INTRODUCTION TO ROKHLIN PROPERTY FOR C^*-ALGEBRAS

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ABSTRACT. This note serves as a material for a 5-hour lecture series presented at 2017 Winter School of Operator Theory and Operator Algebras which is an annual meeting among functional analysis people in Korea. The intent of this note is to introduce Rokhlin property in C^*-algebras which is more delicate and dynamical than the case of von Neumann algebras. All facts in this note are already known to experts, and this note is not a comprehensive summary of the developments in this area which is still rapidly developing but to provide classical facts which somebody must know to start a research in this area.

1. AN OVERVIEW AND TWO IMPORTANT PROBLEMS

In this lecture, a group is always assumed to be discrete. But for the definition of the group action on a C^*-algebra a group needs to be equipped with a suitable topology, or a topological group. Throughout this note $\mathrm{Aut}(A)$ will stand for the group of automorphisms of a C^*-algebra A.

Definition 1.1. An automorphism α on a unital C^*-algebra is called inner if there exists a unitary u in A such that $\alpha = \mathrm{Ad}\, u$ where $\mathrm{Ad}\, u(a) = uau^*$. α is called outer if it is not inner. We denote by $\mathrm{Inn}(A)$ the group of inner automorphisms of A and by $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$ the set of outer automorphisms.

Definition 1.2. Two automorphisms $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$ are conjugate if there is an isomorphism γ from A to B such that $\gamma \circ \alpha = \beta \circ \gamma$. In particular, when $A = B$, then α and β are said to be outer conjugate if there are a unitary u and another automorphism γ such that $\alpha = \mathrm{Ad}\, u \circ \gamma \circ \beta \circ \gamma^{-1}$.

Definition 1.3. Given a single automorphism α on a C^*-algebra A, by considering α^n for every $n \in \mathbb{Z}$, we naturally have a $\mathbb{Z}$-action on A and vice versa. Thus we say that α has the Rokhlin property in this case if for every finite subset F, a positive number ϵ, and any natural number k there exists a partition of unity $\{e_i\}_{i=0}^{k-1} \cup \{f_j\}_{j=0}^{k}$ by projections such that

(1) $\|e_i a - a e_i\| \le \epsilon$ for $i = 0, \dots, k-1$, $\|f_j a - a f_j\| \le \epsilon$ for $j = 0, \dots, k$,
(2) $\|\alpha(e_i) - e_{i+1}\| \le \epsilon$ for $0 \le i \le k-2$, $\|\alpha(f_j) - f_{j+1}\| \le \epsilon$ for $0 \le j \le k-1$,
(3) $\|\alpha(e_{k-1} + f_k) - (e_0 + f_0)\| \le \epsilon$.

Remark 1.4. In view of the classical notion in Dynamical system or Connes' definition on von Neumann algebras, the above definition is weaker than the usual one which means that for any integer k and $\epsilon > 0$ we have a partition of unity consisting of projections $e_0, \dots, e_k$ in A such that $\|\alpha(e_i) - e_{i+1}\| < \epsilon$ for $k = 0, \dots, k$ where $e_{k+1} = e_0$.

Proposition 1.5. *When α has the Rokhlin property, then α is outer. Moreover, for any nonzero integer n α^n is outer.*

Proof. Suppose that α is inner and is of the form $\mathrm{Ad}(u)$. Take $F = \{u\}$, $\epsilon = 1/3$, $k = 1$. Then there are two mutually orthogonal projections f_0, f_1 such that $\|f_i u - u f_i\| \leq 1/3$ for $i = 0, 1$ and $\|\alpha(f_0) - f_1\| \leq 1/3$. Note that $\|f_0 - f_1\| \geq 1$. But

$$2/3 \geq \|f_0 - u f_0 u^*\| + \|u f_0 u^* - f_1\| \geq \|f_0 - f_1\| \geq 1$$

which is a contradiction. We can also show that α^2 is outer; if α^2 is inner, write it as $\mathrm{Ad}(u)$ for some unitary element u. Then take $F = \{u\}$, $\epsilon = 1/4$, $k = 2$. Then there are three mutually orthogonal projections f_0, f_1, f_2 such that

(1) $\|u f_i - f_i u\| < 1/4$ for $i = 0, 1, 2$,
(2) $\|\alpha(f_i) - f_{i+1}\| < 1/4$ for $i = 0, 1$.

Then

$$3/4 \geq \|f_0 - u f_0 u^*\| + \|u f_0 u^* - \alpha^2(f_0)\| + \|\alpha^2(f_0) - \alpha(f_1)\| + \|\alpha(f_1) - f_2\| \geq \|f_0 - f_2\| \geq 1,$$

which is a contradiction. In this way, we can conclude that α^n is outer for any nonzero positive integer n.

For any nonzero integer $m = -n$ where $n > 0$, suppose that α^m is inner, then write $\alpha^m = \mathrm{Ad}(u)$. Then $a = u(\alpha^n(a))u^*$. It follows that $u^* a u = \alpha^n(a)$ which contradicts that α^n is outer. So we are done. □

One of most important and deepest results in the noncommutative dynamical system is the following due to A. Kishimoto.

Theorem 1.6. *Let A be a simple C^*-algebra and α be an automorphism of A. Then, α is outer if and only if for any nonzero hereditary C^*-algebra $B \subset A$ and for any element $a \in \mathcal{M}(A)$ the following holds:*

$$\inf\{\|x a \alpha(x)\| \mid 0 \leq x \in B, \|x\| = 1\} = 0.$$

Definition 1.7. As a $\mathbb{Z}$-action on A, we say that an automorphism α is aperiodic, if α^n is outer for any nonzero integer n.

Then two important problems related to Rokhlin property are the following;

(1) Rokhlin type theorem; characterization of the Rokhlin property of an automorphism or determine the conditions when α has the Rokhlin property.
(2) Classification of automorphisms with the Rokhlin property up to outer conjugacy.

For a single automorphism case, the following is a prototype result as Rokhlin type theorem due to A. Kishimoto. Note that a C^*-algebra A is called *AT* algebra if for every finite subset F and a positive ϵ, there is a unital C^*-subalgebra B of the form $\oplus_{i=1}^n C(\mathbb{T}, M^i_{n_i}(\mathbb{C}))$ for some n such that

$$\mathrm{dist}(F, B) \leq \epsilon.$$

Theorem 1.8. *Let A be a unital simple AT algebra of real rank zero with unique tracial state τ, and α be an approximately inner automorphism of A. Then the following conditions are equivalent:*

(1) *α has the Rokhlin property.*
(2) *α^m is uniformly outer for every nonzero integer m.*
(3) *α extends to an aperiodic automorphism of the weak closure of the GNS representation of τ.*

(4) *The crossed product* $A \rtimes_\alpha \mathbb{Z}$ *has a unique tracial state.*
(5) *The crossed product* $A \rtimes_\alpha \mathbb{Z}$ *has real rank zero.*

Another one in the purely infinite case the following is known due to H. Nakamura. Note that a unital simple C^*-algebra A is said to be purely infinite if for every nonzero element $x \in A$, there exist a, b in A such that $axb = 1$.

Theorem 1.9. *Let A be a separable nuclear unital purely infinite simple C^*-algebra and α be an automorphism of A. Then the following conditions are equivalent.*

(1) α *is aperiodic.*
(2) α *has the Rokhlin property.*

Related to classification, the following is also a milestone result in this direction.

Theorem 1.10. *Let A be a unital simple AT algebra of real rank zero with a unique tracial state. Let α and β be approximately inner automorphism with the Rokhlin property such that $\beta^{-1}\alpha$ is asymptotically inner. Then there exist γ which is asymptotically inner and a unitary u in A such that*

$$\alpha = \mathrm{Ad}(u) \cdot \gamma \cdot \beta \cdot \gamma^{-1}.$$

In particular, α and β are outer conjugate.

Thanks to Kirchberg and Phillips' result, $[\alpha] = [\beta]$ in $\mathrm{KK}(A, A)$ if and only if $\beta^{-1}\alpha$ is asymptotically inner. In addition, the group of asymptotically inner automorphisms coincides with the group of approximately inner automorphisms.

Theorem 1.11. *Let A be a separable nuclear unital purely infinite simple C^*-algebra and α and β be aperiodic automorphisms of A. If α and β are the same class in $KK(A, A)$, there exist γ which is approximately inner and a unitary u in A such that*

$$\alpha = \mathrm{Ad}(u) \cdot \gamma \cdot \beta \cdot \gamma^{-1}.$$

In particular, α and β are outer conjugate.

Now we analyze the methods for classification results. We may call this strategy as Kishimoto's strategy for outer conjugacy. First we phrase so called homotopy lemma which says that an almost central unitary path can be replaced by an almost central and rectifiable one of length smaller than a universal constant without changing its end points.

Theorem 1.12 (Basic Homotopy Lemma)**.** *For any given $\epsilon > 0$ there is a $\delta > 0$ with the following holds ; let u and v be two unitaries in A with certain properties such that*

$$[v]_1 = 0, \|vu - uv\| < \delta, Bott(u, v) = 0.$$

Then there exist a path of unitaries (v_t) such that

(1) $v_0 = v$, $v_1 = 1$,
(2) $\|v_t u - u v_t\| < \epsilon$,
(3) $\mathrm{Length}(v_t) \leq 5\pi + 1$.

Theorem 1.13. *For any finite(compact) subset F of simple unital AT algebra A and $\epsilon > 0$ there exists a finite(compact) subset G of A and $\delta > 0$ satisfying the following conditions; For any unitary path $u \in C[0,1] \otimes A$ such that*

$$\|u_t x - x u_t\| < \delta, \quad x \in G$$

there is a path of unitaries $v \in C[0,1] \otimes A$ such that

$$v_0 = u_0, \quad v_1 = u_1, \quad \|v_t x - x v_t\| < \epsilon, \quad x \in F$$

and the length of v_t is bounded by a universal constant C.

Proof. We may assume that $u_0 = 1$. Since A is an increasing union of A_n's, where $A_n = C(\mathbb{T}) \otimes B_n$ and $z_n = z \otimes id_n$ is the canonical unitary of A_n, we can assume that F consists of z_1 and the unit ball of B_1. For $\epsilon > 0$, we can take $G = F$ and a sufficiently small δ, which comes from Basic Homotopy Lemma as follows: If $u \in C[0,1] \otimes A$ is a path of unitaries with

$$u_0 = 1, \quad \|[u_t, x]\| < \delta \quad x \in F,$$

then there is a path unitaries $v \in (C[0,1] \otimes (A \cap B_1')$ with

$$v_0 = 1, \quad v \approx u, \quad \|[v, 1 \otimes z_1]\| \approx 0 \quad \text{(up to the order of } \delta).$$

In fact, in a matrix picture v_t is 1 in B_1 part because u_t is close to 1 in B_1 and we modify u_t to be precisely 1 in B_1 without affecting z_1. Then we apply Basic Homotopy Lemma to $A \cap B_1'$ and v_1 and z_1 to obtain a path of unitaries w_t in $C[0,1] \otimes (A \cap B_1')$ which connects 1 to v_1 and commutes with z_1 up to ϵ and the length is bounded by $5\pi + 1$ since $Bott(z_1, v_1) = Bott(z_1, v_0 = 1) = 0$. Finally, v_1 is close to u_1, so does w_1 hence we can connect w_1 and u_1 by a short path. □

Theorem 1.14. *(Stability of automorphism with the Rokhlin property) If A is simple unital AT algebra of real rank zero and α be an automorphism with the Rokhlin property, then α has the stability: For a finite or compact subset F of A and ϵ, there exist a compact set G and a positive number $\delta > 0$ satisfying the following: If $v \in C[0,1] \otimes A$ a path of unitaries such that*

$$v(0) = 1, \quad \|[v_t, x]\| < \delta \quad x \in G, t \in [0,1],$$

then there is a unitary $u \in A$ such that

$$\|v(1) - u\alpha(u^*)\| < \epsilon, \quad \|[u, x]\| < \epsilon \quad x \in F.$$

Proof. Once we establish homotopy type lemma, then there is a routine argument as follows. Given (F, ϵ), choose N such that $6\pi/N - 1 < \epsilon/2$ and ϵ_1 such that $\epsilon_1 N$ is very small (In the course of the proof, we will see how small ϵ_1 is.) For $(\cup_{k=0}^{N} \alpha^k(F), \epsilon_1)$ we apply Theorem 1.14 to obtain (G_1, δ_1) satisfying

Now for a path of unitaries v, we define

$$v^{(k)}(s) = \begin{cases} v(s)\alpha(v(s)) \cdots \alpha^{k-1}(v(s)) & k \geq 1, \\ 1 & k = 0 \end{cases}$$

Once we have $v, \alpha, \epsilon_1, N, F$ we can choose (G, δ) such that

$$\|[v(s), y]\| < \delta \quad \forall y \in G$$

implies

$$\|[v^{(k)}(s), x]\| < \min\{\delta_1, \epsilon_1\} \quad x \in G_1 \cup F$$

for $k = 1, \dots, N+1$. (In fact, $G = \cup_{k=0}^{N} \alpha^{-k}(G_1 \cup F)$, $\delta = \frac{\min\{\delta_1, \epsilon_1\}}{N+1}$ would work.)

We apply Theorem 1.14 to $v^{(N)}$ to obtain a path of unitaries $w^0(s)$

(1) $\|[w^0(s), x]\| < \epsilon_1$ for $x \in \cup_{k=0}^{N} \alpha^k(F)$,
(2) $\|w^0(s) - w^0(t)\| < 6\pi|s-t|$.

Similarly, apply Theorem 1.14 to $v^{(N+1)}$ to obtain a path of unitaries $w^1(s)$

(1) $\|[w^1(s), x]\| < \epsilon_1$ for $x \in \cup_{k=0}^{N} \alpha^k(F)$,
(2) $\|w^1(s) - w^1(t)\| < 6\pi|s-t|$.

Since α has the Rokhlin property, there is a partition of unity consisting of projections $e_0, \dots e_{N-1}$, $f_0, \dots, f_N$ which almost commute with

$$\{\alpha^{i+1-N}(w^0(\frac{i}{N-1})) \mid i = 0, \dots, N-1\},$$

$$\{\alpha^{j-N}(w^1(\frac{j}{N})) \mid j = 0, \dots, N\},$$

$\{v^{(k)}(1) \mid k = 1, \dots, N+1\}$, and F up to ϵ_1.

Then we define

$$U = \sum_{i=0}^{N-1} v^{(i+1)}(1)\alpha^{i+1-N}\left(w^0\left(\frac{i}{N-1}\right)^*\right)e_i + \sum_{j=0}^{N} v^{(j+1)}(1)\alpha^{j-N}\left(w^1\left(\frac{j}{N}\right)^*\right)f_j.$$

Note that $\alpha(e_{N-1} + f_N) = e_0 + f_0$. It follows that

$$\begin{aligned}
U\alpha(U^*) &= v(1)e_0 + \\
&\sum_{k=1}^{N-1} v^{(k+1)}(1)\alpha^{k+1-N}(w^0(k/N-1)^*)e_k\alpha^{k+1-N}(w^0(k-1/N-1))\alpha(v^{(k)}(1)^*) + \\
&= \sum_{k=1}^{N} v^{(k+1)}(1)\alpha^{k-N}(w^1(k/N)^*)f_k\alpha^{k-N}(w^1(k-1/N))\alpha(v^{(k)}(1)^*) \\
&+ v(1)f_0 \\
&\approx^{\frac{6\pi}{N-1}\times 2\epsilon_1} v(1)(e_0 + \dots e_{N-1} + f_0 \dots + f_N) = v(1)
\end{aligned}$$

Also U is also almost unitary by the same reasoning, which means if we take the polar decomposition we get the unitary u which we want. □

Then we proceed to show the outer conjugacy of two automorphisms α and β having Rokhlin property by so called Evans-Kishimoto intertwining argument. Suppose that α and β are asymptotically inner, then there is a path of unitaries $(u_t)_{t\geq 0}$ such that $\alpha = \lim_{t\to\infty} \mathrm{Ad}(u_t) \circ \beta$. We take an increasing union of finite sets $B_1 \subset B_2 \subset \cdots$ such that $A = \overline{\cup_{n=1}^{\infty} B_n}$.

Choose $t_1 > 0$ for B_1 such that on B_1 for all $t \geq t_1$

$$\alpha \approx_{\epsilon/2} \mathrm{Ad}(u_t) \circ \beta.$$

Now take $u^{(0)}(t) = u(t)$. We apply the stability result to β for the date $(\emptyset, \epsilon/2, u^{(0)}(t_1\cdot))$ and obtain unitaries u_1 and ω_1 such that

$$u^{(0)}(t_1) = \omega_1 u_1 \beta(u_1^*)$$

where $\|1 - \omega_1\| < \epsilon/2$. Note that

$$\alpha \approx_{\epsilon/2} \operatorname{Ad}\omega_1 \circ \operatorname{Ad} u_1 \circ \beta \circ \operatorname{Ad} u_1^*$$

We define

$$u^{(1)}(t) = u^{(0)}(t + t_1)(u^{(0)}(t_1))^* \tag{1}$$

$$\beta_1 = \operatorname{Ad} u^{(0)}(t_1) \circ \beta \tag{2}$$

Then

$$\lim_{t\to\infty} \operatorname{Ad}(u^{(1)}(t))^* \circ \alpha = \beta_1. \tag{3}$$

We choose $t_2 > t_1 > 0$ such that on B_2 for $t \geq t_2$

$$\beta_1 \approx_{\epsilon/4} \operatorname{Ad}(u^{(1)}(t))^* \circ \alpha$$

Also,

$$\beta_1 \approx_{\delta_1/2} \operatorname{Ad}(u^{(1)}(t))^* \circ \alpha \tag{4}$$

for $G_1 = \beta_1^{-1}(G_1')$ where (G_1', δ_1) comes from the stability result for $(\epsilon/2^3, F_1 = B_1 \cup \{u_1\})$ with respect to β_1. Then we define

$$u^{(2)}(t)^* = u^{(1)}(t + t_2)^* \cdot u^{(1)}(t_2) \tag{5}$$

$$\alpha_2 = \operatorname{Ad} u^{(1)}(t_2)^* \circ \alpha \tag{6}$$

Then

$$\operatorname{Ad} u^{(2)}(t)^* \circ \alpha_2 = \operatorname{Ad} u^{(1)}(t + t_2)^* \circ \alpha$$

As $t \to \infty$, (RHS) goes to β_1.

$$\alpha_2 = \lim_{t\to\infty} \operatorname{Ad} u^{(2)}(t) \circ \beta_1 \tag{7}$$

Again, we apply to the stability result to $(\emptyset, \epsilon/4)$ for $u^{(1)}(t_2\cdot)^*$ for $(\emptyset, \delta_1)$ and obtain ω_2 and u_2 such that

$$u^{(1)}(t_2)^* = \omega_2 u_2 \alpha(u_2^*) \tag{8}$$

where $\|1 - \omega_2\| < \epsilon/4$. Note that

$$\operatorname{Ad}\omega_2 \circ \operatorname{Ad} u_2 \circ \alpha \circ \operatorname{Ad} u_2^* \approx_{\epsilon/4} \beta_1 \tag{9}$$

From (7), we find $t_3 > 0$ such that

$$\|\alpha_2(x) - (\operatorname{Ad} u^{(2)}(t) \circ \beta_1)(x)\| < \epsilon/8 \quad x \in B_3 \tag{10}$$

$$\|\alpha_2(y) - (\operatorname{Ad} u^{(2)}(t) \circ \beta_1)(y)\| < \delta_2/2 \tag{11}$$

for $y \in G_2 = \alpha_1^{-1}(G_2')$ where (G_2', δ_2) comes from the stability result for $(\epsilon/2^4, F_2 = B_2 \cup \{u_2, u_0\})$ with respect to α_1. Then we define

$$u^{(3)}(t) = u^{(2)}(t + t_3) \cdot u^{(2)}(t_3)^* \tag{12}$$

$$\beta_3 = \operatorname{Ad} u^{(2)}(t_3)^* \circ \beta_1 \tag{13}$$

Then

$$\mathrm{Ad}\, u^{(3)}(t) \circ \beta_3 = \mathrm{Ad}\, u^{(2)}(t + t_3) \circ \beta_1$$

As $t \to \infty$, (RHS) goes to α_2.

$$\beta_3 = \lim_{t\to\infty} \mathrm{Ad}\, u^{(3)}(t)^* \circ \alpha_2 \tag{14}$$

Using (4) for $t \geq 0$

$$\|\beta_1(y) - (\mathrm{Ad}\, u^{(1)}(t + t_2)^* \circ \alpha)(y)\| < \delta_1/2$$

for $y \in G_1$. Replacing $u^{(1)}(t+t_2)^*$ and α by $\mathrm{Ad}\, u^{(1)}(t+t_2)^* \,\mathrm{Ad}\, u^{(1)}(t_2)$ and $\mathrm{Ad}\, u^{(1)}(t_2)^*\alpha$ respectively we get $t \geq 0$

$$\|\beta_1(y) - (\mathrm{Ad}\, u^{(2)}(t) \circ \alpha_2)(y)\| < \delta_1/2$$

for $y \in G_1$. In particular, when $t = 0$, $\|\beta_1(y) - \alpha_2(y)\| < \delta_1/2$. Combining these two estimates, we have for $y \in G_1$

$$\|[\beta_1(y), u^{(2)}(t)\| < \delta_1$$

Thus we apply the stability result to obtain unitaries ω_3, u_3 such that

$$\|u^{(2)}(1) - u_3\beta_1(u_3^*)\| < \epsilon/8 \tag{15}$$

$$\|[u_3, x]\| < \epsilon/8 \quad x \in F_1 \tag{16}$$

$$u^{(2)}(t_3) = \omega_3 u_3 \beta_1(u_3^*) \quad , \quad \|1 - \omega_3\| < \epsilon/8 \tag{17}$$

We repeat this process and obtain

$$\beta_1, \alpha_2, \beta_3, \alpha_4, \ldots \tag{18}$$

$$u_1, u_2, u_3, \ldots \tag{19}$$

$$\omega_1, \omega_2, \omega_3, \ldots \tag{20}$$

with required properties. Since $\|[u_n, x]\| < 2^{-n}\epsilon$, $x \in F_n = B_n \cup \{u_k \mid k = n-2, n-4, \ldots\}$ it follows that

$$\lim \mathrm{Ad}(u_{2k+1}u_{2k-1}\cdots u_1) = \gamma_1 \tag{21}$$

$$\lim \mathrm{Ad}(u_{2k}u_{2k-2}\cdots u_2) = \gamma_0 \tag{22}$$

converge as automorphisms. Then from $\alpha_{2k} = \mathrm{Ad}\,\omega_{2k} \circ \mathrm{Ad}\, u_{2k} \circ \alpha_{2k-2}\,\mathrm{Ad}\, u_{2k}^*$ and $\beta_{2k+1} = \mathrm{Ad}\,\omega_{2k+1} \circ \mathrm{Ad}\, u_{2k+1} \circ \beta_{2k-1} \circ \mathrm{Ad}\, u_{2k+1}^*$, it follows that the sequence $\beta_1, \alpha_2, \beta_3$ converges and the limit is obtained as

$$\mathrm{Ad}\, W_1 \circ \gamma_1 \circ \beta \circ \gamma_1^{-1} = \mathrm{Ad}\, W_0 \circ \gamma_0 \circ \alpha \circ \gamma_0^{-1}.$$

2. Examples of actions with the Rokhlin property or the tracial Rokhlin property

Definition 2.1. Let G be a topological group and A a C^*-algebra. An action of G on A is a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$, usually written $g \mapsto \alpha_g$, such that for every $a \in A$, the function $g \mapsto \alpha_g(a)$ is norm continuous. We say that it is point-norm continuous.

It is too much strong to require that $g \mapsto \alpha_g$ be a norm continuous map viewing α_g as the bounded operators on A. For example, take $A = C_0(G)$ and define $\alpha_g(f)(h) = f(g^{-1}h)$ as the translation action. Then for $g \neq h$, it is always that $\|\alpha_g - \alpha_h\| \geq 2$ since we can choose two functions f_1 and f_2 such that $f_1(g^{-1}) = 1$ and $f_2(h^{-1}) = -1$, and $\|f_i\| = 1$ for all i.

Example 2.2. Consider a C^*-algebra O_n generated by partial isometries $s_1, s_2, \dots, s_n$ for $n \in \mathbb{N} \cup \{\infty\}$ satisfying $s_i^* s_i = 1$ for $1 \leq i \leq n$, and $\sum_{j=1}^n s_j s_j^* = 1$. Any n-tuple $(\xi_1, \dots, \xi_n) \in (S_1)^n$ defines an action O_n by sending $(s_1, \dots, s_n)$ to $(\xi_1 s_1, \dots, \xi_n s_n)$.

Definition 2.3. Let G be a finite group and A unital separable C^*-algebra. An action $\alpha : G \to \mathrm{Aut}(A)$ is said to have the Rokhlin property if for every finite subset F of A and a positive number $\epsilon > 0$, there is a set of mutually orthogonal projections $\{e_g\}$ in A such that

(1) $\|e_g a - a e_g\| \leq \epsilon$ for all $a \in F$,
(2) $\|\alpha_h(e_g) - e_{gh}\| \leq \epsilon$ for all $g, h \in G$,
(3) $\sum_{g \in G} e_g = 1$.

For a C^*-algebra A, we set the C^*-algebra of bounded sequence over $\mathbb{N}$ with values in A and the ideal of sequences converging to zero as follows;

$$l^\infty(\mathbb{N}, A) = \{(a_n) \mid \{\|a_n\|\} \text{ bounded}\}$$

$$c_0(\mathbb{N}, A) = \{(a_n) \mid \lim_{n \to \infty} \|a_n\| = 0\}.$$

Then we denote by $A_\infty = l^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$ the sequence algebra of A with the norm of a given by $\limsup_n \|a_n\|$, where $(a_n)_n$ is a representing sequence of a. We can embed A into A_∞ as a constant sequence, and we denote the central sequence algebra of A by

$$A_\infty \cap A'.$$

For an automorphism of α on A, we also denote by α_∞ the induced automorphism on A_∞ and $A_\infty \cap A'$ without confusion.

We save the notation $\lesssim$ for the Cuntz subequivalence of two positive elements; for two positive elements a, b in A $a \lesssim b$ if there is a sequence (x_n) in A such that $\|x_n b x_n^* - a\| \to 0$ as $n \to \infty$. Often when p is a projection, we see that $p \lesssim a$ if and only if there is a projection in the hereditary C^*-subalgebra generated by a which is Murray-von Neumann equivalent to p.

Proposition 2.4 (M. Izumi)**.** *Let $\alpha : G \curvearrowright A$ be an action of a finite group G on a unital separable C^*-algebra A. We say α has the Rokhlin property if there is a partition of unity $\{e_g\}_{g \in G}$ of projections in $A_\infty \cap A'$ such that for all $g, h \in G$*

$$\alpha_{\infty,h}(e_g) = e_{hg}$$

where α_∞ is the induced action.

Proposition 2.5. *Let $\alpha : G \curvearrowright A$ be an action of a finite group G on a unital separable C^*-algebra A. Then α has the Rokhlin property if and only if there exists a unital and equivariant $*$-homomorphism σ from $(C(G), \sigma)$ to $(A_\infty \cap A, \alpha_\infty)$. (Here $\sigma : G \curvearrowright C(G)$ is the G-shift action $\sigma_g(f) = f(g^1 \cdot)$.)*

Consider the Cuntz algebra O_n which is generated by a unit and isometries $s_i's$ with orthogonal images. Then it's well known that $K_0(A)$ is $\mathbb{Z}/(n-1)\mathbb{Z}$, and more explicitly that

$$K_0(O_n) = \{0, [1]_0, 2[1]_0, \dots, (n-2)[1]_0\}.$$

Suppose there is a finite group action $\alpha : G \curvearrowright O_n$. Then from the conditions $1 = \sum_g e_g$ and $\alpha_g(e_1) = e_g$ we have $|G|[e_1]_0 = [1_{O_\infty}]_0$. This means that $|G|m[1]_0 = [1]_0$. It follows that $|G|m - 1$ must be divided by $n-1$ for $m = 1, 2, \dots, n-2$. Thus if $n = 3$, and $|G| - 1$ is odd, then there is no G-action with the Rokhlin property. So the actions with Rokhlin property are very rigid. Thus the following definition was suggested by N.C. Phillips and it was observed that generically there are abundant actions with the tracial Rokhlin property.

Definition 2.6. Let G be a finite group and A unital separable C^*-algebra. An action $\alpha : G \to \mathrm{Aut}(A)$ is said to have the tracial Rokhlin property if for every nonzero positive element $a \in A$ for every finite subset F of A and a positive number $\epsilon > 0$, there is a set of mutually orthogonal projections $\{e_g\}$ in A such that

(1) $\|e_g a - a e_g\| \le \epsilon$ for all $a \in F$,
(2) $\|\alpha_h(e_g) - e_{gh}\| \le \epsilon$ for all $g, h \in G$,
(3) $1 - \sum_{g\in G} e_g$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Then we give the examples of actions with the Rokhlin property and actions with the tracial Rokhlin property based N.C. Phillips. Before analyzing examples, we need some preparations.

Definition 2.7. For a discrete group G and a unital C^*-algebra, we let

$$A[G] = \{\sum_{g\in G} a_g g \mid \text{ it has only finitely many nonzero terms } a_g\text{'s.}\}.$$

This is called a skew group ring with the following algebraic operations

$$\begin{aligned}(ag)(bh) = a(gbg^{-1})gh &= a\alpha_g(b)gh,\\ (ag)^* &= \alpha_{g^{-1}}(a^*)g^{-1}.\end{aligned}$$

Then there is one to one correspondence between a nondegenerate $*$-representation of $A[G]$ on a Hilbert space and a pair of a representation π of A and a group representation V on H which satisfies the following covariant representation.

$$V_g \pi(a) V_g^* = \pi(\alpha_g(a))$$

Definition 2.8. The full crossed product C^*-algebra denoted by $A \rtimes_\alpha G$ is the completion of the skew group ring $A[G]$ with respect to the norm

$$\sup\{\|\pi(\sum_g a_g g)\| \mid \text{nondegenerate } * \text{ representations } \pi\}$$

Proposition 2.9. *Let G be a finite(discrete) group and $v : G \to B(H)$ be a unitary representation, where A is represented as operators on H, such that $\alpha_g(a) = v_g a v_g^*$. We call such a unitary representation implementing unitary representation for α. Then $A \rtimes_\alpha G$ is the norm completion of $\{\sum_g a_g v_g\}$ in $B(H)$.*

When a group G is abelian, then we can think of the Pontryagin dual of G, denoted by $\widehat{G}$, which consists of characters from G to $\mathbb{T}$. Then there is a dual action of $\widehat{G}$ on $A \rtimes_\alpha G$ which is defined by sending $a \to a$ and $v_g \to \gamma(g)v_g$ for $\gamma \in \widehat{G}$. It is an exercise to check that $\widehat{\mathbb{Z}_n} = \mathbb{Z}_n$.

Let $D_n = \otimes_{m=1}^n M_{k(m)}$, $\alpha_n : \mathbb{Z}_2 \curvearrowright D_n$ be the action given by $\otimes_{m=1}^n \mathrm{Ad}(p_m - q_m)$ where $p_m + q_m = 1$ and p_m, q_m are projections in $M_{k(m)}$.

Note that $D_n \rtimes_{\alpha_n} \mathbb{Z}_2$ is isomorphic to the direct sum $M_{t(n)} \oplus M_{t(n)}$ where $t(n) = \prod_{m=1}^n k(m)$. Moreover, the isomorphism sends $a \in D_n$ to (a,a) and sends v to $(v,-v)$ where $v = p_n - q_n$ and p_n, q_n are suitable projections such that $(p_n - q_n) = \otimes_{m=1}^n (p_m - q_m)$. Then we define a connecting homomorphism $\psi_n : D_n \rtimes_{\alpha_n} \mathbb{Z}_2$ to $D_{n+1} \rtimes_{\alpha_{n+1}} \mathbb{Z}_2$ via the above isomorphism by

$$(a,b) \mapsto (a \otimes p_n + b \otimes q_n, b \otimes p_n + a \otimes q_n).$$

If we let α be the automorphism $\otimes_{m=1}^\infty \mathrm{Ad}(p_m - q_m)$ on $D = \otimes_{m=1}^\infty M_{k(m)}$ which is the inductive limit of D_n, then $\alpha = \lim_{n\to\infty} \alpha_n$. Thus we have the following characterization on $D \rtimes_\alpha \mathbb{Z}_2$

Theorem 2.10. *$D \rtimes_\alpha \mathbb{Z}_2$ is the inductive limit of the system $(D_n \rtimes_{\alpha_n} \mathbb{Z}_2, \psi_n)$ which is an AF-algebra.*

Theorem 2.11. *T.F.A.E.*

1. *α has the Rokhlin property*
2. *Infinitely many n, $\mathrm{rank}(p_n) = \mathrm{rank}(q_n)$,*
3. *$\widehat{\alpha}_0 : K_0(D \rtimes_\alpha \mathbb{Z}_2) \to K_0(D \rtimes_\alpha \mathbb{Z}_2)$ is trivial.*

Let $\Lambda(m,n) = \dfrac{\mathrm{rank}(p_{m+1}) - \mathrm{rank}(q_{m+1})}{\mathrm{rank}(p_{m+1}) + \mathrm{rank}(q_{m+1})} \cdots \dfrac{\mathrm{rank}(p_n) - \mathrm{rank}(q_n)}{\mathrm{rank}(p_n) + \mathrm{rank}(q_n)}$ for $n > m$, and $\Lambda(m,\infty) = \lim_{n\to\infty} \Lambda(m,n)$.

Theorem 2.12. *T.F.A.E.*

1. *α has the tracial Rokhlin property,*
2. *$\Lambda(m,\infty) = 0 \quad \forall m$,*
3. *$D \rtimes_\alpha \mathbb{Z}_2$ has a unique tracial state,*
4. *$\widehat{\alpha}$ is trivial on $T(D \rtimes_\alpha \mathbb{Z}_2)$.*

Example 2.13. Let $D = \otimes_{n=1}^\infty M_2$, $\alpha = \otimes_{n=1}^\infty \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $p_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q_n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for every n. By Theorem 2.11, α has the Rokhlin property and $D \rtimes_\alpha \mathbb{Z}_2$ is 2^∞ UHF-algebra.

Example 2.14. Let $D = \otimes_{n=1}^\infty M_{2^n}$ and $\alpha = \otimes_{n=1}^\infty \mathrm{Ad}\begin{pmatrix} I_{2^{n-1}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I_{2^{n-1}-1} \end{pmatrix}$. Then $p_n = \begin{pmatrix} 1_{2^{n-1}+1} & 0 \\ 0 & 0 \end{pmatrix}$ and $q_n = \begin{pmatrix} 0 & 0 \\ 0 & I_{2^{n-1}-1} \end{pmatrix}$. We can check that $\mathrm{rank}(p_n) > \mathrm{rank}(q_n)$ for all n, thus α cannot have the Rokhlin property, but α has the tracial Rokhlin property since $\dfrac{\mathrm{rank}(p_n) - \mathrm{rank}(q_n)}{\mathrm{rank}(p_n) + \mathrm{rank}(q_n)} = \dfrac{2}{2^n}$ and $\Lambda(m,\infty) = 0$ for all m.

3. Rokhlin property and structure theory of C^*-algebras

Now we want to see how the structure of the crossed product C^*-algebra is related to the structure of the original C^*-algebra. For instance, related to Elliott classification program, it is desirable for the crossed product C^*-algebra to have $\mathcal{Z}$-stability, finite nuclear dimension, or the strict comparison property if so is the original one. In particular, $\mathcal{Z}$-stability is the most important since we cannot distinguish K-theoretic invariants between $A\otimes\mathcal{Z}$ and A. In the following we consider a more general property so called D-stability.

Definition 3.1. A unital C^*-algebra D is strongly self-absorbing (aka s.s.a.) if $D \not\cong \mathbb{C}$ and there is an isomorphism $\phi : D \to D \otimes D$ such that ϕ and $\mathrm{id}_D \otimes 1_D$ the first factor imbedding are approximately unitary equivalent, or there is a sequence of unitaries (u_n) in D such that

$$\|u\phi(a)u^* - a \otimes 1_D\| \to 0$$

as n goes to ∞.

It is nice exercise to check that s.s.a implies that approximate inner half flip property which means that $\mathrm{id}_D \otimes 1_D$ and $1_D \otimes \mathrm{id}_D$ are approximately unitarily equivalent in the above sense. It follows that it is nuclear and simple. Moreover, a unital separable s.s.a C^*-algebra is either purely infinite or stably finite with a unique trace. The known examples are UHF-algebra of infinite type, O_2 and O_∞, the Jiang-Su algebra $\mathcal{Z}$.

Definition 3.2. A unital C^*-algebra is called D-absorbing or D-stable if $A \otimes D \cong A$.

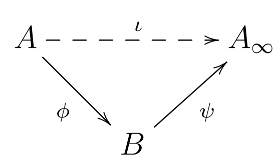
Theorem 3.3. *(Winter and Toms) Let A be unital separable C^*-algebra and D s.s.a. Then A is D-absorbing if and only if there exist a unital $*$-homomorphism $\psi : D \to A_\infty \cap A'$.*

Theorem 3.4. *(Hirshberg-Winter) Let A be a unital separable, simple C^*-algebra and α an action of a finite group G on A. Suppose that α has the Rokhlin property. If A is D-absorbing, so is the crossed product $A \rtimes_\alpha G$.*

We give a proof the above theorem based on Barlak and Szabo's approach.

Definition 3.5. (Barlak-Szabo) We say that $\phi : A \to B$ is sequentially split if there is a $*$-homomorphism $\psi : B \to A_\infty$ such that the following diagram commutes;

(23)



We call ψ in the above diagram a left approximate inverse of ϕ.

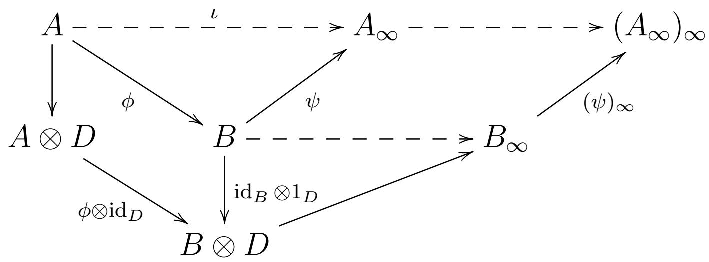
Lemma 3.6. *Let A be separable C^*-algebra and C a unital C^*-algebra. There exists a $*$-homomorphism from C to A_∞ if and only if the first factor imbedding $\mathrm{id}_A \otimes 1_C : A \to A \otimes C$ is sequentially split. Similarly, when $\alpha : G \curvearrowright A, \beta : G \curvearrowright C$ are two G-actions on A, C respectively, there exists an equivariant $*$-homomorphism from (C, β) to $(A_\infty, \alpha_\infty)$ if and only if the first factor imbedding $\mathrm{id}_A \otimes 1_C : (A, \alpha) \to (A \otimes C, \alpha \otimes \beta)$ is sequentially split.*

Lemma 3.7. *If when $\alpha : G \curvearrowright A, \beta : G \curvearrowright B$ are two G-actions on A, B respectively, then if $\phi A \to B$ is sequentially split, then $\phi \rtimes G$ which is defined by sending $\sum_g a_g u_g$ to $\sum_g \phi(a_g) v_g$ is sequentially split and a left approximate inverse of it is $\psi \rtimes G$ where ψ is the left approximate inverse of ϕ.*

Lemma 3.8. *Let the map $\phi : A \to B$ be sequentially split. If B is D-stable, so is A.*

Proof.

(24)



□

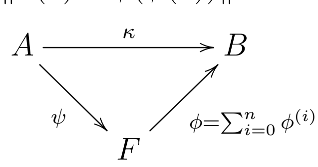
Proof of Theorem 3.4
It is enough to show that the first factor embedding Recall that since $\alpha : G \curvearrowright A$ has the Rokhlin property, we have the equivariant map $\delta : (C(G), \sigma) \to (A_\infty \cap A', \alpha_\infty)$. This means that the first factor embedding $id_A \otimes 1_{C(G)} : (A, \alpha) \to (A \otimes C(G), \alpha \otimes \sigma)$ is sequentially split. By applying Lemma 3.7 we have then $id_A \otimes 1_{C(G)} \rtimes G : A \rtimes_\alpha G \to A \otimes C(G) \rtimes_{\alpha\otimes\sigma} G$ is sequentially split. Now note that in general $A \otimes C(G) \rtimes_{\alpha\otimes\sigma} G$ is isomorphic to $A \otimes \mathcal{K}(l^2(G))$. The latter in our case $A \otimes M_{|G|}(\mathbb{C}) = M_{|G|}(A)$. D-stability is preserved by by tensoring by matrix algebra, thus $M_{|G|}(A)$ is D-stable. It follows that $A \rtimes_\alpha$ is also D-stable by Lemma 3.8.

Now let us turn to finite nuclear dimension. For this we introduce higher dimensional Rokhlin property by Hirshberg-Winter-Zacharias.

Definition 3.9. (Winter-Zacharias). A completely positive map $\phi : A \to B$ between C^*-algebras is said to be of order zero, if for every positive elements $a, b \in A$ with $ab = 0$ it follows that $\phi(a)\phi(b) = 0$. In other words, ϕ is orthogonality-preserving in the sense that $a \perp b$ implies $\phi(a) \perp \phi(b)$.

Definition 3.10. (Winter-Zacharias). Let A and B be two C^*-algebras. Let $\kappa : A\ toB$ be a completely positive map and $n \geq 0$. The map $\kappa$is said to have nuclear dimension n, written $\dim_{nuc}(A) = n$, if n is the smallest natural number satisfying the following property: For every $\mathcal{F} \subset A$ and $\epsilon > 0$, there exist a finite-dimensional C^*-algebra F, a completely positive map $\psi : A \to F$ with$\|\psi\| \leq \|\kappa\|$; completely positive contractive maps of order zero $\phi^{(0)}, \ldots, \phi^{(n)} : F \to B$; (denote $\phi = \phi^{(0)} + \cdots + \phi^{(n)}$) such that (F, ψ, ϕ) yields a finite-dimensional completely positive $(\mathcal{F}, \epsilon)$-approximation of κ, meaning $\|\kappa(a) - \phi(\psi(a))\| < \epsilon$ for all $a \in \mathcal{F}$

(25)



If no such n exists, then κ is said to have infinite nuclear dimension, written $\dim_{\mathrm{nuc}}(\kappa) = \infty$.

Definition 3.11. For a C^*-algebra A, one defines

$$\dim_{\mathrm{nuc}}(A) = \dim_{\mathrm{nuc}}(\mathrm{id}_A : A \to A).$$

Lemma 3.12.

$$\dim_{\mathrm{nuc}}(A) = \dim_{\mathrm{nuc}}(\iota_A : A \to A_\infty)$$

where $\iota_A : A \to A_\infty$ is a natural embedding.

Proof. We use the lifting property of order zero map. ☐

Definition 3.13. (Hirshberg-Winter-Zacharias) A finite group action $\alpha : G \curvearrowright A$ on a unital C^*-algebra has Rokhlin dimension d, written $\dim_{\mathrm{Rok}}(\alpha) = d$, if d is the smallest number with the following properties: Given $\epsilon > 0$ and a finite set $F(\subset A)$, there are positive contractions $\{f_g^{(l)} \mid l = 0, \ldots, d\}_{g\in G}$, or $l+1$ different towers of $|G|$-numbered positive contractions

(1) $\|1 - \sum_{l=0}^d \sum_{g\in G} f_g^{(l)}\| < \epsilon$,
(2) $\|f_g^{(l)} f_h^{(l)}\| < \epsilon$ for $g \neq h$ and $0 \le l \le d$,
(3) $\|\alpha_g(f_h^{(l)}) - f_{gh}^{(l)}\| < \epsilon$,
(4) $\|f_g^{(l)} a - a f_g^{(l)}\| < \epsilon$ for all $g \in G$, $0 \le l \le d$ and $a \in F$.

Remark 3.14. When A is a unital C^*-algebra, $\alpha : G \curvearrowright A$ has the Rokhlin property if and only if $\dim_{\mathrm{Rok}}(\alpha) = 0$.

There is a neat way to express this in terms of the central sequence algebra.

Proposition 3.15. *Let $\alpha : G \curvearrowright A$ be a finite group action on a separable, unital C^*-algebra A and $d \ge 0$. One has $\dim_{\mathrm{Rok}}(\alpha) \le d$ if and only if there exist equivariant order zero maps $\phi^{(0)}, \ldots, \phi^{(d)} : (C(G), \sigma) \to (A_\infty \cap A, \alpha_\infty)$ with $\phi^{(0)}(1) + \cdots + \phi^{(d)}(1) = 1$.*

Now again assume that $\dim_{\mathrm{Rok}}(\alpha) \le d$. One of nice features of $A_\infty \cap A'$ is that there is a canonical $*$-homomorphism

$$(A_\infty \cap A') \otimes A \to A, \quad x \otimes a \mapsto x \cdot a.$$

For every $l = 0, \ldots, d$, we consider equivariant order zero map $\widetilde{\phi}^{(l)}$ given by the composition

$$C(G) \otimes A \xrightarrow{\phi^{(l)} \otimes \mathrm{id}_A} (A_\infty \cap A') \otimes A \xrightarrow{x \otimes a \mapsto x \cdot a} A_\infty$$

Then $\phi^{(0)}(1) + \cdots + \phi^{(d)}(1) = 1$ will give us the commutative diagram of equivariant maps:

$$\begin{array}{ccc} (A, \alpha) & \xrightarrow{\quad\iota_A\quad} & (A_\infty, \alpha_\infty) \\ & {}_{1\otimes \mathrm{id}_A}\searrow \quad \nearrow_{\sum_{i=0}^d \widetilde{\phi}^{(i)}} & \\ & (C(G) \otimes A, \sigma \otimes \alpha) & \end{array}$$

Again by applying the crossed product functor, we get

$$\begin{array}{ccc} A\rtimes_\alpha G & \xrightarrow{\iota_{A\rtimes G}} & (A_\infty \rtimes_{\alpha_\infty} G) \to (A\rtimes_\alpha G)_\infty \\ & \searrow^{(1\otimes \mathrm{id}_A)\rtimes G} \qquad \nearrow_{(\sum_{i=0}^{d} \widetilde{\phi}^{(i)})\rtimes G} & \\ & (C(G)\otimes A)\rtimes_{\sigma\otimes\alpha} G & \\ & \downarrow & \\ & M_{|G|}(A) & \end{array}$$

Since $\dim_{\rm nuc}(M_n(A)) = \dim_{\rm nuc}(A)$ and upward maps are $(d+1)$ order zero maps, we have $(d+1)\times(\dim_{\rm nuc}(A)+1)$ upward maps. Thus

$$\dim_{\rm nuc}(A\rtimes_\alpha G) \le (\dim_{\rm Rok}(\alpha)+1)(\dim_{\rm nuc}(A)+1)-1$$

Corollary 3.16. *Let $\alpha : G \curvearrowright A$ be a finite group action on a unital C^*-algebra A. Assume $\dim_{\rm nuc}(A) < \infty$ and α has the Rokhlin property. Then*

$$\dim_{\rm nuc}(A\rtimes_\alpha G) \le \dim_{\rm nuc}(A).$$

4. Concluding remarks

In this note due to limited time we were not able to cover applications of Rokhlin property to other problems in C^*-algebras, for example fundamental results about the Cuntz algebra O_2, or N. Brown's characterization of AF-embedibility of $A\rtimes_\alpha \mathbb{Z}$. For those we refer the reader to [13] for the former and [2] the latter. Also, we were not able to cover H. Matui and Y. Sato 's weak Rokhlin property of $\mathbb{Z}$-action or amenable group's Rokhlin type property [8, 9]. Recently, G.Szabo launched an intensive study on this subject and refreshed this subject with a newly minded approach not only for Rokhlin type actions but also any strict group actions, but one of most notables is his proof of Powers-Sakai conjecture which says every $\mathbb{R}$-action of a UHF-algebra is approximately inner [15].

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