible cubic column relation, we may assume the polynomial of the column relation is a product of $y = 0$ and a conic after applying an affine transformation of rotation. M is ultimately to be decomposed as a sum of two moment matrices whose column relations are determined by the line $y = 0$ and a conic. Thus, a proper degree-one transformation enables us to assume that M has a single cubic column relation associated to $yc(x, y) = 0$ for a quadratic polynomial c, where c is one of the conics

$$c_1(x, y) = x^2 - A_1 - A_2x - A_3y; \tag{15.1}$$

$$c_2(x, y) = xy - B_1 - B_2x - B_3y - B_4x^2; \tag{15.2}$$

$$c_3(x, y) = y^2 - C_1 - C_2x - C_3y - C_4x^2 - C_5xy, \tag{15.3}$$

for some $A_i, B_i, C_i \in \mathbb{R}$ ($1 \le i \le 5$). A simple test can show that $c_1(x, y) = 0$ is a parabola or a pair of two vertical lines, and $c_2(x, y) = 0$ is a hyperbola; however, $c_3(x, y) = 0$ can be any type of conic. According to the invariance under a degree-one transformation, we can claim that it suffices to consider the case of M with the column relation $c_3(X, Y) = \mathbf{0}$.

If M has a representing measure μ, then we may write $\mu = \mu^{(\ell)} + \mu^{(c)}$, where $\mathrm{supp}\ \mu^{(\ell)}$ is contained in the line $y = 0$ and $\mathrm{supp}\ \mu^{(c)}$ is in the conic $c_3(x, y) = 0$. We thus write

$$M = M[\mu^{(\ell)}] + M[\mu^{(c)}],$$

where each summand is the moment matrix generated by the corresponding measure.

It follows that $M[\mu^{(c)}]$ must have the column relation $c_3(X, Y) = \mathbf{0}$ since the support of $\mu^{(c)}$ must be contained in the curve $c_3(x, y) = 0$. The moment matrix $M[\mu^{(c)}]$ is required to be recursively generated so that at least two of cubic column relations in $M[\mu^{(c)}]$ are linearly dependent; that is, $(xc_3)(X, Y) = \mathbf{0}$ and $(yc_3)(X, Y) = \mathbf{0}$. Thus, $\mathrm{rank}\ M[\mu^{(c)}] \leq 7$.

We now take a crucial observation: Since $\int x^i y^j d\mu^{(\ell)} = 0$ for $i = 0, 1, \ldots, n$ and $j = 1, \ldots, n$, the moment matrix $M[\mu^{(\ell)}]$ looks like

$$M^{(\ell)} := \begin{pmatrix} a_1 & a_2 & 0 & a_3 & 0 & 0 & a_4 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & a_4 & 0 & 0 & a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & a_5 & 0 & 0 & a_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & 0 & a_6 & 0 & 0 & a_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{15.4}$$

for some $a_i \in \mathbb{R}$ $(1 \leq i \leq 7)$. We now set $M^{(c)} := M - M^{(\ell)}$, equivalently,

$$M = M^{(\ell)} + M^{(c)}. \tag{15.5}$$

For the existence of a measure, we may determine $a_1, \ldots, a_5$ concretely; the column relation $c_3(X,Y) = \mathbf{0}$ and $(xc_3)(X,Y) = \mathbf{0}$ must appear in $M - M[\mu^{(\ell)}](= M[\mu^{(c)}])$, and hence the two column relations bring the matrix equation for c_3:

$$\begin{pmatrix} C_1 & C_2 & C_4 & 0 & 0 & 0 & 0 \\ 0 & C_1 & C_2 & C_4 & 0 & 0 & 0 \\ 0 & 0 & C_1 & C_2 & C_4 & 0 & 0 \\ 0 & 0 & 0 & C_1 & C_2 & C_4 & 0 \\ 0 & 0 & 0 & 0 & C_1 & C_2 & C_4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{pmatrix}, \tag{15.6}$$

where $k_i := \beta_{i-1,2} - C_1\beta_{i-1,0} - C_2\beta_{i,0} - C_3\beta_{i-1,1} - C_4\beta_{i+1,0} - C_5\beta_{i,1}$ for $1 \le i \le 5$. We can readily see that this equation has infinitely many solutions with two parameters unless C_1, C_2, and C_4 are 0 simultaneously.

Since a proper degree-one transformation can be found to make $c_3(x,y)$ have a nonzero constant term, we only need to handle the case $C_1 \neq 0$, in which the equation has infinitely many solutions with two parameters a_6 and a_7.

Decomposition: $M = M^{(c)} + M^{(\ell)}$

Example 40 [Decomposition: $M = M^{(c)} + M^{(\ell)}$**]** *Consider* $\beta \equiv \beta^{(6)}$ *with moments:*

$$\begin{array}{llll}
\beta_{00} = 12, & \beta_{10} = 109, & \beta_{11} = 18, & \beta_{20} = 2517, \\
\beta_{11} = 450, & \beta_{02} = 96, & \beta_{30} = 74473, & \beta_{21} = 13050, \\
\beta_{12} = 2388, & \beta_{03} = 432, & \beta_{40} = 2426229, & \beta_{31} = 414450, \\
\beta_{22} = 71916, & \beta_{13} = 12600, & \beta_{04} = 2292, & \beta_{50} = 82752529, \\
\beta_{41} = 13893930, & \beta_{32} = 2351532, & \beta_{23} = 401400, & \beta_{14} = 69528, \\
\beta_{05} = 12168, & \beta_{60} = 2899815357, & \beta_{51} = 481450050, & \beta_{42} = 80325564, \\
\beta_{33} = 13479480, & \beta_{24} = 2279616, & \beta_{15} = 388800, & \beta_{06} = 67236.
\end{array}$$

$M \equiv M(3)(\beta)$ *is positive with a single cubic column relation* $Y^3 = -Y + XY$. *We solve the equation (15.6) and obtain*

$$\begin{array}{lll}
a_1 = -1021 + a_6, & a_2 = -1020 + a_6, & a_3 = -1000 + a_6, \\
a_4 = -960 + a_6, & a_5 = -736 + a_6. &
\end{array}$$

Thus, both $M^{(\ell)}$ *and* $M^{(c)}$ *have now only two parameters* a_6 *and* a_7*.*

Nested Determinants of $M^{(c)}$ and $M^{(\ell)}$

We will use the nested determinants of $M^{(\ell)}$ and $M^{(c)}$ to determine the existence of a representing measure; for the purpose, let $M_{\mathcal{B}} \equiv M(n)_{\mathcal{B}}$ be the compression of the moment matrix $M(n)$ to the columns and the rows in $\mathcal{B}$:

$$\Delta_1^{(\ell)} = \det M^{(\ell)}_{\{1\}}, \qquad \Delta_2^{(\ell)} = \det M^{(\ell)}_{\{1,X\}},$$
$$\Delta_3^{(\ell)} = \det M^{(\ell)}_{\left\{1,X,X^2\right\}}, \qquad \Delta_4^{(\ell)} = \det M^{(\ell)}_{\left\{1,X,X^2,X^3\right\}},$$
$$\Delta_5^{(\ell)} = \det M^{(\ell)}_{\left\{1,X,X^3\right\}};$$
$$\Delta_1^{(c)} = \det M^{(c)}_{\{1\}}, \qquad \Delta_2^{(c)} = \det M^{(c)}_{\{1,X\}},$$
$$\Delta_3^{(c)} = \det M^{(c)}_{\{1,X,Y\}}, \qquad \Delta_4^{(c)} = \det M^{(c)}_{\left\{1,X,Y,X^2\right\}},$$
$$\Delta_5^{(c)} = \det M^{(c)}_{\left\{1,X,Y,X^2,XY\right\}},$$
$$\Delta_{X^3}^{(c)} = \det M^{(c)}_{\left\{1,X,Y,X^2,XY,X^3\right\}}, \qquad \Delta_{X^2Y}^{(c)} = \det M^{(c)}_{\left\{1,X,Y,X^2,XY,X^2Y\right\}},$$
$$\Delta_{X^3,\,X^2Y}^{(c)} = \det M^{(c)}_{\left\{1,X,Y,X^2,XY,X^3,X^2Y\right\}}.$$

Classifications of $M^{(\ell)}$

Lemma 41 *If $M^{(\ell)}$ admits a representing measure, then there are 3 cases:*

(i) $M^{(\ell)}$ admits a 2-atomic representing measure if and only if

$$\Delta_1^{(\ell)} > 0,\ \Delta_2^{(\ell)} > 0,\ \text{ and } \Delta_i^{(\ell)} = 0 \text{ for } i = 3,\ 4,\ 5. \tag{15.7}$$

In this case, the two parameters, a_6 and a_7, are concretely found and fixed.

(ii) $M^{(\ell)}$ admits a 3-atomic representing measure if and only if

$$\Delta_1^{(\ell)} > 0,\ \Delta_2^{(\ell)} > 0,\ \Delta_3^{(\ell)} > 0,\ \text{ and } \Delta_4^{(\ell)} = 0. \tag{15.8}$$

In this case, the last identity will fix a_7 as a rational function of a_6.

(iii) $M^{(\ell)}$ admits a 4-atomic representing measure if and only if

$$\Delta_1^{(\ell)} > 0,\ \Delta_2^{(\ell)} > 0,\ \Delta_3^{(\ell)} > 0,\ \text{ and } \Delta_4^{(\ell)} > 0. \tag{15.9}$$

Classifications of $M^{(c)}$

We can claim that $M^{(c)}$ cannot have another conic column relation besides $c_3(X,Y) = \mathbf{0}$.

Consequently, we always need to assume

$$\Delta_i^{(c)} > 0 \qquad \text{for } i = 1, \ldots, 5; \tag{15.10}$$

that is, $\operatorname{rank}\ M^{(c)}(2) = 5$ and $M^{(c)}(2) \geq 0$, where $M^{(c)}(2)$ is the submatrix of $M^{(c)}$ with moments up to degree 4.

To make $M^{(c)}$ positive, we apply Smul'jan's theorem. We write

$$M^{(c)} = \begin{pmatrix} M^{(c)}(2) & B^{(c)} \\ (B^{(c)})^T & C^{(c)} \end{pmatrix}.$$

It is easy to see that there is $W^{(c)}$ such that $M^{(c)}(2)W^{(c)} = B^{(c)}$. Set

$$C_\flat^{(c)} := (W^{(c)})^T M^{(c)}(2) W^{(c)}.$$

A calculation shows

$$C^{(c)} - C_\flat^{(c)} = \begin{pmatrix} \Delta_{X^3}^{(c)}/\Delta_5^{(c)} & \Delta^{(c)} & 0 & 0 \\ \Delta^{(c)} & \Delta_{X^2Y}^{(c)}/\Delta_5^{(c)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{15.11}$$

where $\Delta^{(c)}$ satisfies $\left(\Delta^{(c)}\Delta_5^{(c)}\right)^2 = \Delta_{X^3}^{(c)}\Delta_{X^2Y}^{(c)} - \Delta_5^{(c)}\Delta_{X^3,X^2Y}^{(c)}$. Thus, with the assumption $\Delta_5^{(c)} > 0$,

$$\begin{aligned} M^{(c)} \geq 0 \iff & C^{(c)} - C_\flat^{(c)} \geq 0 \\ \iff & \left\{ \begin{array}{l} \text{either } \Delta_{X^3}^{(c)} > 0 \text{ and } \Delta_{X^3,X^2Y}^{(c)} \geq 0 \text{ or} \\ \Delta_{X^3}^{(c)} = 0,\ \Delta^{(c)} = 0, \text{ and } \Delta_{X^2Y}^{(c)} \geq 0. \end{array} \right\} \end{aligned} \tag{15.12}$$

References for the coming results:

$$\Delta_{X^3}^{(c)} = 0,\ \Delta^{(c)} = 0,\ \text{ and }\ \Delta_{X^2Y}^{(c)} = 0; \tag{15.13}$$

$$\Delta_{X^3}^{(c)} = 0,\ \Delta^{(c)} = 0,\ \text{ and }\ \Delta_{X^2Y}^{(c)} > 0; \tag{15.14}$$

$$\Delta_{X^3}^{(c)} > 0,\ \Delta^{(c)} = 0,\ \text{ and }\ \Delta_{X^2Y}^{(c)} = 0; \tag{15.15}$$

$$\Delta_{X^3}^{(c)} > 0\ \text{ and }\ \Delta_{X^3,X^2Y}^{(c)} > 0. \tag{15.16}$$

Proposition 42 *Suppose $M \equiv M(3)(\beta) = M^{(\ell)} + M^{(c)}$ (as in (15.5)) is positive with a single cubic column relation of $(yc_3)(XY) = \mathbf{0}$. Let R denote the set of all $(a_6, a_7) \in \mathbb{R}^2$ satisfying $\Delta_i^{(c)} > 0$ for $1 \leq i \leq 5$. Then:*

(i) $M^{(c)}$ is flat if and only if (15.13) holds for some $(a_6, a_7) \in R$;

(ii) In $\mathcal{C}_{M^{(c)}}$, X^3 is linearly dependent and X^2Y is linearly independent if and only if (15.14) holds for some $(a_6, a_7) \in R$;

(iii) In $\mathcal{C}_{M^{(c)}}$, X^3 is linearly independent and X^2Y is linearly dependent if and only if (15.15) holds for some $(a_6, a_7) \in R$;

(iv) In $\mathcal{C}_{M^{(c)}}$, both X^3 and X^2Y are linearly independent if and only if (15.16) holds for some $(a_6, a_7) \in R$.

Since $\operatorname{rank}\ M = 9$, a (minimal) representing measure μ for M must satisfy the variety condition, $9 \leq \operatorname{card}\ \operatorname{supp}\ \mu$; also, we know that $9 \leq \operatorname{card}\ \operatorname{supp}\ \mu \leq 11$. In terms of $\operatorname{card}\ \operatorname{supp}\ \mu$, we present our main results as follows:

Theorem 43 *Suppose $M \equiv M(3)(\beta) = M^{(\ell)} + M^{(c)}$ is positive with a single cubic column relation of $(yc_3)(XY) = \mathbf{0}$. Let R denote the set of all $(a_6, a_7) \in \mathbb{R}^2$ satisfying $\Delta_i^{(c)} > 0$ for $1 \leq i \leq 5$. Then β has a minimal 9-atomic representing measure if and only if either one of the following holds:*

(i) $M^{(\ell)}$ satisfies (15.7) for some $(a_6, a_7) \in R$ (so, a_6 and a_7 are to be fixed) and $M^{(c)}$ satisfies (15.16) for the fixed a_6 and a_7;

(ii) $M^{(\ell)}$ satisfies (15.8) for some $(a_6, a_7) \in R$ (so, a_7 is fixed as a rational function of a_6) and $M^{(c)}$ satisfies either (15.14) or (15.15) (so, a_6 is also fixed), and $M^{(c)}$ also satisfies the variety condition for the fixed $(a_6, a_7) \in R$;

(iii) $M^{(c)}$ satisfies (15.13) for some $(a_6, a_7) \in R$ (so, a_6 and a_7 are to be fixed) and $M^{(\ell)}$ satisfies (15.9) for the fixed a_6 and a_7.

Theorem 44 *Suppose $M \equiv M(3)(\beta) = M^{(\ell)} + M^{(c)}$ is positive with a single cubic column relation of $(yc_3)(XY) = \mathbf{0}$. Let R denote the set of all $(a_6, a_7) \in \mathbb{R}^2$ satisfying $\Delta_i^{(c)} > 0$ for $1 \le i \le 5$. M has a minimal 10-atomic representing measure if and only if either one of the following holds:*

(i) $M^{(\ell)}$ satisfies (15.8) for some $(a_6, a_7) \in R$ (so, a_7 is fixed as a rational function of a_6) and $M^{(c)}$ satisfies (15.16) for some $(a_6, a_7) \in R$;

(ii) $M^{(c)}$ satisfies (15.14) for some $(a_6, a_7) \in R$ (so, a_6 and a_7 are fixed). For the fixed a_6 and a_7, $M^{(c)}$ satisfies the variety condition and $M^{(\ell)}$ satisfies (15.9).

(iii) $M^{(c)}$ satisfies (15.15) for some $(a_6, a_7) \in R$ (so, a_6 is fixed as a real number by $\Delta_{X^2Y}^{(c)} = 0$). For some a_7, $M^{(c)}$ satisfies the variety condition and $M^{(\ell)}$ satisfies (15.9).

Theorem 45 *Suppose $M = M^{(\ell)} + M^{(c)}$ is positive with a single cubic column relation of $(yc_3)(XY) = \mathbf{0}$. Let R denote the set of all $(a_6, a_7) \in \mathbb{R}^2$ satisfying $\Delta_i^{(c)} > 0$ for $1 \le i \le 5$. M has a minimal 11-atomic representing measure if and only if $M^{(\ell)}$ satisfies (15.9) and $M^{(c)}$ satisfies (15.16) for some $(a_6, a_7) \in R$.*

Application to Nonsingular Sextic Moment Problems

Recall that

- $M(1) > 0 \Longrightarrow M(1)$ has infinitely many 3-atomic representing measures.
- $M(2) > 0 \Longrightarrow M(2)$ has infinitely many 6-atomic representing measures.

However, there is an example of a positive definite $M(3)$ such that it has no representing measure [Curto-Fialkow, 1996].

In particular, suppose $M(3)$ is positive definite and we define a moment matrix

$$\widetilde{M} := M(3) - \rho\,\mathbf{v}\mathbf{v}^*.$$

where $\rho > 0$ and $\mathbf{v} = (1, a, b, a^2, ab, b^2, a^3, a^2b, ab^2, b^3)^T$ for some $(a, b) \in \mathbb{R}^2$.

For some cases, it is possible that $\widetilde{M}$ may have a reducible conic column relation for some $\rho > 0$ and $(a, b) \in \mathbb{R}^2$. Then we can apply our main results and will be able to determine the solubility of $M(3)$.

16 Questions

Question 1 *How can we solve sextic moment problems with a single cubic column relation?*

Question 2 *How can we solve nonsingular sextic moment problems?*

Question 3 *Can we realize the numerical conditions appeared in solutions of sextic moment problems as a generic property of the moment sequence?*

Question 4 *Can we get a solution of TMP's related those of the full moment problems (for example, results based on reproducing kernels, recurrence relations, and so on)?*

Question 5 [Algebraic Geometry]

Can we find a concrete discriminant to classify cubic polynomials?

Various Moment Problems

Lecture 3. Applications of moment problems and related topics

Operator Theory and Operator Algebras Winter School
(Mungyeong; December 20-23, 2017)

1 Extremal Sextic Moment Problems

1.1 The Multivariable Vandermonde Matrix

For an algebraic variety $\mathcal{V}(M(n)) = \{(x_1, y_1), \cdots, (x_r, y_r)\}$, define the multivariable Vandermonde matrix W as

$$W = \begin{pmatrix} 1 & x_1 & y_1 & {x_1}^2 & y_1x_1 & {y_1}^2 & \cdots & {x_1}^n & x_1^{n-1}y_1 & \cdots & x_1y_1^{n-1} & y_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_r & y_r & {x_r}^2 & y_rx_r & {y_r}^2 & \cdots & {x_r}^n & x_r^{n-1}y_r & \cdots & x_ry_r^{n-1} & y_r^n \end{pmatrix}$$

We also may label the columns of W as we did for $M(n)$ and let $W_{\mathcal{B}}$ be the compression of W to a basis $\mathcal{B}$ of the column space of $M(n)$.

In the case of one-variable Vandermonde matrices, they are invertible if and only if all the nodes are different. However, this argument is no longer valid for multivariable Vandermonde matrices.

Checking Weak Consistency With Vandermonde Matrices

Recall that:

- **(Weak Consistency)** $p \in \mathcal{P}_n$, $p|_{\mathcal{V}(M(n))} \equiv 0 \Longrightarrow \Lambda(p) = 0$;
- **(Consistency)** $p \in \mathcal{P}_{2n}$, $p|_{\mathcal{V}(M(n))} \equiv 0 \Longrightarrow \Lambda(p) = 0$.

Lemma 1 [Curto-Fialkow-Möller, 2008]

The following are equivalent for β extremal:

(i) β is weakly consistent;

(ii) For any basis $\mathcal{B}$ of $\mathcal{C}_{M(n)}$ (the column space of $M(n)$), $W_{\mathcal{B}}$ is invertible;

(iii) There exists a basis $\mathcal{B}$ of $\mathcal{C}_{M(n)}$ such that $W_{\mathcal{B}}$ is invertible.

Checking Consistency with Vandermonde Matrices

This argument is valid for multidimensional moment problems.

Notations

- Columns of $M(n)$ are labeled with the monomials $\mathbf{X}^j \in \mathcal{P}_n$ ($j \in \mathbb{Z}_+^d$, $|j| \le m$) in degree-lexicographical order;
- $V := \{\mathbf{w}_1, \ldots, \mathbf{w}_s\}$ is a finite subset in $\mathbb{R}^d$;
- $W_m[V]$ is a matrix with s rows and with columns labeled with $\mathbf{X}^j$. (Note that the entry of $W_m[V]$ in the row i $(1 \le i \le s)$ and the column $\mathbf{X}^j$ is $\mathbf{w}_i^j$, and hence $W_m[V]$ is a Vandermonde matrix of points in $\mathbb{R}^d$);
- $U_m[V] := W_m[V]^T$;
- $\tau(m) := \dim \mathcal{P}_m = \binom{m+d}{m}$;
- $p_1, \ldots, p_\tau$ denote the list of monomials in $\mathcal{P}_m$ in degree-lexicographical order.

Given $M(n)(\beta)$, let $\tau \equiv \tau(2n)$, $r = \text{rank } M(n)$, $v = \text{card } \mathcal{V}(M(n))$, and set $L_\beta := (\Lambda_\beta(p_1), \ldots, \Lambda_\beta(p_\tau))^T \in \mathbb{R}^\tau$. Let $\mathcal{B} := \{\mathbf{X}^{i_1}, \ldots, \mathbf{X}^{i_r}\}$ denote a basis of $\mathcal{C}_{M(n)}$. For the case when $V \subseteq \mathcal{V}(M(n))$, let $W_\mathcal{B}[V]$ denote the compression of $W_m[V]$ to columns $\mathbf{X}^{i_1}, \ldots, \mathbf{X}^{i_r}$ and let $U_\mathcal{B}[V] \equiv W_\mathcal{B}[V]^T$.

Theorem 2 [Fialkow, 2008] *If a positive $M(n)$ has a flat extension, then $U_\mathcal{B}[V]$ is invertible for a subset V of $\mathcal{V}(M(n))$.*

Proposition 3 [Fialkow, 2008] *Let $v < \infty$. Then*

(i) β is consistent if and only if $L_\beta \in \text{Ran } U_{2n}[\mathcal{V}(M(n))]$.

(ii) β is weakly consistent if and only if $\text{Ran } M(n) \subseteq \text{Ran } U_n[\mathcal{V}(M(n))]$; equivalently, there exists a matrix Z such that $M(n) = U_n[\mathcal{V}(M(n))]Z$.

1.2 Division Algorithm for Checking Consistency

Theorem 4 (Division Algorithm) *Fix a monomial order* $>$ *on* $\mathbb{Z}_+^n$, *and let* $F = (f_1, \cdots, f_s)$ *be an ordered* s*-tuple of polynomials in* $\mathbb{R}[x_1, \cdots, x_n]$.

Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ *can be written as*

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i, r \in \mathbb{R}[x_1, \cdots, x_n]$, *and either* $r = 0$ *or* r *is a linear combination of monomials, with coefficients in* $\mathbb{R}$, *none of which is divisible by any leading terms of* $f_1, \cdots, f_s$. *We call it* r *a remainder of* f *on division by* F. *Furthermore, if* $a_i f_i \neq 0$, *then we have*

$$\mathrm{multideg}(f) \geq \mathrm{multideg}(a_i f_i).$$

1.3 Solutions to Extremal Sextic Moment Problems
(Based on Curto-Yoo, 2017)

The Case: $\text{rank } M(n) = \text{card } \mathcal{V}(M(n)) = 7$

The basis of $M(3)$ is one of the following:

Case 1. $\mathcal{B}_1 := \{1, X, Y, X^2, XY, Y^2, X^3\}$
Case 2. $\mathcal{B}_2 := \{1, X, Y, X^2, XY, Y^2, X^2Y\}$
Case 3. $\mathcal{B}_3 := \{1, X, Y, X^2, XY, Y^2, XY^2\}$
Case 4. $\mathcal{B}_4 := \{1, X, Y, X^2, XY, Y^2, Y^3\}$

Using the degree-one transformation, $X = \widetilde{Y}$ and $Y = \widetilde{X}$, we know Case 1 and Case 4 are equivalent.

Theorem 5 *Let* $M(3)(\beta) \geq 0$*,* $M(2) > 0$ *and* $\text{rank}\ M(n) = \text{card}\ \mathcal{V}(M(n)) = 7$*.*

Let $\mathcal{V} := \{(x_1, y_1), \cdots, (x_7, y_7)\}$*.*

Let $\mathcal{B}_1 := \{1, X, Y, X^2, XY, Y^2, X^3\}$ *be a basis for* $M(3)$*.*

Then β *has a representing measure if and only if*

- $M(3)$ *is weakly consistent;*
- *For* $0 \leq i + j \leq 2$*,*
  $L(x^i y^j (x^4 - a_{00} - a_{10}x - a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{30}x^3)) = 0$*,*
  where $(a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30})^T = W_{\mathcal{B}_1}^{-1}(x_1^4, \cdots, x_7^4)^T$*.*

Proof. Recall that in the extremal case, $\beta^{(2n)}$ is consistent if and only if it admits a $\mathrm{rank}\ M(n)$-atomic representing measure.

Let $q_k(X,Y)=0$ is the column relation in ith column for $k=8,9,$ and 10.

Since $W_{\mathcal{B}_1}$ is invertible, there exist unique polynomials with the leading monomial x^4 such that vanishes on the variety $\mathcal{V}$, say,

$$r_1(x,y) \;=\; x^4-(a_{00}+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2+a_{30}x^3).$$

Let $\mathcal{I}$ be $\{p\in\mathcal{P}_6 : p|_{\mathcal{V}}\equiv 0\}$. Now by the division algorithm, we may write for any $p\in\mathcal{I}$,

$$p(x,y)=Aq_8+Bq_9+Cq_{10}+Dr_1+r,$$

where $A,B,C\in\mathcal{P}_3$, $D\in\mathcal{P}_2$ and $r(x,y)=c_{00}+c_{10}x+c_{01}y+c_{20}x^2+c_{11}xy+c_{02}y^2+c_{30}x^3$ for some $c_{00},\cdots,c_{02},c_{30}\in\mathbb{R}$.

Claim: $\mathcal{I} = \{fq_8 + gq_9 + hq_{10} + qr_1 : f, g, h \in \mathcal{P}_3, q \in \mathcal{P}_2\}$.

Note that $p|_{\mathcal{V}} \equiv 0 \Longrightarrow r|_{\mathcal{V}} \equiv 0.$

Since $r|_{\mathcal{V}} \equiv 0$, we have the matrix form of a linear system:

$$W_{\mathcal{B}_1} \begin{pmatrix} c_{00} & c_{10} & c_{01} & c_{20} & c_{11} & c_{02} & c_{30} \end{pmatrix}^T = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}^T .$$

Since the matrix in the left hand side is invertible, we know $c_{00} = c_{10} = c_{01} = c_{20} = c_{11} = c_{02} = c_{30} = 0$, which means $r(x, y) = 0$.

Consequently,

$$\beta \text{ is consistent } \Longleftrightarrow \begin{cases} L_\beta(x^i y^j q_k(x, y)) = 0 \ \ (0 \le i + j \le 3;\ k = 8, 9, 10); \\ L_\beta(x^t y^u r_1(x, y)) = 0 \ \ (0 \le t + u \le 2). \end{cases}$$

But it is immediate from the column relations in $M(3)$ and from hypothesis. ■

Solution to Case 2

Theorem 6 *Let* $M(3)(\beta) \geq 0$*,* $M(2) > 0$ ***and*** $\text{rank} \;\; M(n) = \text{card} \;\; \mathcal{V}(M(n)) = 7$*. Let* $\mathcal{B}_2 := \{1, X, Y, X^2, XY, Y^2, X^2Y\}$ *be a basis for* $M(3)$*. Then* β *has a representing measure if and only if* $M(3)$ *is weakly consistent.*

The Case: $\text{rank} \;\; M(n) = \text{card} \;\; \mathcal{V}(M(n)) = 8$

The basis of $M(3)$ is one of the following:

Case 1. $\mathcal{B}_1 := \{1, X, Y, X^2, XY, Y^2, X^3, X^2Y\}$
Case 2. $\mathcal{B}_2 := \{1, X, Y, X^2, XY, Y^2, X^3, XY^2\}$
Case 3. $\mathcal{B}_3 := \{1, X, Y, X^2, XY, Y^2, X^3, Y^3\}$
Case 4. $\mathcal{B}_4 := \{1, X, Y, X^2, XY, Y^2, X^2Y, XY^2\}$
Case 5. $\mathcal{B}_5 := \{1, X, Y, X^2, XY, Y^2, X^2Y, Y^3\}$ (subcase of Case 2)
Case 6. $\mathcal{B}_6 := \{1, X, Y, X^2, XY, Y^2, XY^2, Y^3\}$ (subcase of Case 1)

Theorem 7 *Let* $M(3) \geq 0$, $M(2) > 0$, $\text{rank } M(n) = \text{card } \mathcal{V}(M(n)) = 8$, *and* $\mathcal{V} := \{(x_1, y_1), \cdots, (x_8, y_8)\}$.

Let $\mathcal{B}_1 := \{1, X, Y, X^2, XY, Y^2, X^3, X^2Y\}$ *be a basis for* $M(3)$.

Then β *has a representing measure if and only if*

- $M(3)$ *is weakly consistent;*
- *For* $0 \leq i + j \leq 2$,

  $L(x^i y^j (x^4 - a_{00} - a_{10}x - a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{30}x^3 - a_{21}x^2y)) = 0$,

  $L(x^i y^j (x^3y - b_{00} - b_{10}x - b_{01}y - b_{20}x^2 - b_{11}xy - b_{02}y^2 - b_{30}x^3 - a_{21}x^2y)) = 0$,

  where $(a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30}, a_{21})^T = W_{\mathcal{B}_1}^{-1}(x_1^4, \cdots, x_8^4)^T$

  and $(b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, b_{30}, b_{21})^T = W_{\mathcal{B}_1}^{-1}(x_1^2 y_1, \cdots, x_8^2 y_8)^T$.

Theorem 8 *Let* $M(3)(\beta) \geq 0$, $M(2) > 0$ *and* $\mathrm{rank}\ M(n) = \mathrm{card}\ \mathcal{V}(M(n)) = 8$.

Let $\mathcal{V} := \{(x_1, y_1), \cdots, (x_8, y_8)\}$.

Let $\mathcal{B}_2 := \{1, X, Y, X^2, XY, Y^2, X^3, XY^2\}$ *be a basis for* $M(3)$.

Then β *has a representing measure if and only if*

- $M(3)$ *is weakly consistent;*
- *For* $0 \leq i + j \leq 2$,
  $L(x^i y^j (x^4 - a_{00} - a_{10}x - a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{30}x^3 - a_{12}xy^2)) = 0$,
  where $(a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30}, a_{12})^T = W_{\mathcal{B}_2}^{-1}(x_1^4, \cdots, x_8^4)^T$

For the other Cases, we have a similar result.

2 Any Sequence Has an Interpolating Measure!

For real sequence n-sequence $\beta \equiv \beta^{(m)}$, we define an **interpolating measure** as a (not necessarily positive) Borel measure μ such that $\int \mathbf{x}\, d\mu = \beta_{\mathbf{i}}$ for $|\mathbf{i}| \le m$.

According to Jordan decomposition theorem, we know that every interpolating measure μ has a unique decomposition into a difference $\mu = \mu^+ - \mu^-$ of two positive measures μ^+ and μ^-, at least one of which is finite.

For the univariate case, R. P. Boas showed that any “infinite” sequence of real numbers admits an interpolating measure supported in $[0, \infty)$; that is, one can always find a measure for any sequence of the form $\mu = \mu^+ - \mu^-$ such that both μ^+ and μ^- are positive Borel measures supported in $[0, \infty]$.

Moreover, G. Flessas, K. Burton, and R. R. Whitehead found an algorithm to find such a measure supported in the real line for a “finite” real sequence $\{s_j\}_{j=0}^{2n-1}$. As a generalization of these results, we will see that any truncated moment sequence has an interpolating measure supported in $\mathbb{R}^d$ for any $d \ge 2$.

2.1 V-consistency

A moment sequence $\beta \equiv \beta^{(2n)}$ (or the associated moment matrix $M_d(n) \equiv M_d(n)(\beta^{(2n)})$) is said to be V-**consistent** for a set $V \in \mathbb{R}^d$ if the following holds:

$$p \in \mathcal{P}_{2n},\ p|_V \equiv 0 \Longrightarrow \Lambda(p) = 0. \tag{2.1}$$

This property of the moment sequence guarantees the existence of an interpolating measure. Here is a formal result:

Lemma 9 [Curto-Fialkow-Möller, 2008] *Let $L : \mathcal{P}_{2n} \to \mathbb{R}$ be a linear functional and let $V \subseteq \mathbb{R}^d$. Then the following statements are equivalent:*

(i) There exist $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$ and there exist $\mathbf{w}_1, \ldots, \mathbf{w}_\ell \in V$ such that

$$L(p) = \sum_{k=1}^{\ell} \alpha_k p(\mathbf{w}_k) \text{ for all } p \in \mathcal{P}_{2n}. \tag{2.2}$$

(ii) If $p \in \mathcal{P}_{2n}$ and $p|_V \equiv 0$, then $L(p) = 0$.

If L is the Riesz functional of the moment sequence β, then Lemma 9(ii) is just the $\mathcal{V}(M(n))$-consistency condition of β and $\sum_{k=1}^{\ell} \alpha_k \delta_{\mathbf{w}_k}$ is an interpolating measure for β. While it seems like Lemma 9 gives a concrete solution for β to have an interpolating measure, we should indicate that checking the consistency is a highly nontrivial process. To show that β is V-consistent, it is essential (but, difficult) to find a representation of all the polynomials vanishing on V.

For $M_d(n)$ to have a (positive) representing measure, β must be $\mathcal{V}$-consistent; in the case of extremal cases (that is, $\mathrm{rank}\ M_d(n) = \mathrm{card}\ \mathcal{V}$), it is known that a positive $M_d(n)(\beta)$ is consistent if and only if β admits a unique $\mathrm{rank}\ M_d(n)$-atomic representing measure whose support is exactly $\mathcal{V}$.

In particular, when a positive $M_d(n)$ is invertible, we know $\mathcal{V} = \mathbb{R}^d$ and the only polynomial vanishing on $\mathbb{R}^d$ is the zero polynomial. Thus, $M_d(n)$ is naturally consistent and has an interpolating measure.

2.2 Rank-one Decompositions

After rearranging the terms in (2.2) by the sign of densities, we write a measure μ for a consistent $M_d(n)$ as

$$\mu = \sum_{k=1}^{s} \alpha_k \delta_{\mathbf{w}_k} - \sum_{k=s+1}^{\ell} \alpha_k \delta_{\mathbf{w}_k}, \tag{2.3}$$

where $\alpha_k > 0$ for all $k = 1, \ldots, \ell$; we denote the first summand in (2.3) as μ^+ and the second as μ^-. Due to this fact, a bound of the cardinality of the support of an interpolating measure is established:

Proposition 10 *A minimal measure for a consistent $M_d(n)$ is at most $(2 \deg \mathcal{P}_n)$-atomic.*

Notations Let **w** $= (w_1, \ldots, w_d) \in \mathbb{R}^d$ and let

(i) $\mathbf{v}(\mathbf{w}) := \left(1 \; w_1 \; \cdots \; w_d \; w_1^2 \; w_1 w_2 \; w_1 w_3 \; \cdots \; w_{d-1} w_d \; w_d^2 \; \cdots \; w_1^n \; \cdots \; w_d^n\right)$, which is a row vector corresponding to the monomials $\mathbf{w}^{\mathbf{i}}$ in the degree-lexicographic order.

(ii) $P(\mathbf{w}) := \mathbf{v}(\mathbf{w})^T \mathbf{v}(\mathbf{w})$, which is indeed the rank-one moment matrix generated by the measure $\delta_{\mathbf{w}}$.

For example, if $d = n = 2$ and $\mathbf{w} = (a, b)$, then

$$P(\mathbf{w}) = \begin{pmatrix} 1 & a & b & a^2 & ab & b^2 \\ a & a^2 & ab & a^3 & a^2b & ab^2 \\ b & ab & b^2 & a^2b & ab^2 & b^3 \\ a^2 & a^3 & a^2b & a^4 & a^3b & a^2b^2 \\ ab & a^2b & ab^2 & a^3b & a^2b^2 & ab^3 \\ b^2 & ab^2 & b^3 & a^2b^2 & ab^3 & b^4 \end{pmatrix}$$

Thus, if $M_d(n)$ has an interpolating measure μ supported in a set $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$, then one should be able to write $M_d(n) = \sum_{k=1}^{\ell} d_k P(\mathbf{w}_k)$ for some $d_1, \ldots, d_\ell \in \mathbb{R}$.

2.3 The Existence of an Interpolating Measure

We will verify that any truncated moment matrix turns out to be $\mathbb{R}^d$-consistent after applying proper perturbations, and so it admits an interpolating measure. To prove the main result, we begin with auxiliary results:

Lemma 11 *Assume A and B are matrices of the same size. Then $\operatorname{rank}(A+B) = \operatorname{rank} A + \operatorname{rank} B$ if and only if $\operatorname{Ran} A \cap \operatorname{Ran} B = \{\mathbf{0}\}$ and $\operatorname{Ran} A^T \cap \operatorname{Ran} B^T = \{\mathbf{0}\}$.*

As a special case, one can easily prove:

Lemma 12 *Assume A and B are Hermitian matrices of the same size and $\operatorname{rank} B = 1$. Then $\operatorname{rank}(A+B) = 1 + \operatorname{rank} A$ if and only if $\operatorname{Ran} A \cap \operatorname{Ran} B = \{\mathbf{0}\}$.*

We are ready to introduce a crucial lemma:

Lemma 13 *A point* $\mathbf{w}$ *is in* $\mathcal{V}_{M_d(n)}$ *if and only if the vector* $\mathbf{v}(\mathbf{w})$ *is in* $\mathrm{Ran}\ M_d(n)$*.*

Proof. Assume that $\left\{p_k(\mathbf{X}) \equiv \sum a_{\mathbf{i}}^{(k)} \mathbf{X}^{\mathbf{i}}\right\}_{k=1}^{\ell}$ is the set of polynomials obtained from column relations in $M_d(n)$. Note that $\mathrm{span}\ \{\widehat{p_k}\}_{k=1}^{\ell} = \ker M_d(n)$. Now observe:

$$\begin{aligned}
\mathbf{w} \in \mathcal{V}_{M_d(n)} &\iff p_k(\mathbf{w}) = 0 && \text{for } k = 1, \ldots, \ell \\
&\iff \sum a_{\mathbf{i}}^{(k)} \mathbf{w}^{\mathbf{i}} = 0 && \text{for } k = 1, \ldots, \ell \\
&\iff \langle \widehat{p_k}, \mathbf{v}(\mathbf{w}) \rangle = 0 && \text{for } k = 1, \ldots, \ell \\
&\iff \widehat{p_k} \perp \mathbf{v}(\mathbf{w}) && \text{for } k = 1, \ldots, \ell \\
&\iff \mathbf{v}(\mathbf{w}) \in (\ker M_d(n))^{\perp} = \mathrm{Ran}\ M_d(n). \ \blacksquare
\end{aligned}$$

Theorem 14 *Any truncated moment sequence $\beta \equiv \beta^{(2n)}$ of degree $2n$ has an interpolating measure.*

Proof. Pick a point $\mathbf{w}_1 \notin \mathcal{V}$. Then we know from Lemma 13 that $\mathbf{w} \notin \mathrm{Ran}\ M_d(n)(\beta)$. It also follows from Lemma 12 that $\mathrm{Ran}\ M_d(n) \cap \mathrm{Ran}\ P(\mathbf{w}_1) = \{\mathbf{0}\}$. Therefore, $\mathrm{rank}\ (M_d(n) + P(\mathbf{w}_1)) = 1 + \mathrm{rank}\ M_d(n)$. Next, choose a point $\mathbf{w}_2$ which not in the algebraic variety of $M_d(n) + P(\mathbf{w}_1)$ and we know from the same argument that $\mathrm{rank}\ (M_d(n) + P(\mathbf{w}_1) + P(\mathbf{w}_2)) = 2 + \mathrm{rank}\ M_d(n)$. Keep this process until we obtain an invertible matrix $\widetilde{M} := M_d(n) + \sum_{k=1}^{\ell} P(\mathbf{w}_k)$ for some ℓ. $\widetilde{M}$ is naturally consistent, and so it admits an interpolating measure, say $\tilde{\mu}$. Thus, $M_d(n)$ has an interpolating measure of the form $\tilde{\mu} - \sum_{k=1}^{\ell} \delta_{\mathbf{w}_k}$. ■

Theorem 15 *Any finite sequence has an interpolating measure.*

Proof. It suffices to cover the cases when the given sequence is not the type of $\beta^{(2n)}$. Such a sequence cannot fill up the associated moment matrix, so we use new parameters to complete the moment matrix. If it is possible to make the moment matrix invertible, then the extended moment sequence is consistent. Thus, the given sequence has an interpolating measure. Otherwise, one can follow the same process in the proof of Theorem 14 and verify that the sequence admits an interpolating measure. Lastly, if a sequence begins with zero, then one need take a new nonzero initial moment and repeat the process used in the above. ■

2.4 Interpolating Measures vs Positive Borel Measures

Recall that in the presence of a (positive) representing measure μ for a positive $M_d(n)(\beta)$, a result states that

$$\hat{p} \in \ker M_d(n)(\beta) \iff p(\mathbf{X}) = \mathbf{0} \iff \operatorname{supp}\ \mu \subseteq \mathcal{Z}(p);$$

that is, the algebraic variety of $M_d(n)$ must contain the support of a representing measure. However, the following example shows such an argument is no longer valid for the moment problem about an interpolating measure; consider

$$M_2(1) \equiv M_2(1)\left(\beta^{(2)}\right) = \begin{pmatrix} -1 & -16 & -4 \\ -16 & -94 & -10 \\ -4 & -10 & 2 \end{pmatrix}. \tag{2.4}$$

Note that $M_2(1)$ has a single column relation $X_2 = -(4/3)\mathit{1} + (1/3)X_1$. Indeed, the sequence can be generated by an interpolating measure $\nu = \delta_{(-2,1)} + \delta_{(-2,-2)} - \delta_{(1,1)} - \delta_{(10,1)}$; but, different from the case for a positive measure, $\operatorname{supp}\ \nu \not\subseteq \mathcal{Z}(x_2 + 4/3 - (1/3)x_1) = \mathcal{V}_{\beta^{(2)}}$. Nonetheless, one can still find an interpolating measure supported in the algebraic variety of $M_2(1)$ as follows:

Example 16 *We illustrate how to find a measure of the sequence in (2.4). To find an interpolating measure supported in the algebraic variety of* $M_2(1)$*, select a point* $\left(a, \frac{a-4}{3}\right) \in \mathcal{Z}(x_2 + 4/3 - (1/3)x_1)$ *for some* $a \in \mathbb{R}$*. Using the rank-one decomposition, we write*

$$M_2(1) = \widetilde{M_2(1)} + u \begin{pmatrix} 1 & a & \frac{a-4}{3} \\ a & a^2 & \frac{a(a-4)}{3} \\ \frac{a-4}{3} & \frac{a(a-4)}{3} & \frac{(a-4)^2}{9} \end{pmatrix} \tag{2.5}$$

for some $u \in \mathbb{R}$*. Note that* $\operatorname{rank}\ M_2(1) = 2$ *and we are attracted to guess that a minimal interpolating measure is 2-atomic. To find such a measure, we impose a condition that* $\operatorname{rank}\ \widetilde{M_2(1)} = 1$*; a calculation shows* $\operatorname{rank}\ \widetilde{M_2(1)} = 1$ *if and only if* $u = 162/(a^2 - 32a + 94)$*. If we take* $u = 162/(a^2 - 32a + 94)$*, then*

$$M_2(1) = \frac{-(a-16)^2}{a^2 - 32a + 94} \begin{pmatrix} 1 & \frac{2(8a-47)}{a-16} & \frac{2(2a-5)}{a-16} \\ \frac{2(8a-47)}{a-16} & \frac{4(8a-47)^2}{(a-16)^2} & \frac{4(2a-5)(8a-47)}{(a-16)^2} \\ \frac{2(2a-5)}{a-16} & \frac{4(2a-5)(8a-47)}{(a-16)^2} & \frac{4(2a-5)^2}{(a-16)^2} \end{pmatrix} + \frac{162}{a^2 - 32a + 94} \begin{pmatrix} 1 & a & \frac{a-4}{3} \\ a & a^2 & \frac{a(a-4)}{3} \\ \frac{a-4}{3} & \frac{a(a-4)}{3} & \frac{(a-4)^2}{9} \end{pmatrix}.$$

Therefore, we get an interpolating measure $\mu = \frac{-(a-16)^2}{a^2-32a+94}\delta_{\left(\frac{2(8a-47)}{a-16}, \frac{2(2a-5)}{a-16}\right)} + \frac{162}{a^2-32a+94}\delta_{\left(a, \frac{a-4}{3}\right)}$ *(with* $a^2 - 32a + 94 \neq 0$ *and* $a \neq 16$*), which is supported in* $\mathcal{V}_{M_2(1)}$*.*

3 TMP and Hadamard Product

Preliminaries about the Hadamard Product

Note. (1) $A \circ B = AB \iff$ both A and B are diagonal.

(2) $P(a,b)P(c,d) \neq P(c,d)P(a,b)$ but obviously,

$$P(a,b) \circ P(c,d) = P(c,d) \circ P(a,b) = P(ac,bd).$$

Note. By the result of C. Bayer and J. Teichmann, if $M(n)$ admits one or more representing measures, then we may write

$$M(n) = \sum_{k=1}^{\ell} \rho_k P(x_k, y_k),$$

where $\rho_k > 0$ and $(x_k, y_k) \in \mathbb{R}^2$ for $k = 1, \ldots, \ell \leq \dim \mathcal{P}_{2n}$.

In the presence of a representing measure, we should be able to write

$$M(n) \circ P(a,b) = \sum_{k=1}^{\ell} \rho_k P(ax_k, by_k).$$

Proposition 17 $A \geq 0 \iff A \circ B \geq 0$ *for all* $B \geq 0$*.*

Thus, we know that $M(n) \circ P(a,b) \geq 0$ for any $(a,b) \in \mathbb{R}^2$.

Proposition 18 *Suppose* A *and* B *are square matrices of the same size and both positive. Then*

(i) $\det(A \circ B) \geq (\det A)(\det B)$*;*

(ii) $\text{rank }(A \circ B) \leq (\text{rank } A)(\text{rank } B)$*.*

Moreover, if $A > 0$*, then* $\text{rank }(A \circ B)$ *is equal to the number of nonzero diagonal entries of* B*.*

Thus, we know that $\text{rank }(M(n) \circ P(a,b)) \leq \text{rank } M(n)$ for any $(a,b) \in \mathbb{R}^2$.

Proposition 19 *$A > 0$, $B \geq 0$, $\nu(B) :=$ the number of nonzero main diagonal entries of B:*

(i) There is a permutation P such that $B = P^{-1}(O_{n-\nu(B)} \oplus C)P$, where $C \geq 0$;

(ii) $\mathrm{rank}\ (A \circ B) \geq \nu(B) \geq \mathrm{rank}\ \ B$.

Note If $M(n) > 0$ and $ab \neq 0$, then

$$\frac{(n+1)(n+2)}{2} = \nu(P(a,b)) \leq \mathrm{rank}\ (M(n) \circ P(a,b)) \leq \mathrm{rank}\ \ M(n) = \frac{(n+1)(n+2)}{2};$$

that is, $\mathrm{rank}\ (M(n) \circ P(a,b)) = \mathrm{rank}\ \ M(n)$. (Thus the rank-reduction method via the Hadmard product is not applicable for a positive definite $M(n)$.)

Question 1 *If $M(n)$ is positive but not invertible, what would be the lower bound of $\mathrm{rank}\ (M(n) \circ P(a,b))$?*

Proposition 20 *If both moment matrices $M(\alpha)$ and $M(\beta)$ of the same size admit representing measures, then so does $M(\alpha) \circ M(\beta)$.*

Proof Since $M(\alpha)$ and $M(\beta)$ admit representing measures, we may write

$$M(\alpha) = \sum_{k=1}^{r} \rho_k P(a_k, b_k), \qquad M(\beta) = \sum_{\ell=1}^{s} \tau_\ell P(c_\ell, d_\ell).$$

Thus,

$$M(\alpha) \circ M(\beta) = \sum_{k,\ell} \rho_k \tau_\ell P(a_k c_\ell, b_k d_\ell). \blacksquare$$

Note The converse of Proposition 20 is not true. For example, consider

$$M(\alpha) = M(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proposition 21 *$M(n)$ admit a representing measure if and only if there are some moment matrices A and B having a representing measure respectively such that $M(n) = A \circ B$.*

Question 2 *If $M(n)$ admit a representing measure, then are there moment matrices A and B (both different from $P(1,1)$) having a representing measure respectively such that $M(n) = A \circ B$?*

Note The answer to the question seems to be no for $M(n)$ that has a unique r-atomic representing measure, where r is a prime number.

Theorem 22 *Let $P(a,b) := \begin{pmatrix} 1 & a & b & \cdots & a^n & \cdots & b^n \end{pmatrix}^T \begin{pmatrix} 1 & a & b & \cdots & a^n & \cdots & b^n \end{pmatrix}$. Then $S(n) := M(n) \circ \frac{1}{2}[P(1,1) + P(-1,-1)]$ is the symmetric moment matrix; that is, whose odd-degree moments are all zero. Thus, if $M(n)$ admits a representing measure, then so does $S(n)$.*

Proposition 23 *If $M(n)$ has a column relation $p(X, Y) = \mathbf{0}$, then $M(n) \circ P(a, b)$ has the column relation $p(X/a, Y/b) = \mathbf{0}$ for $ab \neq 0$.*

Question 3 *Suppose a positive $M(n)$ has a single column relation of the form*

$$X^n = p(x, y), \qquad p \in \mathcal{P}_{n-1}.$$

Is there $(a, b) \in \mathbb{R}^2$ ($ab \neq 0$) such that the column Y^n in $\widetilde{M}(n, a, b) := M(n) \circ P(a, b)$ is linearly dependent?

If so, $\widetilde{M}(n, a, b)$ admits a unique recursively generated moment matrix extension. Moreover, if the extension is positive, then $\widetilde{M}(n, a, b)$ has a representing measure, so does $M(n)$.

4 Approximation Method and Convex Analysis

Let $\eta = \dim \mathcal{P}_{2n}$, so $\beta \equiv \beta^{(2n)} \in \mathbb{R}^\eta$. Define convex cones:

$$\begin{aligned}
\mathcal{R}_{d,n} &:= \{\beta \in \mathbb{R}^\eta : \ \beta \text{ has a } K\text{-representing measure}\}, \\
\mathcal{S}_{d,n} &:= \{\beta \in \mathbb{R}^\eta : \ \Lambda_\beta \text{ is } K\text{-positive}\}, \\
\mathcal{F}_{d,n} &:= \{\beta \in \mathbb{R}^\eta : \ M(n)(\beta) \text{ has finitely atomic } K\text{-representing measures}\}
\end{aligned}$$

Note β has a K-representing measure $\Longrightarrow \Lambda_\beta$ is K-positive $\Longrightarrow M(n)(\beta) \geq 0$.

Theorem 24 [Fialkow-Nie, 2010]
For a d-dimensional truncated moment sequence β of degree n,

$$\Lambda_\beta \textit{ is } K\textit{-positive} \Longleftrightarrow \ \beta \in \overline{\mathcal{F}_{d,n}(K)} \ \Longleftrightarrow \ \beta \in \overline{\mathcal{R}_{d,n}(K)}.$$

In other words, Λ_β is K-positive if and only if $\lim_{m\to\infty} \|\beta - \beta^{(m)}\| = 0$ for a sequence $\{\beta^{(m)}\}$ in which each truncated moment sequence $\beta^{(m)}$ has a K-representing measure $\mu^{(m)}$.

Theorem 25 [Fialkow-Nie, 2010]

Let $\beta \equiv \beta^{(m)}$ be a d-dimensional moment sequence and $M(\beta)$ be the moment matrix of β. Then:

(i) $\mathcal{S}_{d,n} = \overline{\mathcal{R}_{d,n}}$.

(ii) If Λ_β is K-positive, then there exists a sequence of positive Borel measure $\{\mu_k\}$, each supported in $K \subseteq \mathbb{R}^d$, having finite moments up to degree m, such that for each $\mathbf{i}$, $|\mathbf{i}\,| \leq m$, $\beta_{\mathbf{i}} = \lim_{k\to\infty} \int \mathbf{x}^{\mathbf{i}}\, d\mu_k$.

Note There is an example such that

$$\beta : K\text{-positive, having no measure, } M(\beta) : \text{ singular.}$$

Question 4 *If Λ_β is strictly K-positive, is $M(\beta)$ positive definite?*

Question 5 *Is there is any example of a K-positive β such that $M(\beta)$ is positive definite but β has no representing measure?*

The Riesz-Haviland Theorem for TMP

Consider an example of a one-dimensional moment sequence:

$$K = \mathbb{R}, \qquad \beta = \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4\} \equiv \{1, 1, 1, 1, 2\}\,.$$

It is known that Λ_β is positive but that β has no representing measure. Thus, the direct analogue of the Riesz-Haviland Theorem for TKMP is not valid.

Theorem 26 [Curto-Fialkow, 2009] *Let $\beta = \beta^{(2n)}$ or $\beta = \beta^{(2n+1)}$ and let K be a closed subset of $\mathbb{R}^d$. β has a K-representing measure if and only if β can be extended to a sequence $\tilde{\beta} \equiv \beta^{(2n+2)}$ such that $\Lambda_{\tilde{\beta}}$ is K-positive.*

As mentioned before, it is often very difficult to verify K-positivity of β, but here is a case with a concrete criterion:

Theorem 27 [Blekherman, 2015] *Given $\beta = \beta^{(2n)}$, it the associated moment matrix $M(n)$ is psd and satisfy $\operatorname{rank}\ M(n) \leq 3n - 3$, then $\Lambda_{\tilde{\beta}}$ is K-positive; thus, $\beta^{(2n-1)}$ has a representing measure.*

5 Core Varieties

Recall the algebraic variety of $M \equiv M(n)$ is $\mathcal{V}(M(n)) = \cap_{p \in \mathcal{P}_n,\, M\hat{p}=\mathbf{0}} \mathcal{Z}(p)$, which we not designate by $\mathcal{V}^{(0)}$. We can easily see that the positivity of M is equivalent to the condition that Λ_β be **square positive**, that is, $\Lambda_\beta(p^2) \geq 0$ $(p \in \mathcal{P}_n)$.

We now introduce $\mathcal{V}^{(1)} := \cap_{p \in \ker \Lambda_\beta,\, p|_{\mathcal{V}^{(0)}} \geq 0} \mathcal{Z}(p)$ as an initial attempt to refine $\mathcal{V}(M(n))$. Now, for $i \geq 0$, let

$$\mathcal{V}^{(i+1)} := \cap_{p \in \ker \Lambda_\beta,\, p|_{\mathcal{V}^{(i)}} \geq 0} \mathcal{Z}(p).$$

We define the **core variety** of β (or of $M(n)$) by $\boldsymbol{V} \equiv \boldsymbol{V}(\beta) := \cap_{i=0}^{\infty} \mathcal{V}^{(i)}$; we also denote this by $\boldsymbol{V}(M)$.

Proposition 28 [Fialkow, 2017]

(i) If β is a representing measure for β, then $\operatorname{supp}\ \mu \subseteq \boldsymbol{V}$.

(ii) If β is a representing measure for β, then $\operatorname{rank}\ M(n) \leq \operatorname{card}\ \boldsymbol{V}$.

(iii) If β is a representing measure for β with $\operatorname{int}(\operatorname{supp}\ \mu) \neq \varnothing$, then $\boldsymbol{V}(\beta) = \mathbb{R}^n$.

A set K in $\mathbb{R}^n$ is called a **determining set** for $\mathcal{P}_m$ if the conditions $p \in \mathcal{P}_m$ and $p|K \equiv 0$ imply $p \equiv 0$. If K has nonempty interior, then K is a determining set, but certain finite sets are also determining sets; see **[Fialkow, 2017]**.

Strict positivity leads to the following existence criterion.

Theorem 29 [Fialkow-Nie, 2010] *For $\beta \equiv \beta^{(m)}$, if K is a determining set for $\mathcal{P}_m$ and Λ_β is strictly K-positive, then β has a K-representing measure.*

The next result provides sufficient conditions for positivity of Λ_β and representing measures for β.

Theorem 30 [Fialkow, 2017] *Let $\beta \equiv \beta^{(2n)}$. If the core variety **V** $\equiv$ **V**(β) is nonempty, then Λ_β is strictly **V**-positive and $\beta^{(2n-1)}$ has a **V**-representing measure. Furthermore, if **V** is nonempty and is either compact or determining set for $\mathcal{P}_{2n}$, then $\beta^{(2n)}$ has a **V**-representing measure.*

Proposition 31 [Fialkow, 2017] *TFAE:*

(i) Λ_β *is strictly positive;*

(ii) $M(n) > 0$ *and* ***V*** $= \mathbb{R}^n$*;*

(iii) ***V*** $= \mathbb{R}^n$*.*

In general, it is difficult to solve nonsingular moment problems, but here is a useful information for a positive definite $M(3)$:

Theorem 32 [Fialkow, 2017] *If* $M(3)$ *is positive definite, then exactly one of the following holds:*

(i) ***V*** $= \mathbb{R}^2$ *and there is a representing measure;*

(ii) card ***V*** $= 10$ *and there is a unique representing measure, whose support is* ***V****;*

(iii) card ***V*** $= 0$ *and there is no representing measure.*

6 Lasserre's Method for Polynomial Optimization

Consider the polynomial optimization problem:

(P) Minimize a polynomial p over a basic closed semi-algebraic set K

$$p_* \equiv p_{\min} := \inf_{\mathbf{x}\in K} \ p(\mathbf{x}),$$

where $K := \left\{\mathbf{x} \in \mathbb{R}^d : h_1(\mathbf{x}) \geq 0, \ldots, h_m(\mathbf{x}) \geq 0\right\}$ and $p, h_1, \ldots, h_m \in \mathbb{R}[\mathbf{x}]$.

For $2t \geq \deg p$, the t-th Lasserre **moment relxation** for (P) is defined by

$$p_t := \inf \ \left\{\Lambda_\beta(p) : \beta \equiv \beta^{(2t)}, \ \beta_0 = 1, \ M_t(\beta) \geq 0\right\}.$$

Then $p_t \leq p_*$, and for $t' \geq t$, $p_{t'} \geq p_t$; thus, $\{p_t\}$ is convergent and $p^{mom} := \lim_{t\to\infty} p_t \leq p_*$. In general, for fixed t, p_t is not necessarily attained at any β.

Assuming that the infimum is attained, at some optimal sequence $\beta \equiv \beta^{\{t\}}$, we seek criteria so that $\Lambda_\beta(p) = p_*$, so that we have finite convergence of $\{p_s\}$ at stage t.

Lasserre's Stopping Criterion

Assume at stage t that $\beta \equiv \beta^{\{w\}}$ has a representing measure μ. Then

$$p_* = p_*\beta_0 = p_* \int 1 \, d\mu \leq \int p \, d\mu = \Lambda_\beta(p) = p_t \leq p_*,$$

so we see the convergence at stage t. Ascertaining the existence of a representing measure for β is difficult in general but Lasserre pays attention to some east-to-check cases such as when $M_t(\beta)$ is flat, that is, $\mathrm{rank}\ M_t(\beta) = \mathrm{rank}\ M_{t-1}(\beta)$. Since β has a $\mathrm{rank}\ M_t(\beta)$-atomic representing measure, the atoms are the global minimizers for p.

We have a bit more general stopping criterion? We need an additional notion: For $\beta \equiv \beta^{(2t)}$ and $0 \leq j \leq m$, the **localizing matrix** $M^{(q_i)} \equiv M^{(q_i)}(\beta) \equiv M_t^{(q_i)}(\beta)$ is defined by $\langle M^{(q_i)} \hat{f}, \hat{g} \rangle = \Lambda_\beta(q_j f g)$ ($f, g \in \mathcal{P}_{t-k_j}$).

Theorem 33 [Fialkow-Nie, 2013] *Suppose* $p_t = \Lambda_\beta(p)$ *for some sequence* $\beta \equiv \beta^{(2t)}$ *for which* $\beta_0 = 1$ *and* $M^{(q_i)}(\beta) \geq 0$ $(0 \leq i \leq m)$*. If* Λ_b *is* K*-positive, then* $p_t = p_*$*.*

7 Embry Truncated Complex Moment Problems

(Based on Jung-Ko-Li-Park, 2003)

Subnormality of an operator

Given a cyclic subnormal bounded operator T on a separable, infinite dimensional, complex Hilbert space $\mathcal{H}$ with cyclic vector x_0, let $\gamma_{ij} := \langle T^{*i}T^j x_0, x_0\rangle$ for any $i, j \in \mathbb{Z}_+$.
The Bran-Halmos' characterization for subnormality of T is the condition that

$$\sum_{0\leq i,j\leq n} \langle T^{*i}T^j p_i(T)x_0, p_j(T)x_0\rangle \geq 0$$

for any $p_i(z) \in \mathbf{P}[z]$ of complex polynomials, $i = 0, 1, \ldots, n$, and for all $n \in \mathbb{N}$. This characterization is equivalent to the fact that $M(n)(\gamma) \geq 0$ for all $n \in \mathbb{N}$, where γ is the truncated complex moment sequence

$$\gamma \equiv \gamma^{(2n)} : \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \ldots, \gamma_{0,2n}, \gamma_{1,2n-1}, \ldots, \gamma_{2n-1,1}, \gamma_{2n,0}$$

with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$. The entry of $M(n) \equiv M(n)(\gamma)$ in row $\bar{Z}^k Z^\ell$ and column $\bar{Z}^i Z^j$ is $\gamma_{\ell+i,j+k}$, that is, $\langle M(n)(\gamma)\bar{Z}^k Z^\ell, \bar{Z}^i Z^j\rangle = \gamma_{\ell+i,j+k}$.

In a parallel approach, Embry described subnormality by the condition that

$$\sum_{0\leq i,j\leq n} \langle T^{*i+j}T^{i+j}p_i(T)x_0, p_j(T)x_0\rangle \geq 0$$

for any $p_i(z) \in \mathbf{P}[\bar{z}, z]$ of complex polynomials, $i = 0, 1, \ldots, n$, and for all $n \in \mathbb{N}$. We will see this condition is equivalent to the fact that $E(n)(\gamma) \geq 0$ for all $n \in \mathbb{N}$, where the **Embry moment matrix** $E(n) \equiv E(n)(\gamma)$ is defined as follows:

As a subcollection of γ, we consider

$$\gamma_E \equiv \{\gamma_{ij}\} \ (0 \leq i + j \leq 2n, \ |i - j| \leq n).$$

We indeed select the following rows and columns of $M(n)$:

$$1, Z, Z^2, \bar{Z}Z, Z^3\bar{Z}Z^2, Z^4, \bar{Z}Z^3, \bar{Z}^2Z^2, Z^5, \ldots$$

Note. $E(n)$ is a submatrix of $M(n)$, that is, when $k \leq \ell$ and $i \leq j$, we see that $\langle E(n)(\gamma)\bar{Z}^kZ^\ell, \bar{Z}^iZ^j\rangle = \langle M(n)(\gamma)\bar{Z}^kZ^\ell, \bar{Z}^iZ^j\rangle$.

For example,

$$E(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix} \qquad \text{and} \qquad E(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

Solution to Embry Truncated Moment Problems

Theorem 34 *Let* $\gamma \equiv \{\gamma_{ij}\}$ *(*$0 \le i+j \le 2n,\ |i-j| \le n$*) be given.*

(i) If n *is an even number, then* γ *has a* $\operatorname{rank}\ E(n)$*-atomic representing measure if and only if* $E(n) \ge 0$ *and* $E(n)$ *admits a flat extensions* $E(n+2)$*.*

(ii) If n *is an odd number, then* γ *has a* $\operatorname{rank}\ E(n)$*-atomic representing measure if and only if* $E(n) \ge 0$ *and* $E(n)$ *admits a flat extensions* $E(n+1)$*.*

Question 6 *Is there any* $M(2)$ *that cannot admit a representing measure but* $E(2)$ *does?*

Question 7 *If* $E(2) > 0$ *has a 4-atomic measure* μ*, then does it admit* $M(2) \ge 0$ *with a 4-atomic measure?*

8 An Operator that Admits a Moment Sequence

Invariant subspace problem

Definition 35 *An operator $T \in \mathcal{L}(\mathcal{H})$* ***admits a moment sequence*** *if there exist nonzero vectors x and y in $\mathcal{H}$ and a (finite, regular, positive) Borel measure μ supported on the spectrum $\sigma(T)$ of T such that for every complex polynomial p,*

$$\langle p(T)x, y\rangle = \int_{\sigma(T)} p(\lambda) d\mu(\lambda).$$

Note If we set $\beta_j := \langle T^j x, y\rangle$ for $j \in \mathbb{Z}_+$, then

$$\langle p(T)x, y\rangle = \int_{\sigma(T)} p(\lambda) d\mu(\lambda) = \Lambda(p),$$

where Λ is the Riesz functional.

Remark If $T \in \mathcal{L}(\mathcal{H})$ has a nontrivial invariant subspace (NIS), then it admits a moment sequence.

Proof. Let $\mu \equiv 0$ on $\sigma(T)$. Also, let $\mathcal{Y}$ in $\mathcal{X}$ be a NIS. Then for any polynomial p

$$p(T)x \in \mathcal{Y} \text{ if } x \in \mathcal{Y}.$$

Thus,

$$\text{there is nonzero } y \in \mathcal{Y}^{\perp} \text{ such that } p(T) \perp y,$$

which leads to

$$\langle p(T)x, y \rangle = 0 = \int_{\sigma(T)} p(\lambda) d\mu(\lambda).$$

Notice that if $p(T)x \notin \mathcal{Y}$, then $\langle p(T)x, y \rangle \neq 0$. ■

Theorem 36 [Atzmon-Goderfroy, 2001] *Suppose $\mathcal{X}$ is a real separable Banach space and T is a bounded linear operator on $\mathcal{X}$ that admits a moment sequence (with associated Borel measure μ supported on $\sigma(T) \subseteq \mathbb{R}$). Then T has a nontrivial invariant subspace.*

Theorem 37 [Foias-Jung-Ko-Pearcy, 2005; Chevreau-Jung-Ko-Pearcy, 2006] *Let $T \in \mathcal{L}(\mathcal{H})$. Then:*

(i) If $T = N + K$ for some normal operator N and some compact operator K, then T admits a moment sequence.

(ii) If $T = S + K$ for some subnormal operator S and some compact operator K, then T admits a moment sequence.

(iii) If T is either nonbiquasitriangular, essentially normal, or hyponormal admits a moment sequence.

Theorem 38 *Suppose $T \in \mathcal{L}(\mathcal{H})$ and $\sigma(T)$ contains at least one isolated point. Then:*

$$T \text{ has an nontrivial invariant subspace} \iff T \text{ admits a moment sequence.}$$

Question 8 *(i) Does every essentially subnormal or essentially hyponormal operator in $\mathcal{L}(\mathcal{H})$ admit a moment sequence?*

(ii) Let T be an invertible operator in $\mathcal{L}(\mathcal{H})$ admitting a moment sequence. Does T^{-1} admit a moment sequence?

(iii) Suppose that $\{T_n\}$ is a sequence of operators in $\mathcal{L}(\mathcal{H})$ admitting a moment sequence and that $\|T - T_0\| \to 0$. Does T_0 admit a moment sequence?

(iv) Does every quasinilpotent operator admits a moment sequence?

9 Subnormal Completion Problem

If $\alpha^{(j)} = \left\{\alpha_m^{(j)}\right\}_{\mathbb{N}_0^d} \in \ell_\infty(\mathbb{N}_0^d)$ for $j = 1, \ldots, d$, then we define the d-variable weighted shift $T = (T_1, \ldots, T_d)$ by

$$T_j \varepsilon_m := \alpha_m^{(j)} \varepsilon_{m+e_j} \ \text{ for } j = 1, \ldots, d.$$

It is easy to check that $T_j T_k = T_k T_j$ if and only if

$$\alpha_{m+e_j}^{(k)} \alpha_m^{(j)} = \alpha_{m+e_k}^{(j)} \alpha_m^{(k)} \ \text{ for } j, k = 1, \ldots, d. \tag{9.1}$$

Given $m \in \mathbb{N}_0^d$, the moment of $\mathcal{C}_\infty = (\alpha^{(1)}, \ldots, \alpha^{(d)})$ of order m is given by

$$s_m = \begin{cases} 1 & \text{if } m = 0_d, \\ (\alpha_{m-e_j}^{(j)})^2 s_{m-e_j} & \text{if } m = (m_1, \ldots, m_d) \text{ where } m_j \neq 0. \end{cases} \tag{9.2}$$

Given a finite collection of positive real numbers

$$\mathcal{C} = \left\{ \alpha_\gamma^{(1)}, \ldots, \alpha_\gamma^{(d)} \right\}_{\gamma \in \Gamma}$$

which satisfies (9.1), where Γ is a lattice set, the **subnormal completion problem in d-variable** entails finding necessary and sufficient conditions for the existence of a subnormal d-variable weighted shift whose initial weights are given by $\mathcal{C}$.

More precisely, given $\mathcal{C}$ as above, we wish to establish the existence of a d-variable weighted shift $T = (T_1, \ldots, T_d)$ such that

$$T_j \varepsilon_\gamma := \alpha_\gamma^{(j)} \varepsilon_{\gamma+e_j} \text{ for } j = 1, \ldots, d \text{ and } \gamma \in \Gamma.$$

The infinite collection of positive numbers $\mathcal{C}_\infty = \left\{ \alpha_\gamma^{(1)}, \ldots, \alpha_\gamma^{(d)} \right\}_{\gamma \in \mathbb{N}_0^d}$ is called a **subnormal completion of $\mathcal{C}$**.

Theorem 39 [Berger's Theorem] *The d-variable weighted shift $T = (T_1, \dots, T_d)$ is subnormal if and only if there is a probability measure ν on the rectangle $R = [0, \kappa_1] \times \cdots \times [0, \kappa_d] \subseteq [0, \infty)^d$, where $\kappa_j = \|T_j\|^2$ for $j = 1, \dots, d$ such that*

$$s_m = \int_R x^m d\nu(x) \quad \text{for } m \in \mathbb{N}_0^d.$$

Cubic Subnormal Completion Problem in 2-variable

Let $\mathcal{C}$ be as in (9.1) with $d = 2$ and $\Gamma = \left\{\gamma \in \mathbb{N}_0^2 : 0 \leq |\gamma| \leq 2\right\}$. The corresponding moment sequence given by (9.2) is $s = \{s_\omega\}_{0 \leq |\omega| \leq 3}$.

Solutions given by Curto-Lee-Yoon and D. Kimsey independently.

10 Interpolation

10.1 Vandermonde Matrix Method for Interpolation

Definition 40 [Indexing Sets]

*(i) A finite set $\Gamma \subseteq \mathbb{N}_0^d$ is called a **lattice set** when for all $\xi \in \Gamma$, there exist $\xi_1 =$ **0**, $\xi_2, \ldots, \xi_k \in \Xi$ and $j_1, \ldots, j_k \in \{1, \ldots, d\}$ such that $\xi_2 = \xi_1 + \mathsf{e}_{j_1}, \ldots, \xi = \xi_k + \mathsf{e}_{j_k}$, where $k = |\xi|$ and e_{j_k} are basis elements for $k = 1, \ldots, d$, that is, every element of ξ is path connected to $\mathbf{0}_d$.*

*(ii) A finite set $\Gamma \subseteq \mathbb{N}_0^d$ is said to be **lower inclusive** if for any $m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$ and $\xi = (\xi_1, \ldots, \xi_d) \in \Gamma$ with $m_j \leq \xi_j$ for $j = 1, \ldots, d$, we get $m \in \Gamma$.*

Note that every lower inclusive set is a lattice set, but the converse is not true.

An index set whose elements correspond to the set of all the monomials in $\mathcal{P}_n^d$ is lower inclusive.

For example,

	Lattice set	Lower inclusive
$\{(0,0),(0,1),(1,0)\}$	Yes	Yes
$\{(0,0),(0,1),(1,1)\}$	Yes	No
$\{(0,0),(1,1),(2,2)\}$	No	No

T. Sauer[1997] investigated the **minimal degree interpolation problem**. This problem is interesting from the fact that interpolating polynomials with small total degree can be stored and computed more easily.

To introduce the problem, we need have some notations and definitions. Let $\mathcal{P}^d$ be the space of polynomials in R^d and let $\mathcal{P}_n^d := \left\{p \in \mathcal{P}^d : \deg p \leq n\right\}$. Given a set of distinct points $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \subseteq \mathbb{R}^d$, the Lagrange interpolation problem with respect to W is **poised** in the subspace $\mathcal{P}(W) \subseteq \mathcal{P}_n^d$ if given any $f : \mathbb{R}^d \to \mathbb{R}$, there exists a unique polynomial $p := L_{\mathcal{P}(W)}(f) \in \mathcal{P}(W)$, where $L_{\mathcal{P}(W)}$ is the interpolating operator of f with respect to $\mathcal{P}(W)$, such that

$$p(\mathbf{x}_i) = f(\mathbf{x}_i), \qquad i = 1, \ldots, k.$$

The minimal degree interpolation problem is to seek a subspace $\mathcal{P}(W) \subseteq \mathcal{P}_n^d$ with n as small as possible so that the Lagrange interpolation problem, with respect to the set of distinct points W, is poised. Moreover, we require that $\mathcal{P}(W)$ is degree reducing, that is, whenever $q \in \mathcal{P}_m^d$ for $m \leq n$, we get $L_{\mathcal{P}(W)}(q) \in \mathcal{P}_m^d$.

T. Sauer[1997] provided an algorithm for a unique minimal degree interpolation subspace $\widetilde{\mathcal{P}(W)} \subseteq \mathcal{P}_n^d$ for a set of distinct points $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \subseteq \mathbb{R}^d$.

Note that Example 42 will illustrate that even though all the points are distinct, multivariable Vandermonde matrices could be singular unlike the case of one-variable.

The set of indices for the monomials generated by Sauer's algorithm, say $\Gamma = \{\lambda_1, \ldots, \lambda_k\} \subseteq \mathbb{N}_0^d$, corresponds to a lower inclusive set. Given a function $f : \mathbb{R}^d \to \mathbb{R}$, the coefficients of the unique interpolating polynomial $p \in \widetilde{\mathcal{P}(W)}$ are evaluated by the following equation:

$$V(\mathbf{w}_1, \ldots, \mathbf{w}_k; \Gamma) \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} f(\mathbf{w}_1) \\ \vdots \\ f(\mathbf{w}_k) \end{pmatrix}, \text{where } V(\mathbf{w}_1, \ldots, \mathbf{w}_k; \Gamma) := \begin{pmatrix} \mathbf{w}_1^{\lambda_1} & \cdots & \mathbf{w}_1^{\lambda_k} \\ \vdots & \ddots & \vdots \\ \mathbf{w}_k^{\lambda_1} & \cdots & \mathbf{w}_k^{\lambda_k} \end{pmatrix}.$$

Note that the multivariable Vandermonde matrix $V(\mathbf{w}_1, \ldots, \mathbf{w}_k; \Gamma)$ is invertible. Thus, the construction of $\widetilde{\mathcal{P}(W)}$ may be considered as selecting some columns of such a multivariable Vandermonde matrix associated to $\mathcal{P}(W)$.

Theorem 41 [Sauer, 1997] *Given a set of distinct points $W = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k\} \subseteq \mathbb{R}^d$, there is a lower inclusive set $\Gamma \subseteq \mathbb{N}_0^d$, with $\operatorname{card} \Gamma = k$, so that $V(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k; \Gamma)$ is invertible.*

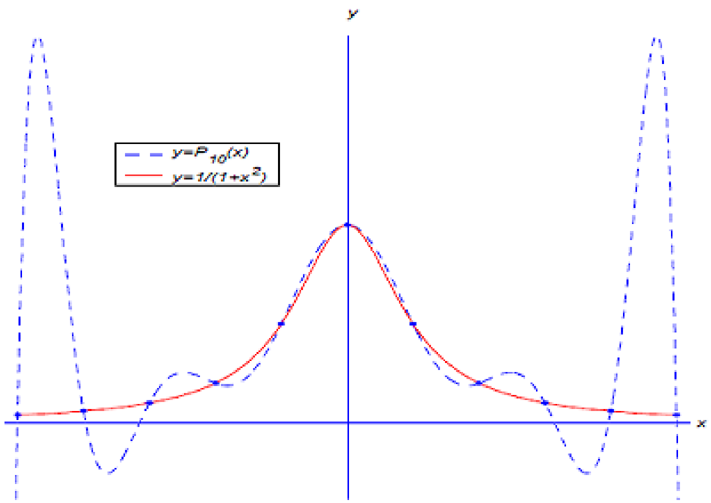
10.2 Runge's Example

Example 42 *Consider to find an interpolating polynomial to approximate the rational function* $f(x) = \frac{1}{1+x^2}$*. We use evenly spaced ten nodes*

$$N = \{(0,0),(\pm 1, 1/2),(\pm 2, 1/5),(\pm 3, 1/10),(\pm 4, 1/17),(\pm 5, 1/26)\}$$

and the Lagrange interpolation formula gives

$$P_{10} = (44200 - 29800x^2 + 8724x^4 - 1079x^6 + 56x^8 - x^{10})/44200$$



Interpolation with a Vandermonde Matrix:

For the set of 11 points $N \equiv \{(x_1, y_1), \cdots, (x_{11}, y_{11})\}$*, consider the multivariable Vandermonde matrix* $V(N)$*:*

$$V(N) = \begin{pmatrix} 1 & x_1 & y_1 & {x_1}^2 & y_1 x_1 & {y_1}^2 & \cdots & {x_1}^4 & x_1^3 y_1 & \cdots & x_1 y_1^3 & y_1^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{11} & y_{11} & {x_{11}}^2 & y_{11} x_{11} & {y_{11}}^2 & \cdots & {x_{11}}^4 & x_{11}^3 y_{11} & \cdots & x_{11} y_{11}^3 & y_{11}^4 \end{pmatrix}$$

The row reduced form of $V(N)$ *is*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1253}{44200} & 0 & -1 & 0 & 0 & 0 & -\frac{5121229}{97682000} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1194 & 0 & 0 & 0 & -\frac{59}{44200} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{12307}{44200} & 0 & 1 & 0 & 0 & 0 & \frac{48701521}{97682000} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{59}{44200} & 0 & 1 & 0 & 0 & 0 & \frac{245177}{97682000} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 11054 & 0 & 1 & 0 & \frac{1253}{44200} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -\frac{25357}{22100} & 0 & 0 & 0 & 1 & 0 & -\frac{46361333}{24420500} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 58 & 0 & 0 & 0 & \frac{1}{44200} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -39660 & 0 & -1 & 0 & -\frac{5527}{22100} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{4193}{2210} & 0 & 0 & 0 & -1 & 0 & \frac{2994338}{1221025} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{44200} & 0 & 0 & 0 & 0 & 0 & -\frac{4193}{97682000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 44200 & 0 & 0 & 0 & \frac{1983}{2210} & 0 \end{pmatrix}$$

We may label the columns in $V(N)$ as we did for $M(n)(\beta)$ and all the column relations have corresponding polynomials; for example, the column relation in the 8th column (X^2Y) can be written as $X^2Y = 1 - Y$, whose corresponding polynomial is $x^2y = 1 - y$, equivalently, $y = 1/(1 + x^2)$. Also, from 15th we have the polynomial $y^4 = -\frac{1253}{44200} + \frac{12307}{44200}y + \frac{59}{44200}x^2 - \frac{25357}{22100}y^2 + \frac{4193}{2210}y^3 - \frac{1}{44200}x^4$, and so on. All these polynomials vanish on N, and so they are interpolating polynomials with respect to N.

Interpolation with a Moment Matrix:

Let us find a two variable polynomial passing through the 11 nodes. First, define the moments

$$\beta_{ij} := \sum_{k=-5}^{5} k^i f(k)^j \left(= \int x^i y^j d\mu \right),$$

where $\mu = \sum_{k=-5}^{5} \delta_{(k,f(k))}$.

Now build the moment matrix $M(3)(\beta)$ with the moments:

$$M(3)(\beta) = \begin{pmatrix} 11 & 0 & \frac{3088}{1105} & 110 & 0 & \frac{786281}{488410} & 0 & \frac{9067}{1105} & 0 & \frac{6846118999}{5396930500} \\ 0 & 110 & 0 & 0 & \frac{9067}{1105} & 0 & 1958 & 0 & \frac{115723}{97682} & 0 \\ \frac{3088}{1105} & 0 & \frac{786281}{488410} & \frac{9067}{1105} & 0 & \frac{6846118999}{5396930500} & 0 & \frac{115723}{97682} & 0 & \frac{13459008802027}{11927216405000} \\ 110 & 0 & \frac{9067}{1105} & 1958 & 0 & \frac{115723}{97682} & 0 & \frac{112483}{1105} & 0 & \frac{1842286051}{5396930500} \\ 0 & \frac{9067}{1105} & 0 & 0 & \frac{115723}{97682} & 0 & \frac{112483}{1105} & 0 & \frac{1842286051}{5396930500} & 0 \\ \frac{786281}{488410} & 0 & \frac{6846118999}{5396930500} & \frac{115723}{97682} & 0 & \frac{13459008802027}{11927216405000} & 0 & \frac{1842286051}{5396930500} & 0 & \frac{28024033625192083}{26359148255050000} \\ 0 & 1958 & 0 & 0 & \frac{112483}{1105} & 0 & 41030 & 0 & \frac{3428999}{488410} & 0 \\ \frac{9067}{1105} & 0 & \frac{115723}{97682} & \frac{112483}{1105} & 0 & \frac{1842286051}{5396930500} & 0 & \frac{3428999}{488410} & 0 & \frac{1670914185763}{11927216405000} \\ 0 & \frac{115723}{97682} & 0 & 0 & \frac{1842286051}{5396930500} & 0 & \frac{3428999}{488410} & 0 & \frac{1670914185763}{11927216405000} & 0 \\ \frac{6846118999}{5396930500} & 0 & \frac{13459008802027}{11927216405000} & \frac{1842286051}{5396930500} & 0 & \frac{28024033625192083}{26359148255050000} & 0 & \frac{1670914185763}{11927216405000} & 0 & \frac{12016344901455416423}{11650743528732100000} \end{pmatrix}$$

We then use the row reduction and identify a column relation:

$$X^2Y = 1 - Y,$$

which is exactly equal to $y = 1/(1 + x^2)$ in the polynomial level.

Invertible Vandermonde Matrices (Based on Curto-Fialkow, 2013)

Lemma 43 *For $N \geq 1$, let $v_1, \ldots, v_N$ be distinct points in $\mathbb{R}^2$, and consider the multivariable Vandermonde matrix $V_N := (v_i^\alpha)_{1 \leq i \leq N,\, \alpha \in \mathbb{Z}_+^2,\, |\alpha| \leq N-1}$, of size $N \times N(N+1)/2$. Then the rank of V_N equals N.*

Corollary 44 *Let **x** $\equiv \{x_1, \ldots, x_m\}$ and **y** $\equiv \{y_1, \ldots, y_n\}$ be sets of distinct real numbers, and consider the grid **x** $\times$ **y** $:= \{(x_i, y_j)\}_{1 \leq i \leq m,\, 1 \leq j \leq n}$ consisting of $N := mn$ distinct points in $\mathbb{R}^2$. Then the generalized Vandermonde matrix $V_{\boldsymbol{x} \times \boldsymbol{y}}$, obtained from V_N by removing all columns indexed by monomials divisible by x^m or y^n, is invertible.*

Corollary 45 *Let $G \equiv$ **x** $\times$ **y** be a grid as in Corollary 44, let $N := mn$, and let $p \in \mathbb{R}[x, y]$ be such that $\deg_x p < m$ and $\deg_y p < n$. Assume also that $p|_G \equiv 0$. Then $p \equiv 0$.*

Proposition 46 *Let $P(x, y) := (x - x_1) \cdots (x - x_d)$ and let $Q(x, y) := (y - y_1) \cdots (y - y_d)$. If $\rho := \operatorname{multideg}(f) \geq d$ and $f|_{\mathcal{V}((P,Q))} \equiv 0$, then there exists $u, v \in \mathcal{P}_{\rho-d}$ such that $f = uP + vQ$.*

10.3 Pencil Problem

According to the Bézout's theorem, two non-degenerate conics may intersect at most 4 different points, naturally so do 3 conics. For example,

$$x^2 = 1,\ y^2 = 1,\ \text{and } x^2 + y^2 = 2$$

intersect at 4 points $\{(\pm 1, \pm 1)\}$ in $\mathbb{R}^2$.

However, if we impose a condition, we may show:

Theorem 47 *Any three conics with different leading terms cannot intersect at 4 distinct points.*

Question 9 *Can we prove the following?:*

> *There is no set of $n+1$ bivariate polynomials of degree n with all different leading terms such that their intersection has n^2 or more distinct points.*

11 Minimal Quadrature Rules on a Parabolic Arc

We describe the minimal quadrature rules of degree 4 for arc-length measure ν on the segment of the parabola $y = x^2$ corresponding to $0 \leq x \leq 1$. Let

$$K := \left\{ (x,y)\mathbb{R}^2 : y = x^2,\ 0 \leq x \leq 1 \right\}.$$

By a K-**quadrature rule** for ν of degree 4 we mean a finite collection of points of K, $(x_0, y_0), \ldots, (x_d, y_d)$, and corresponding positive weights, $\omega_0, \ldots, \omega_d$ such that for every real polynomial $p(x,y)$ with $\deg p \leq 4$,

$$\int_K p(x,y)\, d\nu(x,y) \equiv \int_0^1 p(t,t^2)\sqrt{1+4t^2}\, dt = \sum_{i=0}^{d} \omega_i p(x_i, y_i);$$

a minimal quadrature rule is one for which d is as small as possible.

First, complexify:

$$\gamma_{kj} = \int_K \bar{z}^k z^j \, d\nu = \int_0^1 (t - it)^k (t + it)^j \sqrt{1+4t^2}\, dt, \qquad 0 \leq k + j \leq 4.$$

Each $\gamma_{kj} \in \mathbb{Q}[i, \sqrt{5}, \ln(2+\sqrt{5})]$. Since $M(2)(\gamma)$ has a representing measure (namely, ν), it follows that $M(2)(\gamma) \geq 0$. Also, $\{\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$; moreover $y = x^2$ means

$$Z^2 + 2\bar{Z}Z + \bar{Z}^2 + 2iZ - 2i\bar{Z} = \mathbf{0}.$$

To each $\gamma_{23} \equiv r + is$ ($r, s \in \mathbb{R}$) there exists a unique moment matrix block $B(3)[\gamma_{23}]$ satisfying $\mathrm{Ran}\ M(2)$; moreover, γ_{23} gives rise to a flat extension $M(3)$ if and only if the relation $C_{21} = C_{32}$ holds in the $C(3)$-block.

12 Image Reconstruction

Images are nonstationary two-dimensional signals with edges, textures, and deterministic objects at different locations.

An image can be considered as an element of a vector space, so we may represent it as a linear combination of the elements of any non necessarily orthogonal basis of this space.

Once an image is viewed as finite moments (datum), we may attempt to find a way to represent partial datum (it is the goal of the “truncated” moment problem); an approximation to the full datum can be discovered for the image reconstruction.

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