The spectral theory of commuting pair of operators

Jasang Yoon

The University of Texas Rio Grande Valley

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There are two notions of **the Square Root Problems.**
One is for measures and the other is for operators.
For the the Square Root Problems for measures,
we let μ and ν be probability measures supported in a compact interval in R_+.
Consider the following equation:

$$\int t^n d\mu(t) = \left(\int t^n d\nu(t) \right)^2 \cdots\cdots (1)$$

We are now interested in the following question [CuE, SS]:
Given a measure μ, does there exist a measure ν satisfying (1)?
Also, if such a ν exists, represent ν in terms of μ.
We call this problem the Square Root Problem of measures.

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By [LY2], we notice that (1) can be rewritten as

$$\int t^n d\mu(t) = \int t^n d(\nu * \nu)(t),$$

where $*$ means the convolution [KiYo2].
Thus, the Square Root Problem of measure says that given a measure μ,
does there exist a measure ν such that

$$\mu = \nu * \nu \,?$$

In this sense, ν is called a square root of the measure μ.
If (1) is satisfied, then we say that μ has a square root ν.
Actually, (1) is related to the subnormality and Aluthge transform of operators.

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$\mathcal{H}$: complex Hilbert space

$B(\mathcal{H})$: algebra of bounded operators on $\mathcal{H}$

$S \in \mathcal{B}(\mathcal{H})$ is normal if $S^*S = SS^*$

quasinormal if S commutes with S^*S, i.e., $SS^*S = S^*S^2$

subnormal if $S = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$

and hyponormal if $S^*S \geq SS^*$, that is, $S^*S - SS^* \geq 0$,
where $\geq$ means $\langle (S^*S - SS^*)\, x, x \rangle \geq 0 \; \forall x \in \mathcal{H}$.

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For $k \geq 1$, S is k-hyponormal if

$$M_k(S) := ([S^{*j}, S^i])_{i,j=1}^k \geq 0$$
$$= \begin{pmatrix} [S^*, S] & [S^{*2}, S] & \cdots & [S^{*k}, S] \\ [S^*, S^2] & [S^{*2}, S^2] & \cdots & [S^{*k}, S^2] \\ \vdots & \vdots & \ddots & \vdots \\ [S^*, S^k] & [S^{*2}, S^k] & \cdots & [S^{*k}, S^k] \end{pmatrix} \geq 0,$$

where $[A, B] := AB - BA$.

(1) Background

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Equivalently,

$$\begin{pmatrix} I & S^* & {S^*}^2 & \cdots & {S^*}^k \\ S & S^*S & {S^*}^2S & \cdots & {S^*}^kS \\ S^2 & S^*S^2 & {S^*}^2S^2 & \cdots & {S^*}^kS^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S^k & {S^*}^2S^k & {S^*}^2S^k & \cdots & {S^*}^kS^k \end{pmatrix}_{(k+1)\times(k+1)} \geq 0.$$

(By Choleski's Algorithm [CHO]).

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(Bram-Halmos Criterion for Subnormality) [Bra, Con]:
$S \in \mathcal{B}(\mathcal{H})$: subnormal $\Longleftrightarrow \sum_{i,j} \langle S^i x_j, S^j x_i \rangle \geq 0,$
$\forall x_0, x_1, \cdots, x_k \in \mathcal{H}, \forall k \geq 1.$
The Bram-Halmos criterion can be then rephrased as saying that [CMX]
S is subnormal if and only if S is k-hyponormal for every $k \geq 1$.
Thus,

$$\text{normal} \Longrightarrow \text{quasinormal} \Longrightarrow \text{subnormal}$$
$$\Longrightarrow k\text{-hyponormal} \Longrightarrow \text{hyponormal.}$$

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We consider the polar decomposition of bounded linear operator.

We can write any complex number $z = a + ib$ in polar form using the formulas:
$a = r\cos\theta$ and $b = r\sin\theta$, where $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ and $r = \sqrt{a^2 + b^2}$. In other words,

$$z = r(\cos\theta + i\sin\theta)$$

where $r \geq 0$ and $|\cos\theta + i\sin\theta| = \cos^2\theta + \sin^2\theta = 1$.

The motivation for polar decomposition of bounded linear operator acting on a Hilbert space would be the following equation:
$z = \left(\frac{z}{|z|}\right)|z| = \left(\frac{z}{|z|}\right)\sqrt{z\overline{z}}$ and $|z|^2 = z\overline{z}$ for any nonzero complex number z.

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Let $S \in B(\mathcal{H})$, with the polar decomposition $S \equiv UQ$, where U is a partial isometry and Q is a positive operator. If $\ker U = \ker Q$, then U and Q are unique and $Q := |S| = \sqrt{S^* S}$.

The Aluthge transform of S is the operator

$$\widetilde{S} := |S|^{\frac{1}{2}} U |S|^{\frac{1}{2}},$$

the Duggal transform $\widetilde{S}^D$ of S is

$$\widetilde{S}^D := |S| U.$$

the generalized Aluthge transform $\widetilde{S^\epsilon}$ of S is $\widetilde{S^\epsilon} := |S|^\epsilon U |S|^{1-\epsilon}$, where $0 \leq \epsilon \leq 1$

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For a matrix S, the polar decomposition of S is

$$S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \equiv U|S|.$$

The Aluthge transform $\widetilde{S}$ of S is

$$\widetilde{S} = \begin{pmatrix} \sqrt[4]{2} & 0 \\ 0 & \sqrt[4]{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt[4]{2} & 0 \\ 0 & \sqrt[4]{2} \end{pmatrix} \equiv |S|^{\frac{1}{2}} U |S|^{\frac{1}{2}}.$$

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The singular value decomposition (SVD) is a decomposition of a matrix into the products of a unitary matrix, a diagonal matrix, and another unitary matrix, that is, $A = UDV$, where U and V are unitary and D is a diagonal matrix.

The Cholesky decomposition (CD) is a decomposition of a Hermitian, positive definite matrix into the product of a lower triangular matrix and its conjugate transpose.

That is, for a given symmetric positive definite matrix A, there is a unique factorization of the lower triangular matrix with positive diagonal entries U such that $UU^* = A$.

Example of SVD:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = UDV.$$

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Example of CD:
$A = UU^*$, where

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \text{ and } U = \begin{pmatrix} 2 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{7}}{2} & 0 \\ -\frac{1}{2} & \frac{5\sqrt{7}}{14} & \sqrt{\frac{6}{7}} \end{pmatrix}$$

However, it is not easy to find matrices B, C such that $B^2 = A$ and $C^2 = B$.

We might get B using a numerical algorithm without a long calculation.

However, when I try to find C, I feel that it is not easy to find C.

If A, B are arbitrary positive operators in $\mathcal{B}(\mathcal{H})$, then more.

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1-variable weighted shifts:

$\mathcal{H} \equiv \ell^2(\mathbb{Z}_+)$ with orthonomal basis $\{e_n\}_{n=0}^{\infty}$

$S \equiv W_\alpha \equiv \mathrm{shift}(\alpha_0, \alpha_1, \cdots)$, where $0 < \alpha_n$ (called weight)

$W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ such that $W_\alpha e_n = \alpha_n e_{n+1}$ for all $n \geq 0$, that is,

$$W_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \alpha_0 & 0 & 0 & 0 & \cdots \\ 0 & \alpha_1 & 0 & 0 & \cdots \\ 0 & 0 & \alpha_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+).$$

$W_\alpha^* e_n = \alpha_{n-1} e_{n-1}$ for all $n \geq 0$, where $e_{-1} \equiv \mathbf{0}$ and $\alpha_{-1} \equiv 0$

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For $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \cdots)$, let $\widetilde{W_\alpha}$ (reps. $\widetilde{W_\alpha^D}$) be the Aluthge (resp. Duggal) transform of W_α.

Then $\widetilde{W_\alpha} \equiv \text{shift}(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \cdots)$ and $\widetilde{W_\alpha^D} \equiv \text{shift}(\alpha_1, \alpha_2, \cdots)$.

$$W_\alpha = U_+ D_\alpha \text{ (polar decomposition)}$$

$$\widetilde{W_\alpha} = D_\alpha^{\frac{1}{2}} U_+ D_\alpha^{\frac{1}{2}} \text{ and } \widetilde{W_\alpha^D} = D_\alpha U_+$$

where $U_+ := \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix}$ and $D_\alpha := \begin{pmatrix} \alpha_0 & & \\ & \alpha_1 & \\ & & \ddots \end{pmatrix}$

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The Aluthge transformation was first studied by A. Aluthge in the paper [Alu] in relation with the p-hyponormal and log-hyponormal operators.

Definitions:
$T \in B(H)$ is said to be p-hyponormal, $0 < p \leq 1$, if
$(T^*T)^p \geq (TT^*)^p$
and log-hyponormal, if $\log(T^*T) \geq \log(TT^*)$.

If $p = 1$, T becomes hyponormal and if $p = \frac{1}{2}$, T is called semi-hyponormal.

Semi-hyponormal operators were introduced by Xia [Xia], and p-hyponormal operators have been studied by Aluthge.

Any p-hyponormal operators are q-hyponormal if $q \leq p$.
But there are examples to show that the converse of the above statement is not true [Alu].

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Over the last two decades, Aluthge transform has been studied extensively.

One reason is the connection of Aluthge transformation with the invariant subspace problem. Another one is that Aluthge transformation is very useful in the study of non-normal operators.

Roughly speaking, the Aluthge transform converts an operator into another operator which is closer to being a normal operator.

Since every normal operators has nontrivial invariant subspaces, the Aluthge transform has a natural connection with the invariant subspace problem.

Moreover, S, $\widetilde{S}^D$, and $\widetilde{S}$ have the same spectrum.

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For $n \geq i$, we let $L_i := \bigvee\{e_n : n \geq i\}$ denote the invariant subspace of W_α obtained by removing the first i vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$.

The moment of W_α is defined by

$$\gamma_n \equiv \gamma_n(W_\alpha) := \begin{cases} 1, & \text{if } n = 0; \\ \alpha_0^2 \cdots \alpha_{n-1}^2, & \text{if } n \neq 0. \end{cases}$$

Recall the Berger Theorem [Con, GeWa] for W_α:

W_α is subnormal if and only if there exists a probability measure μ (called the Berger measure associated with W_α) supported on $[0, ||W_\alpha||^2]$ satisfying

$$\gamma_n = \int_0^{||W_\alpha||^2} t^n d\mu(t) \ \ (n = 0, 1, 2, \cdots).$$

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If W_α is subnormal with Berger measure μ, then the Berger measure of $W_\alpha|_{\mathcal{L}_i}$ is $d\mu_{\mathcal{L}_i}(t) = \frac{t^i}{\gamma_i} d\mu(t)$, where $W_\alpha|_{\mathcal{L}_i}$ means the restriction of W_α to $\mathcal{L}_i$ [CuP].

Recall that the Schur product $A \circ B$ of matrices A and B is the entry-wise product, i.e., if $A = (a_{ij})$ and $B = (b_{ij})$ then $A \circ B = (a_{ij} b_{ij})$.

For two bounded sequences $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty}$ and $\beta \equiv \{\beta_k\}_{k=0}^{\infty}$, the Schur product $\alpha \circ \beta$ of α and β is defined by $\alpha \circ \beta := \{\alpha_k \beta_k\}_{k=0}^{\infty}$. Then, for weighted shifts W_α and W_β, we can see that $W_\alpha \circ W_\beta = W_{\alpha \circ \beta} \equiv \text{shift}(\alpha_0 \beta_0, \alpha_1 \beta_1, \cdots)$.

It is known that if W_α and W_β are subnormal, then the Schur product $W_\alpha \circ W_\beta$ is also subnormal [CuP].

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Recall that for W_α, its Aluthge transform $\widetilde{W}_\alpha$ is a weighted shift with weight sequence $\{\sqrt{\alpha_k \alpha_{k+1}}\}_{k=0}^{\infty}$ [LLY1].

In particular, in [KiYo2], it was shown that if W_α and W_β are subnormal weighted shifts with Berger measure ξ_1 and ξ_2, respectively,
then the Berger measure associated with the Schur product $W_\alpha \circ W_\beta$ is the convolution $\xi_1 * \xi_2$.

Note that if we write $\sqrt{\alpha} \equiv \{\sqrt{\alpha_k}\}_{k=0}^{\infty}$ for $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty}$, then since $W_{\sqrt{\alpha}} \circ W_{\sqrt{\alpha}} = W_\alpha$, it follows that
$W_{\sqrt{\alpha}}$ is subnormal $\Longrightarrow$ W_α is subnormal.

Therefore, if $W_{\sqrt{\alpha}}$ is subnormal with Berger measure ν, then W_α has the Berger measure $\nu * \nu$.

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$$\int t^n d\mu(t) = \left(\int t^n d\nu(t)\right)^2 = \int t^n d(\nu * \nu)(t) \cdots\cdots (1)$$

Hence again, (1) is reformulated as the following:
If W_α is subnormal, under what conditions, is $W_{\sqrt{\alpha}}$ subnormal ?
We now consider why this is related to the Aluthge transform.
Note that $\widetilde{W}_\alpha$ can be viewed as the Schur product of two weighted shifts $\widetilde{W}_\alpha = W_{\sqrt{\alpha}} \circ W_{\sqrt{\alpha}}|_{\mathcal{L}_1}$,
where $W_{\sqrt{\alpha}}|_{\mathcal{L}_1} = \mathrm{shift}(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \cdots)$.
Evidently, if $W_{\sqrt{\alpha}}$ is subnormal, then $W_{\sqrt{\alpha}}|_{\mathcal{L}_1}$ is also subnormal.
Moreover, since the Schur product of two subnormal weighted shifts is also subnormal, we can see that

$$W_{\sqrt{\alpha}} \text{ is subnormal} \Longrightarrow \widetilde{W}_\alpha \text{ is subnormal} \cdots\cdots (2)$$

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Therefore, from the viewpoint of the Square Root Problem of measures,
we can say that if W_α is subnormal with Berger measure μ, then

$$\mu \text{ has a square root} \Longrightarrow \widetilde{W}_\alpha \text{ is subnormal} \cdots\cdots (3)$$

We can ask whether the converse of (3) is true.

Hence, by (3), we see that the study of the Square Root Problem of measures is strongly connected to the study of the subnormality and Aluthge transform of operators.

However, we don't know whether the converse of (3) is true.

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If W_α is subnormal with Berger measure μ and μ has a square root ν, then

$$\int t^n d\mu(t) = \left(\int t^n d\nu(t)\right)^2 = \int t^n d(\nu * \nu)(t) \cdots\cdots (1)$$

By the Berger Theorem and (1), we have that $W_{\sqrt{\alpha}}$ is subnormal with Berger measure ν and

$$W_{\sqrt{\alpha}} \text{ is subnormal} \Longrightarrow \widetilde{W}_\alpha \text{ is subnormal} \cdots\cdots (2)$$

Therefore,

$$\mu \text{ has a square root} \Longrightarrow \widetilde{W}_\alpha \text{ is subnormal} \cdots\cdots (3)$$

But

$$\widetilde{W}_\alpha \text{ is subnormal } \overset{???}{\Longrightarrow} \mu \text{ has a square root.}$$

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Question 1:

If W_α is a subnormal weighted shift with Berger measure μ, are the following statements equivalent?

(i) μ has a square root;

(ii) The Aluthge transform $\widetilde{W_\alpha}$ is subnormal.

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For the Square Root Problems for operators,
recall that matrix B is said to be a square root of A if the matrix product B^2 is equal to A $\left(B^2 = A\right)$.

We also recall that an $n \times n$ matrix A is diagonalizable if there is a matrix V and a diagonal matrix D such that $V^{-1}AV = D$. This happens if and only if A has n eigenvectors which constitute a basis for C^n.

In this case, V can be chosen to be the matrix with the n eigenvectors as columns, and thus a square root of A is
$VD^{\frac{1}{2}}V^{-1}$.
Indeed, we get

$$\left(VD^{\frac{1}{2}}V^{-1}\right)^2 = \left(VD^{\frac{1}{2}}V^{-1}\right)\left(VD^{\frac{1}{2}}V^{-1}\right) = VDV^{-1} = A \cdots\cdots (4)$$

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Motivated by (4), we are interested in the following question:
For $S \in \mathcal{B}(\mathcal{H})$, if S^2 has a property,

$$\text{when does } S \text{ have the same property ?} \cdots\cdots (5)$$

Similarly, for a positive operator S, we can ask what is $\sqrt{S}$?
Furthermore, if S has a property,

$$\text{when does } \sqrt{S} \text{ have the same property ?} \cdots\cdots (6)$$

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We call (5) and (6) the Square Root Problems of operators.

(5) and (6) are related to the following long-open problems in operator theory:
(a) characterize the subnormal operators having a square root;
(b) classify all subnormal operators whose square roots are also subnormal (cf. [KiYo4], [OlTh], [Wog].

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Recall:

$$\int t^n d\mu(t) = \left(\int t^n d\nu(t)\right)^2 = \int t^n d(\nu * \nu)(t) \cdots\cdots (1)$$

If we consider the above Square Root Problems, that is, (1), (5), and (6), to the case of commuting pairs of subnormal operators, then these are also strongly related to the Lifting Problem for Commuting Subnormals (LPCS) which is another long-open problem in operator theory.

The LPCS asks for finding necessary and sufficient conditions for a pair of commuting subnormal operators on a Hilbert space to admit commuting normal extensions.

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Recall:

If ξ_1 and ξ_2 are two measures (over $\mathbb{R}_+$ for example), the convolution of ξ_1 and ξ_2 is defined by

$$(\xi_1 * \xi_2)(E) = \int 1_E(x+y)\,\xi_1(x)\,\xi_2(y)$$

or

$$(\xi_1 * \xi_2)(E) = \int 1_E(xy)\,\xi_1(x)\,\xi_2(y)$$

for any measurable set E in $\mathbb{R}_+$.

So the convolution of two measures is a measure.

For example, we use convolution of measures in probability theory.

If a random variable X has the probability distribution P and a random variable Y has the probability Q, X independent from Y, then the distribution of $X + Y$ is $P * Q$.

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Example:

X is the set of numbers of “head" after 3 flips of a fair coin, that is,

$X = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

Then, we have:

$Prob(x = 0) = \frac{1}{8}$; $Prob(x = 1) = \frac{3}{8}$; $Prob(x = 2) = \frac{3}{8}$; $Prob(x = 3) = \frac{1}{8}$.

The discrete probability distribution P for X : $\frac{1}{8}$ for $x = 0$, $\frac{3}{8}$ for $x = 1$, $\frac{3}{8}$ for $x = 2$, and $\frac{1}{8}$ for $x = 3$.

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A special case is when P and Q are absolutely continuous with respect to the Lebesgue measure, i.e. $dP(x) = f(x)dx$ and $dQ(y) = g(y)dy$.

In that case $P * Q$ has a density which is the convolution of the two densities: $f * g$ (this time it's a convolution of functions, which results in a function).

So $d(P * Q)(z) = (f * g)(z)dz$, where

$$(f * g)(z) = \int_{\mathbb{R}_+} f(x)g(z - x)dx = \int_{\mathbb{R}_+} f(z - y)g(y)dy$$

or

$$(f * g)(z) = \int_{\mathbb{R}_+} f(x)g(zx^{-1})dx = \int_{\mathbb{R}_+} f(zy^{-1})g(y)dy$$

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Theorem 2:
$\left(\int t^n d\nu(t)\right)^2 = \int t^n d(\nu * \nu)(t)$
Proof:
Recall that if ξ_1 and ξ_2 are probability measures on $\mathbb{R}_+$,
then the convolution of ξ_1 and ξ_2 (denoted by $\xi_1 * \xi_2$) is defined by:
For every Borel set $E \subset \mathbb{R}_+$,

$$(\xi_1 * \xi_2)(E) := (\xi_1 \times \xi_2)(p^{-1}(E)),$$

where $p : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $p(s,t) = st$.
Since p is a continuous function, $p^{-1}(E)$ is a
$\xi_1 \times \xi_2$-measurable set,
so that the convolution $\xi_1 * \xi_2$ is a well-defined measure on $\mathbb{R}_+$.

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(continue Proof)

Moreover, $\xi_1 * \xi_2$ is also a probability measure because

$$\begin{aligned}(\xi_1 * \xi_2)(\mathbb{R}_+) &= (\xi_1 \times \xi_2)(p^{-1}(\mathbb{R}_+)) = (\xi_1 \times \xi_2)(\mathbb{R}_+ \times \mathbb{R}_+) \\ &= \xi_1(\mathbb{R}_+)\xi_2(\mathbb{R}_+) = 1.\end{aligned}$$

Observe that by the Fubini Theorem,

$$\begin{aligned}&\left(\int t^n d\nu(t)\right)^2 = \iint s^n t^n d\nu(s) d\nu(t) \\ &= \int s^n t^n d\left(\nu \times \nu\right)\left(s, t\right) = \iint s^n t^n d(\nu \times \nu)(p^{-1}(st)) \quad \cdots\cdots (7) \\ &= \int t^n d(\nu * \nu)(t).\end{aligned}$$

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Theorem 3:

If W_α and W_β are subnormal weighted shifts with Berger measure ξ_1 and ξ_2, respectively,
then $W_\alpha \circ W_\beta$ is also subnormal and the Berger measure associated with the Schur product $W_\alpha \circ W_\beta$ is the convolution $\xi_1 * \xi_2$.

Proof:

The subnormality of $W_\alpha \circ W_\beta$ comes from the following facts:
For $k \geq 1$, if W_α and W_β are k-hyponormal, then $W_\alpha \circ W_\beta$ is also k-hyponormal.
By the Bram-Halmos criterion, if $k \longrightarrow \infty$, then
W_α and W_β are subnormal $\Longrightarrow W_\alpha \circ W_\beta =: W_{\alpha\circ\beta}$ is subnormal.

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(continue Proof)

We now want to show that $\xi_1 * \xi_2$ is the Berger measure of $W_{\alpha\circ\beta}$.

Let $\gamma_{k_1}(W_{\alpha\circ\beta})$ be the moment of order $k_1 \geq 0$ for the subnormal weighted shift $W_{\alpha\circ\beta}$.

Note that for $k_1 \geq 1$

$$\begin{aligned} &\gamma_{k_1}(W_{\alpha\circ\beta}) = (\alpha_0\beta_0)^2 \cdots (\alpha_{k_1-1}\beta_{k_1-1})^2 \\ &= \left(\alpha_0^2 \cdots \alpha_{k_1-1}^2\right)\left(\beta_0^2 \cdots \beta_{k_1-1}^2\right) = \gamma_{k_1}(W_\alpha)\gamma_{k_1}(W_\beta) \end{aligned} \quad \cdots\cdots(8)$$

where $\gamma_{k_1}(W_\alpha)$ (resp. $\gamma_{k_1}(W_\beta)$) is the moment of order k_1 for W_α (resp. W_β).

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(continue Proof)
Recall Berger Theorem:
If $\mathbf{T} \equiv W_\alpha$, then W_α is subnormal if and only if there exists a probability measure ξ_α supported in $[0, \|W_\alpha\|^2]$ such that $\gamma_{k_1}(W_\alpha) = \int s^{k_1} d\xi_\alpha(s)$ for all $k_1 \geq 1$.

Recall the continuous function p from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}_+$ such that $p(s_1, s_2) = s_1 s_2$.

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(continue Proof)
Then, by (7) and (8), we have that for $k_1 \geq 0$,

$$\gamma_{k_1}(W_{\alpha\circ\beta}) = \gamma_{k_1}(W_\alpha)\gamma_{k_1}(W_\beta)$$
$$= \left(\int s_1^{k_1} d\xi_1(s_1)\right)\left(\int s_2^{k_1} d\xi_2(s_2)\right)$$
$$= \int s_1^{k_1} s_2^{k_1} d(\xi_1 \times \xi_2)(s_1, s_2) = \int (s_1 s_2)^{k_1} d\left(\left[(\xi_1 \times \xi_2)(p^{-1})\right]\right)(s_1 s_2)$$
$$= \int (s_1 s_2)^{k_1} d(\xi_1 * \xi_2)(s_1 s_2) = \int t^{k_1} d(\xi_1 * \xi_2) \ (t = s_1 s_2).$$

It follows from the Berger Theorem that $W_{\alpha\circ\beta}$ has the Berger measure $\xi_1 * \xi_2$.

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Recall the following result:

Proposition 4 [CuEx]:

Let $\mu = \sum_{i=0}^{N} \alpha_i \delta_{x_i}$ be a finitely atomic probability measure supported in $[0, 1]$,

where $0 \leq x_0 < x_1 < \cdots < x_N = 1$ and $\alpha_i > 0$ for $i = 0, \cdots, N$.

If μ has a square root ν, i.e. $\mu = \nu * \nu$, then

$$\operatorname{supp}(\nu) = \begin{cases} \{0\} \cup ([\sqrt{x_1}, 1] \cap \operatorname{supp}(\mu)) & (x_0 = 0) \\ [\sqrt{x_0}, 1] \cap \operatorname{supp}(\mu) & (x_0 \neq 0) \end{cases} \quad \cdots\cdots (9)$$

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Theorem 5:

If $\phi = \sum_{i=1}^{m} \alpha_i \delta_{x_i}$ and $\varphi = \sum_{j=1}^{n} \beta_j \delta_{y_j}$ are probability measures, then

$$\phi * \varphi = \sum_{i,j} \alpha_i \beta_j \delta_{\{x_i y_j\}}.$$

Proof:

Recall $p : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $p(s,t) = st$. Note

$$\begin{aligned}
(\phi * \varphi)(\mathbb{R}_+) &= (\phi \times \varphi)(p^{-1}(\mathbb{R}_+)) = (\phi \times \varphi)(\mathbb{R}_+ \times \mathbb{R}_+) \\
&= \phi(\mathbb{R}_+)\varphi(\mathbb{R}_+) = (\sum_{i=1}^{m} \alpha_i \delta_{x_i})(\sum_{j=1}^{n} \beta_j \delta_{y_j}) \\
&= \sum_{i,j} \alpha_i \beta_j \delta_{\{x_i y_j\}}.
\end{aligned}$$

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Example 6: Let a and b be such that $0 < a < b^2 < b < 1$.
Then, $0 < a^2 < ab < a < b^2 < b < 1$.
Let $\mu = \alpha_0\delta_{a^2} + \alpha_1\delta_{ab} + \alpha_2\delta_a + \alpha_3\delta_{b^2} + \alpha_4\delta_b + \alpha_5\delta_1$, where

$$\sum_{i=0}^{5} \alpha_i = 1,\ \alpha_i > 0 \text{ with } \alpha_1^2 = 4\alpha_0\alpha_3,\ \alpha_2^2 = 4\alpha_0\alpha_5, \text{ and } \alpha_4^2 = 4\alpha_3\alpha_5.$$

Note that we can always find infinitely many $\alpha_0, \cdots, \alpha_5$ satisfying the above relations.

For example, $\alpha_0 = \frac{1}{9}, \alpha_1 = \frac{2}{9}, \alpha_2 = \frac{2}{9}, \alpha_3 = \frac{1}{9}, \alpha_4 = \frac{2}{9}, \alpha_5 = \frac{1}{9}$ satisfy the relations.
Let $\nu = \sqrt{\alpha_0}\delta_a + \sqrt{\alpha_3}\delta_b + \sqrt{\alpha_5}\delta_1$.

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(continue Example 6)

Then, by Theorem 5 we have

$$\nu * \nu = (\sqrt{\alpha_0}\delta_a + \sqrt{\alpha_3}\delta_b + \sqrt{\alpha_5}\delta_1) * (\sqrt{\alpha_0}\delta_a + \sqrt{\alpha_3}\delta_b + \sqrt{\alpha_5}\delta_1)$$
$$= \alpha_0\delta_{a^2} + 2\sqrt{\alpha_0\alpha_3}\delta_{ab} + 2\sqrt{\alpha_0\alpha_5}\delta_a + \alpha_3\delta_{b^2} + 2\sqrt{\alpha_3\alpha_5}\delta_b + \alpha_5\delta_1$$
$$= \alpha_0\delta_{a^2} + \alpha_1\delta_{ab} + \alpha_2\delta_a + \alpha_3\delta_{b^2} + \alpha_4\delta_b + \alpha_5\delta_1$$
$$= \mu.$$

However, we obtain

$$\mathrm{supp}(\nu) = \{a, b, 1\} \neq \{a, b^2, b, 1\}$$
$$= [a, 1] \cap \{a^2, ab, a, b^2, b, 1\} = [\sqrt{x_0}, 1] \cap \mathrm{supp}(\mu)$$

which shows that the second row of the set equality in (9) in Proposition 4 does not hold.

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Theorem 7:

Let $\mu = \sum_{i=0}^{N} \alpha_i \delta_{x_i}$ be a finite atomic probability measure supported in $[0, 1]$,
where $0 \leq x_0 < x_1 < \cdots < x_N = 1$ and $\alpha_i > 0$ for $i = 0, \cdots, N$.

If μ has a square root ν, i.e. $\mu = \nu * \nu$, then

$$\begin{aligned} &\{x_{N-1}, 1\} \subseteq \mathrm{supp}(\nu) \\ &\subseteq \begin{cases} \{0\} \cup ([\sqrt{x_1}, 1] \cap \mathrm{supp}(\mu)) & (x_0 = 0) \\ [\sqrt{x_0}, 1] \cap \mathrm{supp}(\mu) & (x_0 \neq 0). \end{cases} \end{aligned} \qquad \cdots\cdots (10)$$

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Proof:
Let $\nu = \sum_{j=0}^{k} \beta_j \delta_{y_j}$, where $y_0 < y_1 < \cdots < y_k$. Then, by Theorem 5, we have

$$\begin{aligned} \nu * \nu &= \left(\sum_{j=0}^{k} \beta_j \delta_{y_j}\right) * \left(\sum_{j=0}^{k} \beta_j \delta_{y_j}\right) \\ &= \beta_0^2 \delta_{y_0^2} + \beta_0 \beta_1 \delta_{y_0 y_1} + \cdots + \beta_k^2 \delta_{y_k^2}. \end{aligned}$$

Note

$$\operatorname{supp}(\nu * \nu) = \left\{ y_0^2, y_0 y_1, \ldots, y_{k-1} y_k, y_k^2 \right\}.$$

For convenience, we will use the following notation:

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(continue proof)

$$\mathcal{PO}\left(\mathrm{supp}(\nu * \nu)\right) = \left\{ \begin{array}{ccccccc} y_0^2 & < & y_0 y_1 & < & \cdots & < & y_0 y_k \\ & & \wedge & & & & \wedge \\ & & y_1^2 & < & \cdots & < & y_1 y_k \\ & & & & & & \wedge \\ & & & & \vdots & & \vdots \\ & & & & \wedge & & \wedge \\ & & & & y_{k-1}^2 & < & y_{k-1} y_k \\ & & & & & & \wedge \\ & & & & & & y_k^2 \end{array} \right\}$$

$$\cdots\cdots(\#)$$

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(continue proof)
where $<$ is the standard inequality of real numbers which produces a partial order on $\mathrm{supp}(\nu * \nu)$.

Since $0 \leq x_0 < x_1 < \cdots < x_N = 1$ and $\mu = \nu * \nu$, we have

$$y_0^2 = x_0,\ y_{k-1}y_k = x_{N-1}, \text{ and } y_k^2 = x_N = 1.$$

Hence, we have $y_{k-1} = x_{N-1}$ and $y_k = 1$. Therefore, we obtain the first inclusion in (10).
Let $x_0 = 0$.

(2) Some results

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(continue proof)

Let $x_0 = 0$.

Since $y_0 = 0$, we have

$$\mathcal{PO}\left(\mathrm{supp}(\nu * \nu)\right) = \left\{ \begin{array}{ccccccc} 0 & < & y_1^2 & < & \cdots & < & y_1 \\ & & & & & & \wedge \\ & & & & \vdots & & \vdots \\ & & & & \wedge & & \wedge \\ & & & & y_{k-1}^2 & < & y_{k-1} \\ & & & & & & \wedge \\ & & & & & & 1 \end{array} \right\}.$$

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Since $\mu = \nu * \nu$, it follows that

$$\mathrm{supp}(\nu)\backslash\{0\} = \{y_1, \cdots, y_k\} \subseteq \mathrm{supp}(\mu)\backslash\{0\}.$$

Since $y_1^2 = x_1$, we have

$$\begin{aligned}&\mathrm{supp}(\nu)\backslash\{0\} = \{y_1, \cdots, y_k\} \\ &\subseteq [\sqrt{x_1}, 1] \cap \mathrm{supp}(\mu)\backslash\{0\}\end{aligned} \quad \cdots\cdots (11).$$

Let $x_0 \neq 0$. Then, we get that $\mathrm{supp}(\nu * \nu)$ has the partial order shown in $(\#)$. Since $y_0^2 = x_0$, we have

$$\begin{aligned}&\mathrm{supp}(\nu) = \{y_0, \cdots, y_k\} \\ &\subseteq [y_0, 1] \cap \mathrm{supp}(\mu) = [\sqrt{x_0}, 1] \cap \mathrm{supp}(\mu)\end{aligned} \quad \cdots\cdots (12).$$

Therefore, by (11) and (12), we have the second inclusion in (10), as desired.

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Recall:

Question 1:

If W_α is a subnormal weighted shift with Berger measure μ, are the following statements equivalent?

(i) μ has a square root;

(ii) The Aluthge transform $\widetilde{W_\alpha}$ is subnormal.

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Lemma 8: Let μ be a finitely atomic probability measure that always has its largest atom at 1 in a compact interval in $\mathbb{R}_+$.

(a) If μ has 2 atoms, i.e., $\mu = a\delta_p + b\delta_1$ $(0 \le p < 1)$, then μ has a square root if and only if $p = 0$;

(b) If μ has 3 atoms, i.e., $\mu = a\delta_p + b\delta_q + c\delta_1$ $(0 \le p < q < 1)$, then μ has a square root if and only if $p = q^2$ and $b^2 = 4ac$;

(c) If μ has 4 atoms, i.e., $\mu = a\delta_p + b\delta_q + c\delta_r + d\delta_1$ $(0 \le p < q < r < 1)$, then μ has a square root if and only if $p = 0$, $q = r^2$, and $c^2 = 4bd$.

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Proof: Since the proofs for (a) and (b) are similar to the one of (c), we only want to show (c).

$(\Longrightarrow)$

Assume that μ has a square root.

Then, by Theorem 7, we have

$$\begin{aligned} &\{x_{N-1}, 1\} \subseteq \operatorname{supp}(\nu) \\ &\subseteq \begin{cases} \{0\} \cup ([\sqrt{x_1}, 1] \cap \operatorname{supp}(\mu)) & (x_0 = 0) \\ [\sqrt{x_0}, 1] \cap \operatorname{supp}(\mu) & (x_0 \neq 0). \end{cases} \quad \cdots\cdots (10) \end{aligned}$$

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(continue Proof)

Thus, if $p = 0$ ($\mu = a\delta_p + b\delta_q + c\delta_r + d\delta_1$), then $\mathrm{supp}(v) = \{0, r, 1\}$ or $\{r, 1\}$, because $0 \le x_0 = p < x_1 = q < r < x_N = 1$.

If $\mathrm{supp}(v) = \{r, 1\}$, then by Theorem 5, we have

$$\mathrm{supp}(\mu) = (\mathrm{supp}(\nu))^2 = \{r^2, r, 1\},$$

a contradiction.

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(continue Proof)
If $\mathrm{supp}(v) = \{0, r, 1\}$, then

$$\mathrm{supp}(\mu) = (\mathrm{supp}(\nu))^2 = \{0, r^2, r, 1\}.$$

In this case, we have $p = 0$ and $q = r^2$.

If instead $p \neq 0$ ($\mu = a\delta_p + b\delta_q + c\delta_r + d\delta_1$), then $\mathrm{supp}(v) = \{r, 1\}$ or $\{q, r, 1\}$.
In this case, $(\mathrm{supp}(\nu))^2$ is different from $\mathrm{supp}(\mu)$.
Thus, the case $p \neq 0$ cannot occur.
Therefore, we must have that $p = 0$, $q = r^2$ and $\mathrm{supp}(v) = \{0, r, 1\}$.

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(continue Proof)

We now write

$$\nu = x\delta_0 + y\delta_r + z\delta_1 \ (0 < x, y, z < 1; x + y + z = 1).$$

Since $\nu * \nu = \mu$, it follows from Theorem 5 that $c^2 = 4bd$, that is,

$$(x\delta_0 + y\delta_r + z\delta_1) * (x\delta_0 + y\delta_r + z\delta_1) = a\delta_0 + b\delta_{r^2} + c\delta_r + d\delta_1$$
$$\Longrightarrow a = x^2 + 2xy + 2xz; b = y^2; c = 2zy; d = z^2$$
$$\Longrightarrow c^2 = 4bd.$$

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(continue Proof)

$(\Longleftarrow)$

Assume that $p = 0$, $q = r^2$, and $c^2 = 4bd$. Put

$$\nu := \left(1 - \sqrt{b} - \sqrt{d}\right)\delta_0 + \sqrt{b}\delta_r + \sqrt{d}\delta_1$$

Then, we have

$$\begin{aligned}
&\nu * \nu \\
&= (1 - \sqrt{b} - \sqrt{d})(1 + \sqrt{b} + \sqrt{d})\delta_0 + b\delta_{r^2} + 2\sqrt{b}\sqrt{d}\delta_r + d\delta_1 \\
&= (1 - b - d - 2\sqrt{b}\sqrt{d})\delta_0 + b\delta_{r^2} + c\delta_r + d\delta_1 = \mu
\end{aligned}$$

and μ has a square root.

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Theorem 9: Let W_α be a subnormal weighted shift with finitely atomic Berger measure μ having at most 4 atoms.
Then, μ has a square root if and only if the Aluthge transform $\widetilde{W}_\alpha$ of W_α is subnormal.

Proof: ($\Longrightarrow$) If μ has a square root, then by (3), $\widetilde{W}_\alpha$ is subnormal, as desired.

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(continue Proof)

($\Longleftarrow$) Briefly stated, our strategy to prove the converse is as follows:

(a) Compute $\mu_{\mathcal{L}_1}$ and note that the weight sequence of $\widetilde{W}_\alpha$ is a square root of that of $W_\alpha \circ W_\alpha|_{\mathcal{L}_1}$.

(b) Observe that ν is a square root of $\mu * \mu_{\mathcal{L}_1}$.

(c) Predict $\operatorname{supp}(\nu)$ based on Theorem 7 and (b).

(d) Compute the equation $\nu * \nu = \mu * \mu_{\mathcal{L}_1}$ to obtain our desired results.

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(continue Proof)
We suppose that $\widetilde{W_\alpha}$ is subnormal.

Case 1: Let μ has 2 atoms, then $\mu = a\delta_p + (1-a)\,\delta_1$, where $0 < a < 1$ and $0 \leq p < 1$.
Recall that
if W_α is a subnormal weighted shift with Berger measure μ and $\mathcal{L}_j := \bigvee\{e_k : k \geq j\}$ is the invariant subspace obtained by removing the first j vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$,
then the Berger measure $\mu_{\mathcal{L}_j}$ of $W_\alpha|_{\mathcal{L}_j}$ is given by (cf. Cu)

$$d\mu_{\mathcal{L}_j}(t) = \frac{t^j}{\gamma_j} d\mu(t) \cdots\cdots (14)$$

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(continue Proof)

Thus, in particular, the Berger measure ξ of $W_\alpha|_{\mathcal{L}_1}$ is $d\xi = \frac{t}{\alpha_0^2}d\mu$, where $\alpha := \{\alpha_k\}_{k=0}^{\infty}$.

Then, by (14) we have

$$d\mu_{\mathcal{L}_1}(t) = \frac{t}{\gamma_1(\mu)}d\mu(t) = \frac{ap\delta_p + (1-a)\delta_1}{ap + (1-a)}.$$

Note that $W_\alpha \circ W_\alpha|_{\mathcal{L}_1}$ is a weighted shift with $\{\alpha_k\alpha_{k+1}\}_{k=0}^{\infty}$.

Thus, the weight sequence of $\widetilde{W_\alpha}$ is a square root of that of $W_\alpha \circ W_\alpha|_{\mathcal{L}_1}$.

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(continue Proof)
Hence, if $\widetilde{W}_\alpha$ has the Berger measure ν, then since $W_\alpha \circ W_\alpha|_{\mathcal{L}_1}$ has the Berger measure $\mu * \mu_{\mathcal{L}_1}$, it follows that ν is a square root of $\mu * \mu_{\mathcal{L}_1}$.
Since

$$\operatorname{supp}(\mu * \mu_{\mathcal{L}_1}) = \left\{p^2, p, 1\right\},$$

it follows from Theorem 5 that $\operatorname{supp}(\nu) = \{p, 1\}$.
We write

$$\nu = x\delta_p + (1-x)\,\delta_1,$$

where $0 < x < 1$.

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(continue Proof)
If $p \neq 0$, then the equation

$$\nu * \nu = \mu * \mu_{\mathcal{L}_1}$$
$$\Longrightarrow x^2 \delta_{p^2} + 2x(1-x)\,\delta_p + (1-x)^2\,\delta_1 = \frac{a^2 p \delta_{p^2} + a(1-a)(1+p)\delta_p + (1-a)^2 \delta_1}{ap + (1-a)}$$
$$\Longrightarrow x = \frac{a\sqrt{p}}{\sqrt{ap+(1-a)}};\, (1-x) = \frac{1-a}{\sqrt{ap+(1-a)}};\, 2\sqrt{p} = 1 + p$$
$$\Longrightarrow p = 1$$

which is a contradiction.

Thus, we should have $p = 0$. Therefore, by Lemma 8 (a), μ has a square root.

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(continue Proof)
Case 2: Let μ has 3 atoms, then

$$\mu = a\delta_p + b\delta_q + (1 - a - b)\,\delta_1$$

where $0 < a, b < 1$ and $0 \leq p < q < 1$.
Recall:

$$d\mu_{\mathcal{L}_j}(t) = \frac{t^j}{\gamma_j} d\mu(t) \cdots\cdots (14)$$

By (14), note that

$$d\mu_{\mathcal{L}_1}(t) = \frac{ap\delta_p + bq\delta_q + (1 - a - b)\,\delta_1}{ap + bq + (1 - a - b)}.$$

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(continue Proof)
Thus, by Theorems 5 and 7, we have

$$\operatorname{supp}(\mu * \mu_{\mathcal{L}_1}) = \left\{p^2, pq, q^2, p, q, 1\right\}$$

and $\operatorname{supp}(\nu) = \{p, q, 1\}$. We write

$$\nu = x\delta_p + y\delta_q + (1 - x - y)\delta_1,$$

where $0 < x, y, x + y < 1$.
If $p = 0$, then the equation $\nu * \nu = \mu * \mu_{\mathcal{L}_1}$ implies $q = 1$ which drives a contradiction.

If $p \neq 0$, then the equation $\nu * \nu = \mu * \mu_{\mathcal{L}_1}$ implies $p = q^2$ and $b^2 = 4ac$. Therefore, by Lemma 8 (b), μ has a square root.

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(continue Proof)
Case 3: Let μ has 4 atoms, then

$$\mu = a\delta_p + b\delta_q + c\delta_r + d\delta_1,$$

where $0 < a, b, c, d < 1$, $a + b + c + d = 1$ and $0 \leq p < q < r < 1$.
Thus, by (14) we have

$$d\mu_{\mathcal{L}_1}(t) = \frac{1}{ap + bq + cr + d}\left(ap\delta_p + bq\delta_q + cr\delta_r + d\delta_1\right).$$

Since

$$\operatorname{supp}\left(\mu * \mu_{\mathcal{L}_1}\right) = \left\{p^2, q^2, r^2, pq, pr, qr, p, q, r, 1\right\},$$

it follows from Theorems 5 and 7 that $\operatorname{supp}(\nu) = \{p, q, r, 1\}$.

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(continue Proof)
Write
$$\nu = x\delta_p + y\delta_q + z\delta_r + w\delta_1,$$
where $0 < x, y, z, w < 1$ and $x + y + z + w = 1$.
We suppose $p \neq 0$.
Then, the equation $\nu * \nu = \mu * \mu_{\mathcal{L}_1}$ implies
$$\begin{aligned} &x^2 = \tfrac{a^2 p}{E}, \quad y^2 = \tfrac{b^2 q}{E}, \quad z^2 = \tfrac{c^2 r}{E}, \quad w^2 = \tfrac{d^2}{E}, \\ &2xy = \tfrac{ab(p+q)}{E}, \quad 2xz = \tfrac{ac(p+r)}{E}, \quad 2yz = \tfrac{bc(q+r)}{E}, \\ &2xw = \tfrac{ad(p+1)}{E}, \quad 2yw = \tfrac{bd(q+1)}{E}, \quad \text{and} \quad 2zw = \tfrac{cd(r+1)}{E}, \end{aligned}$$
where $E := ap + bq + cr + d$.

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(continue proof)
Note that

$$\begin{aligned} 2xw &= \frac{ad(p+1)}{E} \Longrightarrow 2\left(\frac{a\sqrt{p}}{\sqrt{E}}\right)\left(\frac{d}{\sqrt{E}}\right) = \frac{ad\,(p+1)}{E} \\ &\Longrightarrow 2\sqrt{p} = (p+1) \end{aligned}$$

which gives $p = 1$, a contradiction.
Thus, we should have $p = 0$.
Recall: $\mathrm{supp}\,(\mu * \mu_{\mathcal{L}_1}) = \{p^2, q^2, r^2, pq, pr, qr, p, q, r, 1\}$.
In turn, if $r \neq \sqrt{q}$, then the equation $2yz = \frac{bc(q+r)}{E}$ implies

$$2\,\frac{b\sqrt{q}}{\sqrt{E}}\frac{c\sqrt{r}}{\sqrt{E}} = \frac{bc\,(q+r)}{E} \Longrightarrow 2\sqrt{q}\sqrt{r} = (q+r)$$

which gives $r = q$, a contradiction.

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(continue proof)
Thus, we must have $r = \sqrt{q} \Longrightarrow q = r^2$.
In this case, the equation $\nu * \nu = \mu * \mu_{\mathcal{L}_1}$ eventually implies

$$y = \frac{br}{\sqrt{E}}, \ \ w = \frac{d}{\sqrt{E}}, \ \ z = \frac{c\,(r+1)}{2\sqrt{E}}, \ \ z^2 + 2yw = \frac{c^2 r + bd\left(r^2+1\right)}{E}$$

which gives

$$\frac{c^2(r+1)^2}{4E} + \frac{2brd}{E} = \frac{c^2 r + bd(r^2+1)}{E} \Longrightarrow \left(c^2 - 4bd\right)(r-1)^2 = 0.$$

But since $0 < r < 1$, we have $c^2 = 4bd$.
Thus, we have $p = 0$, $q = r^2$, and $c^2 = 4bd$.
Therefore, by Lemma 8 (c), μ has a square root.
Therefore, our proof is now complete.

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Lemma 10:
Let $\mu = \sum_{i=0}^{4} \alpha_i \delta_{x_i}$ be a finite atomic probability measure supported in $[0, 1]$,
where $0 \leq x_0 < x_1 < \cdots < x_4 = 1$, $\alpha_i > 0$, and $\sum_{i=0}^{4} \alpha_i = 1$.

Then, μ has a square root if and only if
$x_0 \neq 0$, $x_0 = x_3^4$, $x_1 = x_3^3$, $x_2 = x_3^2$, $\frac{\alpha_1}{\sqrt{\alpha_0}} = \frac{\alpha_3}{\sqrt{\alpha_4}}$, and
$\alpha_2 = \frac{\alpha_1^2}{4\alpha_0} + 2\sqrt{\alpha_0 \alpha_4}$.

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Lemma 11: Let $\mu = \sum_{i=0}^{4} \alpha_i \delta_{x_i}$ be a finite atomic probability measure supported in $[0, 1]$, where $0 \leq x_0 < x_1 < \cdots < x_4 = 1$, $\alpha_i > 0$, and $\sum_{i=0}^{4} \alpha_i = 1$.

Then, μ has a square root if and only if $x_0 \neq 0$, $x_0 = x_3^4$, $x_1 = x_3^3$, $x_2 = x_3^2$, $\frac{\alpha_1}{\sqrt{\alpha_0}} = \frac{\alpha_3}{\sqrt{\alpha_4}}$, and $\alpha_2 = \frac{\alpha_1^2}{4\alpha_0} + 2\sqrt{\alpha_0 \alpha_4}$.

Lemma 12: Let W_α be a subnormal weighted shift with finite atomic Berger measure μ given as in Lemma 11.

If $\widetilde{W}_\alpha$ is subnormal, then we have that $x_0 \neq 0$ and

$$x_1^2 = x_0 x_2 \text{ or } x_1^2 = x_0 x_3.$$

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Lemma 13: Let W_α be a subnormal weighted shift with finite atomic Berger measure μ given as in Lemma 11.
If $\widetilde{W}_\alpha$ is subnormal, then we have that $x_0 \neq 0$ and

$$x_1^2 = x_0 x_2 \text{ or } x_1^2 = x_0 x_3.$$

Theorem 14: Let W_α be a subnormal weighted shift with finitely atomic Berger measure μ having at most 5 atoms.

Then, μ has a square root if and only if the Aluthge transform $\widetilde{W}_\alpha$ of W_α is subnormal.

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Proof of Theorem 14:
We need Lemmas 11, 12, 13 including the below partial order relation:

$$\mathcal{PO}\left(supp(\nu * \nu)\right) = \left\{ \begin{array}{ccccccc} y_0^2 & < & y_0 y_1 & < & \cdots & < & y_0 y_k \\ & & \wedge & & & & \wedge \\ & & y_1^2 & < & \cdots & < & y_1 y_k \\ & & & & & & \wedge \\ & & & & \vdots & & \vdots \\ & & & & \wedge & & \wedge \\ & & & & y_{k-1}^2 & < & y_{k-1} y_k \\ & & & & & & \wedge \\ & & & & & & y_k^2 \end{array} \right\}.$$

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Question 15:
If W_α is a subnormal weighted shift with Berger measure μ, are the following statements equivalent?
(i) μ has a square root;
(ii) The Aluthge transform $\widetilde{W}_\alpha$ is subnormal.

Question 16: If $\widetilde{W}_\alpha$ is subnormal, is $W_{\sqrt{\alpha}}$ subnormal?

Question 17: For $S \in B(\mathcal{H})$ and $S \geq 0$, is it true that $\sqrt{S}$ is subnormal if and only if $\widetilde{S}$ is subnormal?

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Question 18: Extend Question 15 to a multivariable version.

Question 19: What is a correction definition of p-hyponormal of a pair $\mathbf{T} \equiv (T_1, T_2)$?

For Question 18, we need to define a correct meaning of polar decomposition for a pair $\mathbf{T} \equiv (T_1, T_2)$.

Also we need to define a proper Aluthge transform $(\widetilde{T_1, T_2})$.

Furthermore, we need to extend Theorem 7 for a multivariable version.

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Thank you for your attention!

Polar decompositions for commuting pairs and invariant subspaces

Jasang Yoon

The University of Texas Rio Grande Valley

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1

Outline of talk**:**

(1) Quasinormal

(2) Quasinormals of a commuting pair

(3) Aluthge transforms of a commuting pair

(4) Invariant subspaces

(5) References

2

$\mathcal{H}$: complex Hilbert space

$B(\mathcal{H})$: algebra of bounded operators on $\mathcal{H}$

Definitions: $S \in \mathcal{B}(\mathcal{H})$ is normal if $S^*S = SS^*$

quasinormal if S commutes with S^*S, i.e., $SS^*S = S^*S^2$

subnormal if $S = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$

and hyponormal if $S^*S \geq SS^*$.

3

Known: The only form of quasinormal

1-variable weighted shift is $r \cdot U_+ = \mathrm{shift}(r, r, \cdots)$, where $r \in \mathbb{R}_+$.

For the 1-variable case, we have

$$\text{normal} \Longrightarrow \text{quasinormal} \Longrightarrow \text{subnormal} \Longrightarrow \text{hyponormal}.$$

(2) Quasinormals of a commuting pair

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Consider n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$.

For $i, j, k \in \{1, 2, \cdots, n\}$, $\mathbf{T}$ is called matricially quasinormal if each T_i commutes with each $T_j^* T_k$.

$\mathbf{T}$ is (jointly) quasinormal if each T_i commutes with each $T_j^* T_j$.

$\mathbf{T}$ is spherically quasinormal if each T_i commutes with $\sum_{j=1}^{n} T_j^* T_j$.

For n-tuple case, we note that

normal $\Longrightarrow$ matrically quasinormal $\Longrightarrow$ (jointly) quasinormal $\Longrightarrow$ spherically quasinormal $\Longrightarrow$ subnormal.

5

Let $S \in B(\mathcal{H})$, with the polar decomposition $S \equiv U|S|$, where U is a partial isometry with $\ker U = \ker S$ and $|S| := \sqrt{S^* S}$.

The Aluthge transform of S is the operator

$$\widetilde{S} := |S|^{\frac{1}{2}} U |S|^{\frac{1}{2}},$$

the Duggal transform $\widetilde{S}^D$ of S is

$$\widetilde{S}^D := |S| U.$$

the generalized Aluthge transform $\widetilde{S^{\epsilon}}$ of S is $\widetilde{S^{\epsilon}} := |S|^{\epsilon} U |S|^{1-\epsilon}$, where $0 \leq \epsilon \leq 1$.

6
For $i = 1, 2$, consider the polar decomposition $T_i \equiv U_i\,|T_i|$.
Then for a pair $\mathbf{T} = (T_1, T_2)$,
we can define toral polar decomposition of (T_1, T_2) as follows:

$$\mathbf{T} := (U_1|T_1|, U_2|T_2|)\,.$$

In this case, the generalized toral Aluthge transform of $\mathbf{T}$ is defined by [KiYo7].

$$\widetilde{\mathbf{T}}^{\epsilon} := (\widetilde{T_1^{\epsilon}}, \widetilde{T_2^{\epsilon}}) \equiv (|T_1|^{\epsilon}U_1|T|^{1-\epsilon}, |T_2|^{\epsilon}U_2|T|^{1-\epsilon})\;\;(0 \le \epsilon \le 1)\,.$$

We now look at spherical polar decomposition of (T_1, T_2):
Consider

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} : \mathcal{H} \to \mathcal{H} \bigoplus \mathcal{H}$$

7

Since T is an operator from $\mathcal{H}$ into $\mathcal{H} \bigoplus \mathcal{H}$,

T has a standard singular-operator polar decomposition $T = VP$, that is,

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P,$$

where $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a partial isometry from $\mathcal{H}$ to $\mathcal{H} \bigoplus \mathcal{H}$ and

$$P = (T^* T)^{\frac{1}{2}} = \sqrt{T_1^* T_1 + T_2^* T_2}$$

is a positive operator on $\mathcal{H}$. Also, we have $(V_1^*, V_2^*) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 1$ on

$$(\ker T_1 \cap \ker T_2)^\perp = (\ker P)^\perp = (\ker V_1 \cap \ker V_2)^\perp.$$

8
Define the spherical polar decomposition as follows:

$$\mathbf{T} = (T_1, T_2) = (V_1 P, V_2 P).$$

Hence, we can define
the generalized spherical Aluthge transform $\widehat{\mathbf{T}}^{\epsilon}$ of $\mathbf{T}$ as follows $(0 \leq \epsilon \leq 1)$:

$$\widehat{\mathbf{T}}^{\epsilon} := (\widehat{T_1^{\epsilon}}, \widehat{T_2^{\epsilon}}) \equiv (P^{\epsilon} V_1 P^{1-\epsilon}, P^{\epsilon} V_2 P^{1-\epsilon}) : \mathcal{H} \bigoplus \mathcal{H} \to \mathcal{H}.$$

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Theorem 1: Assume that $(T_1, T_2) \equiv (V_1 P, V_2 P)$, where $P = (T_1^* T_1 + T_2^* T_2)^{1/2}$, and let

$$\widehat{(T_1, T_2)} \equiv (\widehat{T_1}, \widehat{T_2}) := (\sqrt{P} V_1 \sqrt{P}, \sqrt{P} V_2 \sqrt{P}).$$

Assume also that (T_1, T_2) is commutative. Then

(i) (V_1, V_2) is a (joint) partial isometry; more precisely, $V_1^* V_1 + V_2^* V_2$ is the projection onto $\mathrm{ran}\ P$;

(ii) $\widehat{(T_1, T_2)}$ is commutative.

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Proof (i) An easy computation reveals that

$$\begin{aligned} P^2 &= T_1^* T_1 + T_2^* T_2 = (V_1 P)^* (V_1 P) + (V_2 P)^* (V_2 P) \\ &= P(V_1^* V_1 + V_2^* V_2) P, \end{aligned}$$

and therefore $(V_1^* V_1 + V_2^* V_2)|_{\text{ran } P}$ is the identity operator on ran P, as desired.

To prove (ii), consider the product

$$\widehat{T_1}\widehat{T_2} = \sqrt{P} V_1 \sqrt{P} \sqrt{P} V_2 \sqrt{P} = \sqrt{P} V_1 P V_2 \sqrt{P}.$$

Then

$$\begin{aligned} \widehat{T_1}\widehat{T_2}\sqrt{P} &= \sqrt{P} T_1 T_2 = \sqrt{P} T_2 T_1 = (\sqrt{P} V_2 P V_1 \sqrt{P})\sqrt{P} \\ &= (\sqrt{P} V_2 \sqrt{P})(\sqrt{P} V_1 \sqrt{P})\sqrt{P} = \widehat{T_2}\widehat{T_1}\sqrt{P}. \end{aligned}$$

11

(continue Proof)

It follows at once that $\widehat{T_1}\widehat{T_2} - \widehat{T_2}\widehat{T_1}$ vanishes on ran P, as desired.

On the other hand, $\widehat{T_1}\widehat{T_2} - \widehat{T_2}\widehat{T_1}$ vanishes on ker P.

Since $\mathcal{H} = \ker P \bigoplus \overline{(\mathrm{Ran} P^*)} = \ker P \bigoplus \overline{(\mathrm{Ran} P)}$
(because $P^* = P$),
we easily see that $\widehat{T_1}\widehat{T_2} - \widehat{T_2}\widehat{T_1} = 0$;
that is $(\widehat{T_1, T_2})$ is commutative.

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Theorem 2: Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair of operators. Then, the spherical Duggal transform $\widehat{\mathbf{T}}^D$ is also commuting.

In general, the generalized spherical Aluthge transform $\widehat{\mathbf{T}}^{\epsilon}$ is also commuting.

Remark 3: In comparison with the generalized spherical Aluthge transform, the generalized toral Aluthge Transform is not commuting.

13

2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$:

Consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$

and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$.

Define the 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \text{ and } T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$.

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$$T_1 = \begin{pmatrix} W_0 & & & \\ & W_1 & & \\ & & W_2 & \\ & & & \ddots \end{pmatrix}, \quad W_n = shift(\alpha_{0n}, \alpha_{1n} \cdots)$$

and

$$T_2 = \begin{pmatrix} 0 & & & \\ D_0 & 0 & & \\ & D_1 & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \quad D_n = \begin{pmatrix} \beta_{0n} & & & \\ & \beta_{1n} & & \\ & & \beta_{2n} & \\ & & & \ddots \end{pmatrix}$$

15

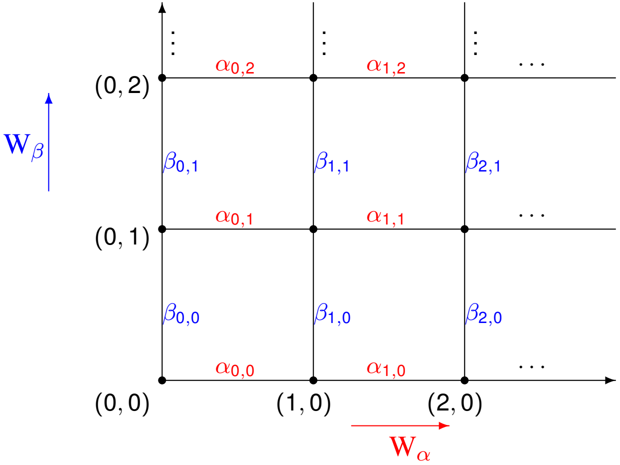

Figure: Weight diagram for 2-variable weighted shift $W_{(\alpha,\beta)}$

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Consider the ordered orthonormal basis with lexicographic order

$$E := \{e_{(0,0)}, e_{(0,1)}, e_{(1,0)}, e_{(0,2)}, e_{(1,1)}, e_{(2,0)}, \cdots\}.$$

Then, the matrix representation of T_i $(i = 1, 2)$ with respect to the ordered basis E are

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \alpha_{(0,0)} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & \alpha_{(0,1)} & 0 & 0 & \cdots \\ 0 & 0 & \alpha_{(1,0)} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \alpha_{(0,2)} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

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and

$$T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \beta_{(0,0)} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & \beta_{(0,1)} & 0 & 0 & \cdots \\ 0 & 0 & \beta_{(1,0)} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \beta_{(0,2)} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

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Lemma 4: Let $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ be a 2-variable weighted shift.

Then

$$\widetilde{T}_1 e_{\mathbf{k}} = \sqrt{\alpha_{\mathbf{k}}\alpha_{\mathbf{k}+\varepsilon_1}}\, e_{\mathbf{k}+\varepsilon_1}$$

and

$$\widetilde{T}_2 e_{\mathbf{k}} = \sqrt{\beta_{\mathbf{k}}\beta_{\mathbf{k}+\varepsilon_2}}\, e_{\mathbf{k}+\varepsilon_2}$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$.

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Theorem 5: Let $W_{(\alpha,\beta)}$ be a commuting 2-variable weighted shift. Then

$$\begin{aligned} \widetilde{W}_{(\alpha,\beta)} &\equiv \left(\widetilde{T}_1, \widetilde{T}_2\right) \text{ is commuting} \\ &\Longleftrightarrow \alpha_{\mathbf{k}+\varepsilon_2}\alpha_{\mathbf{k}+\varepsilon_1+\varepsilon_2} = \alpha_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}+2\varepsilon_2} \end{aligned}$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$.

20

Proof: For $\mathbf{k} \in \mathbb{Z}_+^2$

$$\begin{aligned}
\widetilde{T}_2 \widetilde{T}_1 e_{\mathbf{k}} &= \sqrt{\alpha_{\mathbf{k}} \alpha_{\mathbf{k}+\varepsilon_1} \beta_{\mathbf{k}+\varepsilon_1} \beta_{\mathbf{k}+\varepsilon_1+\varepsilon_2}} e_{\mathbf{k}+\varepsilon_1+\varepsilon_2} \\
&= \sqrt{(\alpha_{\mathbf{k}} \beta_{\mathbf{k}+\varepsilon_1}) \alpha_{\mathbf{k}+\varepsilon_1} \beta_{\mathbf{k}+\varepsilon_1+\varepsilon_2}} e_{\mathbf{k}+\varepsilon_1+\varepsilon_2} \\
&= \sqrt{(\beta_{\mathbf{k}} \alpha_{\mathbf{k}+\varepsilon_2}) \alpha_{\mathbf{k}+\varepsilon_1} \beta_{\mathbf{k}+\varepsilon_1+\varepsilon_2}} e_{\mathbf{k}+\varepsilon_1+\varepsilon_2} \quad \text{(by commuting)} \\
&= \sqrt{\beta_{\mathbf{k}} \alpha_{\mathbf{k}+\varepsilon_1} (\beta_{\mathbf{k}+\varepsilon_2} \alpha_{\mathbf{k}+2\varepsilon_2})} e_{\mathbf{k}+\varepsilon_1+\varepsilon_2} \quad \text{(by commuting)} \\
&= \sqrt{\beta_{\mathbf{k}} \beta_{\mathbf{k}+\varepsilon_2} (\alpha_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}+2\varepsilon_2})} e_{\mathbf{k}+\varepsilon_1+\varepsilon_2}.
\end{aligned}$$

On the other hand,

$$\widetilde{T}_1 \widetilde{T}_2 e_{\mathbf{k}} = \sqrt{\beta_{\mathbf{k}} \beta_{\mathbf{k}+\varepsilon_2} \alpha_{\mathbf{k}+\varepsilon_2} \alpha_{\mathbf{k}+\varepsilon_1+\varepsilon_2}} e_{\mathbf{k}+\varepsilon_1+\varepsilon_2}.$$

It follows that $\widetilde{T}_1 \widetilde{T}_2 = \widetilde{T}_2 \widetilde{T}_1$ if and only if

$$\alpha_{\mathbf{k}+\varepsilon_2} \alpha_{\mathbf{k}+\varepsilon_1+\varepsilon_2} = \alpha_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}+2\varepsilon_2}.$$

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Theorem 6: Let $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ be a 2-variable weighted shift. Then the spherical Aluthge transform $\widehat{W_{(\alpha,\beta)}}$ is a pair of weighted shifts with the following weights

$$\widehat{T}_1 e_{(k_1,k_2)} = \alpha_{(k_1,k_2)} \frac{(\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1+1,k_2)})^{1/4}}{(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)})^{1/4}} e_{(k_1+1,k_2)}$$

and

$$\widehat{T}_2 e_{(k_1,k_2)} = \beta_{(k_1,k_2)} \frac{(\alpha^2_{(k_1,k_2+1)} + \beta^2_{(k_1,k_2+1)})^{1/4}}{(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)})^{1/4}} e_{(k_1,k_2+1)}$$

for all $(k_1, k_2) \in \mathbb{Z}^2_+$.

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Proof: Let $P := \sqrt{T_1^* T_1 + T_2^* T_2}$. Then, we have the weights of $\widehat{W}_{(\alpha,\beta)}$ as follows:

$$Pe_{(k_1,k_2)} = \sqrt{\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2}\, e_{(k_1,k_2)}.$$

Now, $T_1 = V_1 P$ implies

$$V_1 \sqrt{P} e_{(k_1,k_2)} = T_1 \left(\sqrt{P}\right)^{-1} e_{(k_1,k_2)} = \frac{\alpha_{(k_1,k_2)} e_{(k_1,k_2)+\epsilon_1}}{\left(\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2\right)^{1/4}}$$

and similarly for T_2 and V_2, that is,

$$V_2 \sqrt{P} e_{(k_1,k_2)} = T_2 \left(\sqrt{P}\right)^{-1} e_{(k_1,k_2)} = \frac{\beta_{(k_1,k_2)} e_{(k_1,k_2)+\epsilon_2}}{\left(\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2\right)^{1/4}}.$$

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(continue Proof)
In other words, (V_1, V_2) is a 2-variable weighted shift. Now, let us compute $\widehat{T}_1 := \sqrt{P} V_1 \sqrt{P}$ and $\widehat{T}_2 := \sqrt{P} V_2 \sqrt{P}$, respectively. Acting on $e_{(k_1,k_2)}$, we have

$$\widehat{T}_1 e_{(k_1,k_2)} = \sqrt{P} V_1 \sqrt{P} e_{(k_1,k_2)} = \sqrt{P} \left(\frac{\alpha_{(k_1,k_2)} e_{(k_1,k_2)+\epsilon_1}}{\left(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}\right)^{1/4}} \right)$$

$$= \frac{\alpha_{(k_1,k_2)}}{\left(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}\right)^{1/4}} \left(\alpha^2_{(k_1,k_2)+\epsilon_1} + \beta^2_{(k_1,k_2)+\epsilon_1}\right)^{1/4} e_{(k_1,k_2)+\epsilon_1}.$$

and

$$\widehat{T}_2 e_{(k_1,k_2)} = \frac{\alpha_{(k_1,k_2)}}{(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)})^{1/4}} (\alpha^2_{(k_1,k_2)+\epsilon_2} + \beta^2_{(k_1,k_2)+\epsilon_2})^{1/4} e_{(k_1,k_2)+\epsilon_2}.$$

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Theorem 7: If $\mathbf{T} = W_{(\alpha,\beta)}$, then

(i) the toral Duggal transform $\widetilde{\mathbf{T}}^D$ is commuting if and only if for all $k_1, k_2 \geq 0$,

$$\alpha_{(k_1,k_2+1)}\alpha_{(k_1+1,k_2+1)} = \alpha_{(k_1+1,k_2)}\alpha_{(k_1,k_2+2)}.$$

(ii) the spherical Duggal transform $\widehat{W_{(\alpha,\beta)}}^D$ is a pair of weighted shifts with the following weights

$$\widehat{\alpha}^D_{(k_1,k_2)} := \alpha_{(k_1,k_2)}\sqrt{\frac{\alpha^2_{(k_1+1,k_2)}+\beta^2_{(k_1+1,k_2)}}{\alpha^2_{(k_1,k_2)}+\beta^2_{(k_1,k_2)}}}$$

$$\text{and } \widehat{\beta}^D_{(k_1,k_2)} := \beta_{(k_1,k_2)}\sqrt{\frac{\alpha^2_{(k_1,k_2+1)}+\beta^2_{(k_1,k_2+1)}}{\alpha^2_{(k_1,k_2)}+\beta^2_{(k_1,k_2)}}}.$$

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$T \in \mathcal{B}(\mathcal{H})$ is said to be p-hyponormal, $0 < p \leq 1$, if
$(T^*T)^p \geq (TT^*)^p$
and log-hyponormal, if $\log(T^*T) \geq \log(TT^*)$.

If $p = 1$, T becomes hyponormal and if $p = \frac{1}{2}$, T is called semi-hyponormal.

Recall that: Let $T \in B(\mathcal{H})$.

i) [Alu] For $0 < p < \frac{1}{2}$, if T is p-hyponormal, then $\widetilde{T}$ is $p + \frac{1}{2}$-hyponormal.

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ii) [Alu] For $\frac{1}{2} \le p < 1$, if T is p-hyponormal, then $\widetilde{T}$ is 1-hyponormal.

iii) [Tan] If T is invertible and T is log-hyponormal, then $\widetilde{T}$ is $\frac{1}{2}$-hyponormal.

iv) [LLY1] For $k \ge 2$, the Aluthge transform of weighted shifts needs not preserve the k-hyponormality.

v) [Ex] The Aluthge transform of a subnormal weighted shift need not be subnormal.

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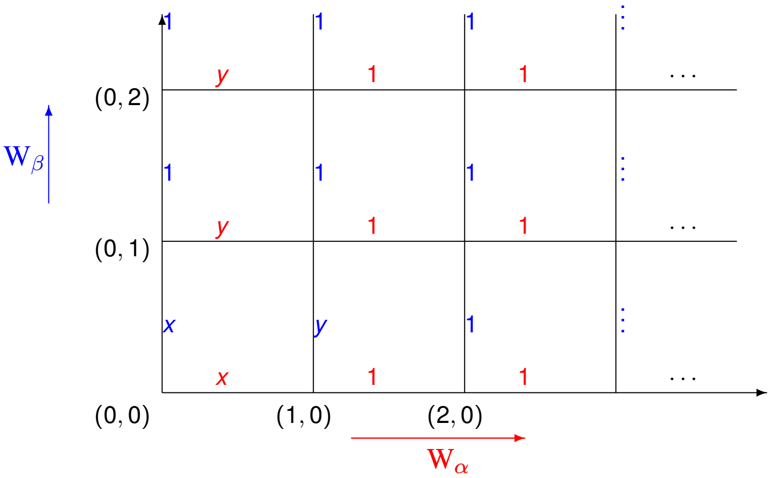

Figure: Weight diagram for Example 8

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Example 8 For $0 < x, y < 1$, let $W_{(\alpha,\beta)}$ be the 2-variable weighted shift in above Figure. Then

(i) $W_{(\alpha,\beta)}$ is subnormal $\Longleftrightarrow x \leq s(y) := \sqrt{\frac{1}{2-y^2}}$;

(ii) $W_{(\alpha,\beta)}$ is hyponormal $\Longleftrightarrow x \leq h(y) := \sqrt{\frac{1+y^2}{2}}$;

(iii) $\widetilde{W}_{(\alpha,\beta)}$ is hyponormal $\Longleftrightarrow x \leq TH(y) := \frac{1+y}{2}$;

(iv) $\widehat{W}_{(\alpha,\beta)}$ is hyponormal

$\Longleftrightarrow x \leq SH(y) := \frac{2\left(1+y^2-y^4\right)}{(1+\sqrt{2})(1+y^2)\left(\sqrt{1+y^2}-y^2\right)}$.

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(continue Example 8)

$s(y) \leq h(y) \leq SH(y)$ and $TH(y) < h(y)$ for all $0 < y < 1$, while $TH(y) < s(y)$ on $(0, q)$ and $TH(y) > s(y)$ on $(q, 1)$, where $q \cong 0.52138$.

Then $W_{(\alpha,\beta)}$ is hyponormal but $\widetilde{W}_{(\alpha,\beta)}$ is not hyponormal if $0 < TH(y) < x \leq h(y)$,
and $\widehat{W}_{(\alpha,\beta)}$ is hyponormal but $W_{(\alpha,\beta)}$ is not hyponormal if $0 < h(y) < x \leq SH(y)$.

Remark 9: Example 8 shows that the spherical Aluthge transform may turn the given $W_{(\alpha,\beta)}$ a more nicely behaved 2-variable weighted shift.

30

It is known that $T \in B(\mathcal{H})$ is quasinormal if and only if $T = \widetilde{T}$ if and only if $T = \widetilde{T}^D$.

We use $\mathfrak{C}_0$ to denote the set of commuting pairs of operators.

Theorem 10: Let $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{C}_0$. The following statements are equivalent.

(i) $\mathbf{T}$ is spherically quasinormal.

(ii) $\widehat{(T_1, T_2)} = (T_1, T_2)$.

(iii) $\widehat{(T_1, T_2)}^D = (T_1, T_2)$.

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Proof of Theorem 10:

Recall $\mathbf{T} = (T_1, T_2) = (V_1 P, V_2 P)$ and $P = \sqrt{T_1^* T_1 + T_2^* T_2}$.

Claim: For $i = 1, 2$, T_i commutes with P if and only if V_i commutes with P.

Proof of Claim: If T_i commutes with P, then $V_i P^2 = (V_i P)P = T_i P = PT_i = P(V_i P)$, and as a consequence $(V_I P - PV_i)P = 0$; that is, V_i commutes with P on $\operatorname{ran} P$.

On the other hand, $V_i P - PV_i$ vanishes on $\ker P$.
$(\because \ker P = \ker V_1 \cap \ker V_2)$
Since $\mathcal{H} = \ker P \bigoplus \overline{(\operatorname{ran} P^*)} = \ker P \bigoplus \overline{(\operatorname{ran} P)}$ (because $P^* = P$), it now easily follows that V_i commutes with P.
The converse is trivial. Thus, we prove **Claim**.

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(continue Proof)

(i) $\Longrightarrow$ (ii):

Suppose that **T** is spherically quasinormal.

Since for $i = 1, 2$, T_i commutes with $P^2 = T_1^* T_1 + T_2^* T_2$, then for $i = 1, 2$ T_i commutes with P (by the continuous functional calculus for P).

Observe now that

$$\begin{aligned} (\widehat{T_1, T_2}) &= \left(\sqrt{P} V_1 \sqrt{P}, \sqrt{P} V_2 \sqrt{P}\right) \sqrt{P} \\ &= \left(\sqrt{P} T_1, \sqrt{P} T_2\right) = (T_1, T_2) \sqrt{P}, \end{aligned}$$

so that

$$(\widehat{T_1, T_2}) = (T_1, T_2) \text{ on } \overline{\operatorname{ran} \sqrt{P}} \cdots\cdots (1)$$

33

(continue Proof)

On the other hand, since $\ker P = \ker T_1 \bigcap \ker T_2$,
it follows easily that

$$\widehat{(T_1, T_2)} = (T_1, T_2) \text{ on } \overline{\ker P} \cdots\cdots (2)$$

Since $\mathcal{H} = \left(\overline{\operatorname{ran} P}\right) \bigoplus \ker P$, we can combine (1) and (2) to prove that $\widehat{(T_1, T_2)} = (T_1, T_2)$.

34
(continue Proof)
(ii) $\Rightarrow$ (iii): Note

$$
\begin{aligned}
\widehat{\mathbf{T}} &= \mathbf{T} \Longrightarrow \left(\sqrt{P}V_1\sqrt{P}, \sqrt{P}V_2\sqrt{P}\right) = (V_1 P, V_2 P) \\
&\Longrightarrow \left(\sqrt{P}T_1, \sqrt{P}T_2\right) = \left(T_1\sqrt{P}, T_2\sqrt{P}\right) \\
&\Longrightarrow T_i \text{ commutes with } \sqrt{P}\ (i = 1, 2) \\
&\Longrightarrow T_i \text{ commutes with } P\ (i = 1, 2) \\
&\Longrightarrow V_i \text{ commutes with } P\ (i = 1, 2)\ \text{(by } \textbf{Claim}\text{)} \\
&\Longleftrightarrow \widehat{\mathbf{T}}^D = \mathbf{T}.
\end{aligned}
$$

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(continue Proof)

(iii) $\Rightarrow$ (i): Assume that $\widehat{\mathbf{T}}^D = \mathbf{T}$.

It follows from above that V_i commutes with P $(i = 1, 2)$.

As a consequence, T_i commutes with P, which implies that T_i commutes with P^2 $(i = 1, 2)$.

Therefore, $\mathbf{T}$ is spherically quasinormal, as desired.

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Theorem 11: Let $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ be a 2-variable weighted shift.

Then the following statements are equivalent.

(i) $\mathbf{T} \equiv (T_1, T_2)$ is spherically quasinormal (T_i commutes with $T_1^* T_1 + T_2^* T_2$);

(ii) There exists a constant $c > 0$ such that for all $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$,

$$\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2 = c;$$

(iii) $T_1^* T_1 + T_2^* T_2 = c \cdot I$.

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Definition: A commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is a spherical isometry if $T_1^* T_1 + T_2^* T_2 = I$.

Corollary 12: A 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) = W_{(\alpha,\beta)}$ is a spherical isometry if and only if

$$\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 1$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$.

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Corollary 13 A 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ is spherically quasinormal if and only if there exists $c > 0$ such that $\frac{1}{\sqrt{c}}\mathbf{T}$ is a spherical isometry, that is, $T_1^* T_1 + T_2^* T_2 = I$.

We pause to recall an important result about spherical isometries.

Theorem: 14 [EsPu] Any spherical isometry is subnormal.

Combining Corollary 13 and Theorem 14, we easily obtain the following result.

Theorem 15: Any spherically quasinormal 2-variable weighted shift is subnormal.

39

In [JKP], I.B. Jung, E. Ko, and C. Pearcy proved that an operator $T \in \mathcal{B}(\mathcal{H})$ with dense range has a nontrivial invariant subspace if and only if $\widetilde{T}$ does.

The invariant subspace problem (1932, J. Von Neumann)

Let $\mathcal{X}$ be a a complex Banach space with $\dim(\mathcal{X}) \geq 2$ and $T \in B(\mathcal{X})$.

Does T have a non-trivial $(\neq \{0\}, \mathcal{X})$ invariant subspace (NIS)?

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1934, J Von Neumann (unpublished), 1966, Aronszaju & Smith (Ann of Math):
T : compact operator $\Longrightarrow$ T has NIS.

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T is subnormal $\Longrightarrow$ T has NIS.

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A bounded operator on the classical Banach space ℓ_1 having only the trivial invariant subspaces.

1987, S. Brown (Ann of Math):
T is hyponormal with $\mathrm{int}\,(\sigma\,(T)) \neq \varnothing \Longrightarrow$ T has NIS.

Problem: Prove or disprove ISP for hyponormal operators

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Recall:

For $T = U|T| = UP \in \mathcal{B}(\mathcal{H})$

$$\begin{array}{ccccc} \mathcal{H} & & \overrightarrow{T = UP} & & \mathcal{K} \\ \| & & & & \| \\ \ker T = \ker U = \ker P & & & & \ker T^* \\ \oplus & & & & \oplus \\ \overline{ranP} = (\ker T)^{\perp} = \overline{ranT^*} & & \overrightarrow{U} \atop \overleftarrow{U^*} & & \overline{ranT} \end{array} \quad \cdots\cdots (2)'$$

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Theorem 16: Let $T = U|T| \in \mathcal{B}(\mathcal{H})$ be an operator with dense range.
Then, T has a NIS if and only if $\widetilde{T}^D$ does,
where $\widetilde{T}^D$ is the Duggal transform for T.

Proof:

i) If $\ker T = \{0\}$, then U is unitary and $|T|$ is a quasi-affinity.
(Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be a quasi-affinity if it has a trivial kernel and dense range)
Since

$$U\widetilde{T}^D = U|T|U = TU,$$

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(Continue Proof)
$\widetilde{T}^D$ and T are unitarily equivalent. So $\mathrm{Lat}\,(T) = \mathrm{Lat}\left(\widetilde{T}^D\right)$, where $\mathrm{Lat}(T)$ be the set of invariant subspaces for T and $\mathrm{Lat}\left(\widetilde{T}^D\right)$ for $\widetilde{T}^D$.
ii) If $\ker T \neq \{0\}$, T has a nontrivial invariant subspace. Since $\ker T = \ker U$, we have that

$$\widetilde{T}^D(\ker T) = |T|U(\ker T) = 0,$$

i.e., $\widetilde{T}^D(\ker T) \subset \ker T$. Hence $\widetilde{T}^D$ also has a NIS.

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Lemma 17: If $T \in \mathcal{B}(\mathcal{H})$ with dense range, then $\widetilde{T}^\epsilon \neq 0$ $(0 < \epsilon \leq 1)$.

Proof: If $\widetilde{T}^\epsilon = 0$, then
$|T|^\epsilon U|T|^{1-\epsilon}(\mathcal{H}) = 0 \Longrightarrow U|T|^{1-\epsilon}(\mathcal{H}) \subseteq \ker(|T|^\epsilon)$.
Thus, we have

$$\begin{aligned} T(\mathcal{H}) &= U|T|^{1-\epsilon}(|T|^\epsilon(\mathcal{H})) \subseteq U|T|^{1-\epsilon}(\mathcal{H}) \subseteq \ker(|T|^\epsilon) \\ &\Longrightarrow T(\mathcal{H}) \subseteq \ker(|T|^\epsilon) = \ker T. \end{aligned}$$

Since T has dense range, $\ker T = \mathcal{H}$, i.e., $T = 0$.
This is a contradiction to the fact that T has dense range.
Therefore, we have $\widetilde{T}^\epsilon \neq 0$.

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Theorem 18: Consider $T \in \mathcal{B}(\mathcal{H})$ with a dense range.
Then, for $0 \leq \epsilon < 1$, T has a nontrivial invariant subspace if and only if $\widetilde{T}^{\epsilon}$ dose.

Proof:

If $\ker T = \{0\}$. Then, for $0 \leq \epsilon \leq 1$, $|T|^{\epsilon}$, $|T|^{1-\epsilon}$, and T are all quasi-affinities and U is unitary, because of $(2)^{/}$.

Let a set $\overline{A}$ mean the smallest closed set containing A.

$(\Longrightarrow)$

Let $\mathcal{N}$ be a nontrivial invariant subspace for T.
Then, $\overline{(|T|^{\epsilon}\mathcal{N})}$ is nontrivial, indeed, $|T|^{\epsilon}\mathcal{N} \neq \{0\}$ because $\mathcal{N} \neq \{0\}$ and $|T|^{\epsilon}$ is a quasi-affinity.
Also, $\overline{(|T|^{\epsilon}\mathcal{N})} \neq \mathcal{H}$ because

$$U|T|^{1-\epsilon}(|T|^{\epsilon}\mathcal{N}) = U|T|\mathcal{N} = T\mathcal{N} \subseteq \mathcal{N} \neq \mathcal{H}$$

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(Continue Proof)
and $U|T|^{1-\epsilon}$ is a quasi-affinity. Hence, $\overline{(|T|^{\epsilon}\mathcal{N})} \neq \{0\}, \mathcal{H}$.
Now

$$\begin{aligned} \widetilde{T}^{\epsilon}(|T|^{\epsilon}\mathcal{N}) &= |T|^{\epsilon}U|T|^{1-\epsilon}(|T|^{\epsilon}\mathcal{N}) = |T|^{\epsilon}U|T|\mathcal{N} \\ &= |T|^{\epsilon}T\mathcal{N} \subseteq |T|^{\epsilon}\mathcal{N} \subseteq \left(\overline{|T|^{\epsilon}\mathcal{N}}\right). \end{aligned}$$

Hence, we have that $\widetilde{T}^{\epsilon}\left(\overline{|T|^{\epsilon}\mathcal{N}}\right) \subseteq \left(\overline{|T|^{\epsilon}\mathcal{N}}\right)$, and so
$\widetilde{T}^{\epsilon}$ has a nontrivial invariant subspace.
$(\Longleftarrow)$
We let $\mathcal{M}$ be a nontrivial invariant subspace for $\widetilde{T}^{\epsilon}$.
Then, $\overline{(U|T|^{1-\epsilon}\mathcal{M})} \neq \mathcal{H}$ since

$$|T|^{\epsilon}(U|T|^{1-\epsilon}\mathcal{M}) = |T|^{\epsilon}U|T|^{1-\epsilon}\mathcal{M} = \widetilde{T}^{\epsilon}\mathcal{M} \subseteq \mathcal{M} \neq \mathcal{H}$$

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(Continue Proof)
and $|T|^{\epsilon}$ is a quasi-affinity.
Also $\overline{(U|T|^{1-\epsilon}\mathcal{M})} \neq \{0\}$ since $\mathcal{M} \neq \{0\}$, $|T|^{1-\epsilon}$ is a quasi-affinity, and U is unitary.
Hence, $\overline{(U|T|^{1-\epsilon}\mathcal{M})}$ is nontrivial.
Now we have that

$$\begin{aligned} T(U|T|^{1-\epsilon}\mathcal{M}) &= U|T|(U|T|^{1-\epsilon}\mathcal{M}) = U|T|^{1-\epsilon}(|T|^{\epsilon}U|T|^{1-\epsilon}\mathcal{M}) \\ &= U|T|^{1-\epsilon}\widetilde{T}^{\epsilon}\mathcal{M} \subseteq U|T|^{1-\epsilon}\mathcal{M} \\ &\subseteq \overline{(U|T|^{1-\epsilon}\mathcal{M})}. \end{aligned}$$

Hence, $T\left(\overline{(U|T|^{1-\epsilon}\mathcal{M})}\right) \subseteq \overline{(U|T|^{1-\epsilon}\mathcal{M})}$, and so T has a nontrivial invariant subspace.

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(Continue Proof)

Suppose that $\ker T \neq \{0\}$.

Since $\ker T \neq \{0\}$ and $T \neq 0$, we have that $\ker T$ is a nontrivial invariant subspace for T.

By Lemma 17, we obtain that

$$\ker \widetilde{T}^{\epsilon} \neq \mathcal{H} \cdots\cdots (3)$$

On the other hand, since

$$\widetilde{T}^{\epsilon}(\ker |T|^{1-\epsilon}) = |T|^{\epsilon} U |T|^{1-\epsilon}(\ker |T|^{1-\epsilon}) = 0,$$

we have that $\ker |T|^{1-\epsilon} \subseteq \ker \widetilde{T}^{\epsilon}$.

Since $\ker |T|^{1-\epsilon} = \ker |T| = \ker T \neq \{0\}$, we have that

$$\ker \widetilde{T}^{\epsilon} \neq \{0\} \cdots\cdots (4)$$

By (3) and (4), we have that $\widetilde{T}^{\epsilon}$ has a nontrivial invariant subspace.

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Recall toral polar decomposition of $\mathbf{T} \equiv (T_1, T_2)$

$$\mathbf{T} := (U_1|T_1|, U_2|T_2|) .$$

and the generalized toral Aluthge transform of $\mathbf{T}$

$$\widetilde{\mathbf{T}}^{\epsilon} := (\widetilde{T_1^{\epsilon}}, \widetilde{T_2^{\epsilon}}) \equiv (|T_1|^{\epsilon} U_1 |T|^{1-\epsilon}, |T_2|^{\epsilon} U_2 |T|^{1-\epsilon}) \;\; (0 \le \epsilon \le 1) .$$

Recall the spherical polar decomposition of $\mathbf{T}$

$$\mathbf{T} = (V_1 P, V_2 P), \text{ where } P = (T_1^* T_1 + T_2^* T_2)^{\frac{1}{2}} .$$

and generalized spherical Aluthge transform of $\mathbf{T}$

$$\widehat{\mathbf{T}}^{\epsilon} := (\widehat{T_1^{\epsilon}}, \widehat{T_2^{\epsilon}}) \equiv (P^{\epsilon} V_1 P^{1-\epsilon}, P^{\epsilon} V_2 P^{1-\epsilon}) \;\; (0 \le \epsilon \le 1) .$$

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Recall that:

Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair of operators.

Then,

(i) the spherical Aluthge transform $\widehat{\mathbf{T}}$ is also commuting.

(ii) the spherical Duggal transform $\widehat{\mathbf{T}}^D$ is also commuting.

(iii) the generalized spherical Aluthge transform $\widehat{\mathbf{T}}^{\epsilon}$ is also commuting.

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Proof of (iii):
Since $\ker \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \ker P$, we have

$$\ker P = \ker V_1 \cap \ker V_2 \cdots\cdots (5)\,.$$

Since **T** is commuting, $V_1 P V_2 = V_2 P V_1$ on $(\mathrm{ran}\ P)$, and $\mathcal{H} = \ker P \oplus \overline{(\mathrm{ran}\ P)}$, by (5),

$$V_1 P V_2 = V_2 P V_1 \cdots\cdots (6)$$

Now, it follows from (6) that

$$[\widehat{T}_1^\epsilon, \widehat{T}_2^\epsilon] = P^\epsilon (V_1 P V_2 - V_2 P V_1) P^{1-\epsilon} = 0.$$

Therefore, $\widehat{\mathbf{T}}^\epsilon$ is commuting.

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Theorem 19: Let $\mathbf{T} \equiv (T_1, \cdots, T_n)$ be a commuting n-tuple of operators with dense ranges.
Then, $\widehat{\mathbf{T}}$ has a common nontrivial invariant subspace if and only if $\mathbf{T}$ does.
Proof:
Case 1: $\mathbf{T}$ is a commuting n-tuple of quasi-affinities.
$(\Longrightarrow)$

$$\begin{array}{cccc} \psi: & \mathrm{Lat}\left(\widehat{\mathbf{T}}\right) & \longrightarrow & \mathrm{Lat}(\mathbf{T}) \\ & \mid & & \mid \\ & \mathcal{M} & & \psi(\mathcal{M}) = \overline{\left(V_1 P V_2 \cdots P V_n \sqrt{P} \mathcal{M}\right)} \end{array}$$

Want: <u>ψ is well-defined</u> and if $\mathcal{M}$ is nontrivial, then $\overline{\left(V_1 P V_2 \cdots P V_n \sqrt{P} \mathcal{M}\right)}$ is also nontrivial.

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(Continue Proof)

Let $\mathcal{M}$ be a common nontrivial invariant subspace for $\widehat{\mathbf{T}}$.
Since $\mathcal{M} \in \mathrm{Lat}(\sqrt{P}V_i\sqrt{P})$ for $i = 1, 2, \cdots, n$, we know

$$\sqrt{P}(V_1PV_2\cdots PV_n\sqrt{P}\mathcal{M}) \subseteq \mathcal{M},$$

indeed,

$$\begin{aligned}
&\sqrt{P}(V_1PV_2\cdots PV_n\sqrt{P}\mathcal{M}) \\
&= (\sqrt{P}V_1\sqrt{P})(\sqrt{P}V_2\sqrt{P})\cdots(\sqrt{P}V_n\sqrt{P})\mathcal{M} \\
&\subseteq (\sqrt{P}V_1\sqrt{P})(\sqrt{P}V_2\sqrt{P})\cdots(\sqrt{P}V_{n-1}\sqrt{P})\mathcal{M} \subseteq \cdots \subseteq \mathcal{M}.
\end{aligned}$$

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(Continue Proof)

Since $\sqrt{P}$ has dense range, $V_1 PV_2 \cdots PV_n \sqrt{P}\mathcal{M}$ can not be a dense set in $\mathcal{H}$, i.e.,

$$\overline{\left(V_1 PV_2 \cdots PV_n \sqrt{P}\mathcal{M}\right)} \neq \mathcal{H}.$$

Since $T_i T_j = T_j T_i$ for $i, j = 1, 2, \cdots, n$, we observe

$$V_i PV_j P - V_j PV_i P = (V_i PV_j - V_j PV_i)P$$

and $V_i PV_j = V_j PV_i$ on $\mathrm{ran} P$. Since P has dense range, we thus have for $i, j = 1, 2, \cdots, n$,

$$V_i PV_j = V_j PV_i \cdots\cdots (7)$$

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(Continue Proof)
Next, we want to show that $\overline{\left(V_1 P V_2 \cdots P V_n \sqrt{P}\mathcal{M}\right)} \neq \{0\}$.
Assume that $V_1 P V_2 \cdots P V_n \sqrt{P}\mathcal{M} = \{0\}$. Note

$$T_1(V_2 P V_3 \cdots P V_n \sqrt{P}\mathcal{M}) = V_1 P V_2 \cdots P V_n \sqrt{P}\mathcal{M} = \{0\}.$$

Since T_1 is one-to-one, we have that

$$V_2 P V_3 \cdots P V_n \sqrt{P}\mathcal{M} = \{0\} \cdots\cdots (8)$$

Repeating this process, we have $V_n \sqrt{P}\mathcal{M} = \{0\}$. Also, by (7) and (8), we have $V_i \sqrt{P}\mathcal{M} = \{0\}$ for $i = 1, 2, \cdots, n$.

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(Continue Proof)

Since $V = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix}$ is an isometry, we have

$$\sqrt{P}\mathcal{M} \subseteq \ker(V_1) \cap \cdots \cap \ker(V_n) = \ker V = \{0\},$$

which is a contradiction because $\sqrt{P}$ is one-to-one and $\mathcal{M}$ is nontrivial. Thus, we have

$$V_1 P V_2 \cdots P V_n \sqrt{P}\mathcal{M} \neq \{0\}.$$

Therefore, $\overline{\left(V_1 P V_2 \cdots P V_n \sqrt{P}\mathcal{M}\right)}$ is nontrivial. Recall

$$\psi : \mathrm{Lat}(\widehat{\mathbf{T}}) \to \mathrm{Lat}(\mathbf{T}) \text{ by } \psi(\mathcal{M}) = \overline{\left(V_1 P V_2 \cdots P V_n \sqrt{P}\mathcal{M}\right)}.$$

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(Continue Proof)

By (7) again, we obtain

$$T_1\left(V_1PV_2\cdots PV_n\sqrt{P}\mathcal{M}\right)=V_1P\left(V_1PV_2\cdots PV_n\sqrt{P}\mathcal{M}\right)$$
$$=V_1PV_2\cdots PV_n\left(PV_1\right)\sqrt{P}\mathcal{M}=\left(V_1PV_2\cdots PV_n\sqrt{P}\right)\left(\sqrt{P}V_1\sqrt{P}\mathcal{M}\right)$$
$$\subseteq V_1PV_2\cdots PV_n\sqrt{P}\mathcal{M}\subseteq\overline{\left(V_1PV_2\cdots PV_n\sqrt{P}\mathcal{M}\right)}.$$

Similarly, we can show that for $i=1,2,\cdots,n$,

$$T_i\left(V_1PV_2\cdots PV_n\sqrt{P}\mathcal{M}\right)\subseteq\overline{\left(V_1PV_2\cdots PV_n\sqrt{P}\mathcal{M}\right)}.$$

By the previous argument, if $\mathcal{M}$ is a common nontrivial invariant subspace for $\widehat{\mathbf{T}}$, then $\psi(\mathcal{M})=\overline{(V_1PV_2\cdots PV_n\sqrt{P}\mathcal{M})}$ is also a common nontrivial invariant subspace for $\mathbf{T}$.

Hence ψ is well-defined and the desired result; that is, if $\mathcal{M}\in\mathrm{Lat}(\widehat{\mathbf{T}})$ is nontrivial, then $\psi(\mathcal{M})\in\mathrm{Lat}(\mathbf{T})$ is also nontrivial.

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(Continue Proof)

($\Longleftarrow$)

$$\begin{array}{cccc} \phi: & \mathrm{Lat}(\mathbf{T}) & \longrightarrow & \mathrm{Lat}\left(\widehat{\mathbf{T}}\right) \\ & | & & | \\ & \mathcal{N} & & \phi(\mathcal{N}) = \overline{\left(\sqrt{P}\mathcal{N}\right)} \end{array}$$

Want: ϕ is well-defined and if $\mathcal{N}$ is nontrivial, then $\overline{\left(\sqrt{P}\mathcal{N}\right)}$ is also nontrivial.

Let $\mathcal{N}$ be a common nontrivial invariant subspace for $\mathbf{T} \equiv (T_1, \cdots, T_n)$. Then, we have

$$T\mathcal{N} = \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} \mathcal{N} = \begin{pmatrix} V_1\sqrt{P} \\ \vdots \\ V_n\sqrt{P} \end{pmatrix} \sqrt{P}\mathcal{N} \subseteq \bigoplus_{i=1}^{n} \mathcal{H}_i \cdots\cdots (9)$$

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(Continue Proof) Now, let us show that $\overline{\left(\sqrt{P}\mathcal{N}\right)}$ is nontrivial.
Since $T_1, T_2, \cdots, T_n$ are all quasi-affinities, we have

$$\begin{aligned}\ker\sqrt{P} &= \ker P = \ker(T_1^*T_1 + \cdots + T_n^*T_n) \\ &= \ker(T_1) \cap \cdots \cap \ker(T_n) = \{0\}.\end{aligned}$$

Thus, $\sqrt{P}$ is one-to-one, and so $\overline{\left(\sqrt{P}\mathcal{N}\right)} \neq \{0\}$.
On the other hand, suppose that $\overline{\left(\sqrt{P}\mathcal{N}\right)} = \mathcal{H}$. Since T_i has dense range for all $i = 1, 2, \cdots, n$, $V = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix}$ is an onto isometry (by $(2)^{/}$).
Since $\sqrt{P}$ has dense range, for all $i = 1, 2, \cdots, n$, $V\sqrt{P}$ maps dense sets in $\mathcal{H}$ into dense sets in $\bigoplus_{i=1}^{n} \mathcal{H}_i$.

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(Continue Proof)
Hence, by (9), we have

$$\overline{(T\mathcal{N})} \subseteq \bigoplus_{i=1}^{n} \mathcal{N}_i \neq \bigoplus_{i=1}^{n} \mathcal{H}_i \text{ and } \overline{(T\mathcal{N})} = \bigoplus_{i=1}^{n} \mathcal{H}_i \cdots\cdots (10),$$

where $\mathcal{N}_i = \mathcal{N}$ and $\mathcal{H}_i = \mathcal{H}$.
Hence, (10) drives a contradiction. Thus, $\sqrt{P}\mathcal{N}$ can not be a dense set, that is, we have $\overline{\left(\sqrt{P}\mathcal{N}\right)} \neq \mathcal{H}$. Therefore, $\overline{\left(\sqrt{P}\mathcal{N}\right)}$ is nontrivial.
Recall

$$\phi : \mathrm{Lat}(\mathbf{T}) \to \mathrm{Lat}(\widehat{\mathbf{T}}) \text{ given by } \phi(\mathcal{N}) = \overline{\left(\sqrt{P}\mathcal{N}\right)}.$$

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(Continue Proof)
Then, ϕ is well-defined, in fact, for a common invariant subspace $\mathcal{N}$ for $\mathbf{T}$, we have for $i = 1, 2, \cdots, n$

$$\begin{aligned} \widehat{T}_i\left(\sqrt{P}\mathcal{N}\right) &= \left(\sqrt{P}V_i\sqrt{P}\right)\left(\sqrt{P}\mathcal{N}\right) = \sqrt{P}V_iP\mathcal{N} \\ &= \sqrt{P}T_i\mathcal{N} \subseteq \sqrt{P}\mathcal{N} \subseteq \overline{\left(\sqrt{P}\mathcal{N}\right)}. \end{aligned}$$

Thus, $\phi(\mathcal{N}) = \overline{\left(\sqrt{P}\mathcal{N}\right)}$ is a common invariant subspace for $\widehat{\mathbf{T}}$.
Therefore, we have that there is a mapping $\phi : \mathrm{Lat}(\mathbf{T}) \to \mathrm{Lat}(\widehat{\mathbf{T}})$ such that, if $\mathcal{N} \in \mathrm{Lat}(\mathbf{T})$ is nontrivial, then $\phi(\mathcal{N}) \in \mathrm{Lat}(\widehat{\mathbf{T}})$ is also nontrivial.

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(Continue Proof)
Claim 1: If $\ker(T_i) \neq \{0\}$ for some $i \in \{1, 2, \cdots, n\}$, then $\ker(T_i)$ is a common nontrivial invariant subspace for **T**.

Proof of Claim 1: Clearly, $\ker T_i \in \mathrm{Lat}(T_i)$. By the commutativity of **T**, for $j = 1, 2, \cdots, n$,

$$T_i(T_j(\ker(T_i))) = T_j T_i(\ker(T_i)) = 0.$$

Thus, for $j = 1, 2, \cdots, n$,

$$T_j(\ker(T_i)) \subseteq \ker(T_i) \cdots\cdots (11).$$

Hence, $\ker T_i \neq \{0\}, \mathcal{H}$ is a common invariant subspace for **T** and we prove **Claim 1**.

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(Continue Proof)

Case 2: Suppose $\ker(T_i) \neq \{0\}$ for some $i \in \{1, 2, \cdots, n\}$.
Since T_i and T_j commute for $j = 1, 2, \cdots, n$, by the above **Claim 1**, we have

$$T_j(\ker(T_i)) \subseteq \ker(T_i) \cdots\cdots (11)$$

Therefore, $\ker(T_i)$ is a common invariant subspace for **T**.
On the other hand, we consider two subcases,
that is, $\ker(P) \neq \{0\}$ or $\ker(P) = \{0\}$.

If $\ker(P) \neq \{0\}$, then $\ker(\sqrt{P}) = \ker(P) \neq \{0\}$. Since $T_j \neq 0$ for all $j = 1, 2, \cdots, n$, $\ker(P) \neq \mathcal{H}$. Thus, we have $\ker(P) \neq \{0\}, \mathcal{H}$, so that

$$\widehat{T}_j\left(\ker(\sqrt{P})\right) = \sqrt{P} V_j \sqrt{P}\left(\ker(\sqrt{P})\right) \subseteq \ker(\sqrt{P}).$$

Hence, $\widehat{\mathbf{T}}$ has a common invariant subspace.

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(Continue Proof)
If $\ker(P) = \{0\}$, then $\ker(\sqrt{P}) = \{0\}$, so that $\sqrt{P}$ has a dense range.

If $\overline{\left(\sqrt{P}\,(\ker(T_i))\right)} = \mathcal{H}$, then
$V_i\sqrt{P}\sqrt{P}(\ker(T_i)) = T_i(\ker(T_i)) = 0 \Longrightarrow V_i\sqrt{P}\,(\mathcal{H}) = 0$,
so that, $V_i = 0$, that is, $T_i = 0$. Thus, this drives a contradiction to $T_i \neq 0$. Therefore, $\overline{\left(\sqrt{P}\,(\ker(T_i))\right)} \neq \mathcal{H}$.

If $\overline{\left(\sqrt{P}\,(\ker(T_i))\right)} = \{0\}$, then $\ker(T_i) = \{0\}$ (because of $\ker(\sqrt{P}) = \{0\}$) which is contradictive to our assumption.
Thus, $\overline{\left(\sqrt{P}\,(\ker(T_i))\right)} \neq \{0\}$. Therefore, we have

$$\overline{\left(\sqrt{P}\,(\ker(T_i))\right)} \neq \{0\}, \mathcal{H}.$$

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(Continue Proof)
Now, we have for $j = 1, 2, \cdots, n$,

$$\begin{aligned}\widehat{T}_j\left(\sqrt{P}\,(\ker(T_i))\right) &= \left(\sqrt{P}V_j\sqrt{P}\right)\left(\sqrt{P}\,(\ker(T_i))\right)\\ &= \left(\sqrt{P}V_j\right)(P\,(\ker(T_i))) = \sqrt{P}T_j\,(\ker(T_i))\\ &\subseteq \sqrt{P}\,(\ker(T_i))\ \text{(by (11))}\\ &\subseteq \overline{\left(\sqrt{P}\,(\ker(T_i))\right)}.\end{aligned}$$

Hence, $\overline{\left(\sqrt{P}\,(\ker(T_i))\right)}$ is a common nontrivial invariant subspace for $\widehat{\mathbf{T}}$.
Therefore, we have the desired result.

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A commuting n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ is said to doubly commute

if $T_iT_j = T_jT_i$ and $T_iT_j^* = T_j^*T_i$ for all $i, j = 1, 2, \cdots, n$ and $i \neq j$.

Lemma 20: Let $\mathbf{T} = (T_1, \cdots, T_n) = (U_1|T_1|, \cdots, U_n|T_n|)$ be a doubly commuting n-tuple of injective operators.

Then, we have for $i, j = 1, 2, \cdots, n$ and $i \neq j$

(a) $|T_i|\,\big|T_j\big| = |T_j||T_i|$, (b) $U_iU_j = U_jU_i$, and (c) $|T_i|^{\frac{1}{2}}U_j = U_j|T_i|^{\frac{1}{2}}$.

Lemma 21: If $\mathbf{T} = (T_1, \cdots, T_n) = (U_1|T_1|, \cdots, U_n|T_n|)$ is a doubly commuting n-tuple of operators, then $\widetilde{\mathbf{T}}$ is commuting n-tuple of operators.

Proof:

Note that for $i, j = 1, 2, \cdots, n$

$$\widetilde{T}_i\widetilde{T}_j = |T_i|^{\frac{1}{2}}U_i|T_i|^{\frac{1}{2}}|T_j|^{\frac{1}{2}}U_j|T_j|^{\frac{1}{2}} = |T_j|^{\frac{1}{2}}U_j|T_j|^{\frac{1}{2}}|T_i|^{\frac{1}{2}}U_i|T_i|^{\frac{1}{2}} = \widetilde{T}_j\widetilde{T}_i.$$

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Theorem 22: Let $\mathbf{T} \equiv (T_1, \cdots, T_n)$ be a doubly commuting n-tuple of quasi-affinities.

Then, $\widetilde{\mathbf{T}}$ has a common nontrivial invariant subspace if and only if $\mathbf{T}$ does.

Proof:

$(\Longrightarrow)$

$$\begin{array}{cccc} \rho: & \mathrm{Lat}(\widetilde{\mathbf{T}}) & \longrightarrow & \mathrm{Lat}(\mathbf{T}) \\ & \Big| & & \Big| \\ & \mathcal{K} & & U_1 \cdots U_n |T_1|^{\frac{1}{2}} \cdots |T_n|^{\frac{1}{2}} \mathcal{K} \end{array}$$

Then ρ is well-defined and if $\mathcal{K}$ is nontrivial, then $U_1 \cdots U_n |T_1|^{\frac{1}{2}} \cdots |T_n|^{\frac{1}{2}} \mathcal{K}$ is also nontrivial.

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$(\Longleftarrow)$

$$\begin{array}{cccc} \varphi: & \mathrm{Lat}(\mathbf{T}) & \longrightarrow & \mathrm{Lat}(\widetilde{\mathbf{T}}) \\ & | & & | \\ & \mathcal{L} & & |T_1|^{\frac{1}{2}}|T_2|^{\frac{1}{2}}\cdots|T_n|^{\frac{1}{2}}\mathcal{L} \end{array}$$

Then φ is well-defined and if $\mathcal{L}$ is nontrivial, then $|T_1|^{\frac{1}{2}}|T_2|^{\frac{1}{2}}\cdots|T_n|^{\frac{1}{2}}\mathcal{L}$ is also nontrivial.

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Theorem 23: Let $\mathbf{T} \equiv (T_1, \cdots, T_n)$ be a commuting n-tuple of operators with dense ranges.
Then, $\widehat{\mathbf{T}}^{\epsilon}$ has a common nontrivial invariant subspace if and only if $\mathbf{T}$ does.
Proof: Want to show that there are mappings

$$\begin{array}{ccccc} \exists\alpha : & \mathrm{Lat}\left(\widehat{\mathbf{T}}^{\epsilon}\right) & \longrightarrow & \mathrm{Lat}\left(\mathbf{T}\right) & \\ & | & & | & \\ & \mathcal{M} & & \alpha(\mathcal{M}) & \\ & | & & | & \\ & \beta(\mathcal{N}) & \longleftarrow & \mathcal{N} & : \exists\beta \end{array},$$

such that if $\mathcal{M} \in \mathrm{Lat}\left(\widehat{\mathbf{T}}^{\epsilon}\right)$ (resp. $\mathcal{N} \in \mathrm{Lat}\left(\mathbf{T}\right)$) is nontrivial, then $\alpha(\mathcal{M}) \in \mathrm{Lat}\left(\mathbf{T}\right)$ (resp. $\beta(\mathcal{N}) \in \mathrm{Lat}\left(\widehat{\mathbf{T}}^{\epsilon}\right)$) is also nontrivial, where $\alpha(\mathcal{M}) := \overline{\left(V_1 P V_2 \cdots P V_n P^{1-\epsilon}\left(\mathcal{M}\right)\right)}$ and $\beta(\mathcal{N}) := \overline{P^{\epsilon}\left(\mathcal{N}\right)}$.

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Thank you for your attention!

Taylor spectra and open problems

Jasang Yoon

The University of Texas Rio Grande Valley

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Outline of this talk:

(1) Generalized Aluthge transforms for commuting pairs

(2) Taylor spectra

(3) Open problems

(4) References

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$\mathcal{H}$: complex Hilbert space

$B(\mathcal{H})$: algebra of bounded operators on $\mathcal{H}$

Let $S \in B(\mathcal{H})$, with the polar decomposition $S \equiv U|S|$, where U is a partial isometry with $\ker U = \ker S$ and $|S| := \sqrt{S^* S}$.

The Aluthge transform of S is the operator

$$\widetilde{S} := |S|^{\frac{1}{2}} U |S|^{\frac{1}{2}}.$$

(1) Generalized Aluthge transforms for commuting pairs

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The generalized Aluthge transform $\widetilde{S^{\epsilon}}$ of S is

$$\widetilde{S^{\epsilon}} := |S|^{\epsilon} U |S|^{1-\epsilon},$$

where $0 < \epsilon < 1$,
and the Duggal transform $\widetilde{S}^{D}$ of S is

$$\widetilde{S}^{D} := |S| U.$$

(1) Generalized Aluthge transforms for commuting pairs

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Recall toral polar decomposition of $\mathbf{T} \equiv (T_1, T_2)$

$$\mathbf{T} := (U_1|T_1|, U_2|T_2|)\,.$$

and the generalized toral Aluthge transform of $\mathbf{T}$

$$\widetilde{\mathbf{T}}^{\epsilon} := (\widetilde{T_1^{\epsilon}}, \widetilde{T_2^{\epsilon}}) \equiv (|T_1|^{\epsilon}U_1|T|^{1-\epsilon}, |T_2|^{\epsilon}U_2|T|^{1-\epsilon}) \ (0 \le \epsilon \le 1)\,.$$

Recall the spherical polar decomposition of $\mathbf{T}$

$$\mathbf{T} = (V_1P, V_2P), \text{ where } P = (T_1^*T_1 + T_2^*T_2)^{\frac{1}{2}}\,.$$

and generalized spherical Aluthge transform of $\mathbf{T}$

$$\widehat{\mathbf{T}}^{\epsilon} := (\widehat{T_1^{\epsilon}}, \widehat{T_2^{\epsilon}}) \equiv (P^{\epsilon}V_1P^{1-\epsilon}, P^{\epsilon}V_2P^{1-\epsilon}) \ (0 \le \epsilon \le 1)\,.$$

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Recall that:
Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair of operators. Then,
(i) the spherical Aluthge transform $\widehat{\mathbf{T}}$ is also commuting.
(ii) the spherical Duggal transform $\widehat{\mathbf{T}}^D$ is also commuting.
(iii) the generalized spherical Aluthge transform $\widehat{\mathbf{T}}^\epsilon$ is also commuting.

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In [JKP2], I.B. Jung, E. Ko and C. Pearcy proved that T and $\widetilde{T}$ have the same spectrum.

In [CJL], M. Cho, I.B. Jung, and W.Y. Lee also proved that T and $\widetilde{T}^D$ have the same spectrum.

We next show that these results may be extended to the toral and spherical (generalized spherical) Aluthge transform.

For this, we introduce the Taylor spectrum and Taylor essential spectrum of commuting n-tuples $\mathbf{T} = (T_1, \cdots, T_n)$.

For additional facts about this notion of a joint spectrum, the reader is referred to ([Cu1], [Appl], [Cu3]).

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Let $\Lambda \equiv \Lambda_n[e]$ be the complex exterior algebra on n generators $e_1, \dots, e_n$ with identity $e_0 \equiv 1$,
multiplication denoted by $\wedge$ (wedge product) and complex coefficients, subject to the collapsing property

$$e_i \wedge e_j + e_j \wedge e_i = 0 \ (1 \leq i, j \leq 1) \text{ and } e_i \wedge e_i = 0.$$

The elements $e_{j_1} \wedge \dots \wedge e_{j_k}$, $(1 \leq j_1 < \dots < j_k \leq n)$ form a basis for Λ^k, where
$\Lambda^0 = \langle e_0 \rangle \cong \mathbb{C}$, $\Lambda^1 = \langle e_1 \rangle \oplus \cdots \oplus \langle e_n \rangle$,
$\Lambda^2 = \langle e_1 \wedge e_2 \rangle \oplus \cdots \oplus \langle e_{n-1} \wedge e_n \rangle$, and $\Lambda^n = \langle e_1 \wedge \cdots \wedge e_n \rangle$.

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The exterior algebra over $\mathbb{C}$ is then given by

$$\Lambda = \{\textstyle\sum_J \alpha_J e_J : e_J = e_{j_1} \wedge \ldots \wedge e_{j_k} \text{ and } \alpha_J \in \mathbb{C}\}.$$

$\Lambda \equiv \Lambda_n[e]$ is graded, that is, $\Lambda = \oplus_{i=0}^{n} \Lambda^i$, with $\Lambda^i \wedge \Lambda^k \subset \Lambda^{i+k}$.
Moreover, $\dim \Lambda^k = \binom{n}{k}$, so that, as a vector space over $\mathbb{C}$,

Λ^k is isomorphic to $\mathbb{C}^{\binom{n}{k}} := \underbrace{\mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{\binom{n}{k}\text{ - sums}}$.

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Some properties of wedge product.

(i) (k-form) $\wedge$ (ℓ-form) $\to$ ($k+\ell$)-form, ($k, \ell \in \mathbb{Z}_+$),
(ii) $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$,
(iii) $e_i \wedge e_j = -e_j \wedge e_i$ and $e_i \wedge e_i = 0$,
(iv) $(e_i \wedge e_j) \wedge e_k = e_i \wedge (e_j \wedge e_k)$,
(v) $\omega \wedge \alpha\eta = \alpha(\omega \wedge \eta) = \alpha\omega \wedge \eta$ if α is a 0-form, i.e., $\alpha \in \mathbb{C}$.

(2) Taylor spectra

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forms	Geometric meaning	basis
0	$\Lambda^0 = \langle e_0 \rangle \cong \mathbb{C}$	1
1	$\Lambda^1 = \langle e_1 \rangle \oplus \cdots \oplus \langle e_n \rangle$	$e_1, \ldots, e_n$
2	$\Lambda^2 = \langle e_1 \wedge e_2 \rangle \oplus \cdots \oplus \langle e_{n-1} \wedge e_n \rangle$	$e_1 \wedge e_2, \ldots, e_{n-1} \wedge e_n$
$\vdots$	$\vdots$	$\vdots$
n	$\Lambda^n = \langle e_1 \wedge \cdots \wedge e_n \rangle$	$e_1 \wedge \cdots \wedge e_n$

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Denote $\Lambda \equiv \Lambda_n[e] = \oplus_{i=0}^{n} \Lambda^i$ and we call $\Lambda_n[e]$ the **exterior algebra** on n generators with inner product

$$\langle e_I, e_J \rangle := \begin{cases} 0 & \text{if } I \neq J \\ 1 & \text{if } I = J \end{cases},$$

where $I, J \subseteq \{1, 2, \ldots, n\}$, $e_I \equiv e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$, $e_J \equiv e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_\ell}$, $\{i_1, \ldots, i_k\}, \{j_1, \ldots, j_\ell\} \subseteq \{1, \ldots, n\}$.
$(\Lambda_n[e], \langle , \rangle)$ is Hilbert space with orthonormal basis

$$\{e_{0,} e_{1,} e_{2,} \ldots, e_n, e_1 \wedge e_{2,} \ldots, e_1 \wedge \ldots \wedge e_n\} = \{e_I : I \subseteq \{1, \ldots, n\}\}.$$

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If $S \in B(\mathcal{X})$, one keeps the same symbol S to denote the operator defined on $\Lambda_n[e, \mathcal{X}]$ by $S\left(\sum_I x_I e_I\right) = \sum_I S\left(x_I\right) e_I$.
Let $E_i : \Lambda_n[e, \mathcal{X}] \to \Lambda_n[e, \mathcal{X}]$ be given by $e_I \longmapsto e_i \wedge e_I$ and we call it the **creation operator.**
We will now compute E_i^* relative to the above mentioned inner product.
Any form $e_I \in \Lambda_n[e, \mathcal{X}]$ can be uniquely decomposed as
$e_I = e_i \wedge \xi' + \xi''$,
where ξ', ξ'' have no e_i contribution. Then

$$\begin{aligned}\langle E_i^* e_I, e_J\rangle &= \langle e_I, E_i e_J\rangle \\ &= \langle e_i \wedge \xi', e_i \wedge e_J\rangle + \left\langle \xi'', e_i \wedge e_J \right\rangle = \langle \xi', e_J\rangle .\end{aligned}$$

Therefore, $E_i^* e_I = \xi'$.

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Claim: $E_i^* E_j + E_j E_i^* = \delta_{ij}$.

Proof of Claim: If $i = j$, then

$$
\begin{aligned}
(E_i^* E_j + E_j E_i^*)(e_I) &= (E_i^* E_i + E_i E_i^*)(e_i \wedge \xi' + \xi'') \\
&= E_i^* E_i \left(\xi''\right) + E_i E_i^* (e_I) \\
&= E_i^*(e_i \wedge \xi'') + E_i \xi' \\
&= \xi'' + e_i \wedge \xi' = e_I.
\end{aligned}
$$

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If $i \neq j$, then

$$\begin{aligned}
&(E_i^* E_j + E_j E_i^*)(e_i \wedge \xi' + \xi'') \\
&= E_i^*(e_j \wedge e_i \wedge \xi' + e_j \wedge \xi'') + E_j(\xi') \\
&= -E_i^*(e_i \wedge e_j \wedge \xi' + e_j \wedge \xi'') + E_j(\xi') \\
&= -e_j \wedge \xi' + e_j \wedge \xi' = 0.
\end{aligned}$$

Moreover, E_i is a partial isometry
$\left(\because \quad E_i E_i^* E_i = E_i(I - E_i E_i^*) = E_i - E_i^2 E_i^* = E_i\right)$

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Given a normed space (Banach space) $\mathcal{X}$; the exterior algebra over $\mathcal{X}$ is defined to be

$$\begin{aligned} &\Lambda[\mathcal{X}] = \Lambda_n[e] = \Lambda_n[e, \mathcal{X}] \\ &= \left\{ \sum_J x_J e_J : e_J = e_{j_1} \wedge \ldots \wedge e_{j_k} \text{ and } x_J \in \mathcal{X} \right\}. \end{aligned}$$

The subspace

$$\begin{aligned} &\Lambda^i = \Lambda^i[\mathcal{X}] = \Lambda^i[e, \mathcal{X}] \\ &= \left\{ \sum_{|J|=k} x_J e_J : e_J = e_{j_1} \wedge \ldots \wedge e_{j_k} \text{ and } x_J \in \mathcal{X} \right\} \end{aligned}$$

and $\Lambda^i[e, \mathcal{X}]$ can be identified with $\mathcal{X} \oplus \mathcal{X} \oplus \cdots \oplus \mathcal{X}$.

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Koszul Complex

Set $\Lambda_n[e, \mathcal{X}] = \mathcal{X} \otimes_{\mathbb{C}} \Lambda_n[e] = \mathcal{X} \otimes_{\mathbb{C}} \oplus_{i=0}^{n} \Lambda^i = \oplus_{i=0}^{n} \mathcal{X} \otimes_{\mathbb{C}} \Lambda^i$, where $\Lambda_n[e]$ is in page 11.

Let $\mathbf{T} \equiv (T_1, \cdots, T_n)$ and $D_{\mathbf{T}} := \sum_{i=1}^{n} T_i \otimes E_i$, where T_i is an operator on X and

$$D_{\mathbf{T}} : \begin{array}{ccc} \Lambda(\mathcal{X}) & \to & \Lambda(\mathcal{X}) \\ | & & | \\ x_I \otimes e_I & & \sum_{i=1}^{n} T_i x_I \otimes e_i \wedge e_I \end{array}.$$

(note that $\sum_I x_I e_I = \sum_I x_I \otimes e_I$)
Then $D_{\mathbf{T}} \circ D_{\mathbf{T}} = 0$.

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Claim: $D_{\mathbf{T}} \circ D_{\mathbf{T}} = 0$

Proof of Claim:

$D_{\mathbf{T}} \circ D_{\mathbf{T}}(x_I \otimes e_I) = \sum_{i,j=1}^{n} T_i T_j x_I \otimes E_i E_j e_I$

$= \sum_{i<j}^{n} T_i T_j x_I \otimes E_i E_j e_I + \sum_{i>j}^{n} T_i T_j x_I \otimes E_i E_j e_I + \sum_{i=j}^{n} T_i T_j x_I \otimes E_i E_j e_I$

$= \sum_{i<j}^{n} T_i T_j x_I \otimes E_i E_j e_I - \sum_{i<j}^{n} T_i T_j x_I \otimes E_i E_j e_I = 0$
(because $T_i T_j = T_j T_i$)

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From the above **Claim** given above, we have $\mathrm{Ran} D_{\mathbf{T}} \subseteq \mathrm{Ker} D_{\mathbf{T}}$. Thus this naturally leads to a **cochain complex** (because $\mathrm{Ker} D_{\mathbf{T}}^{i+1} / \mathrm{Ran} D_{\mathbf{T}}^{i}$: cohomology for all $i \in \{0, 1, \cdots, n-1\}$), called the **Koszul complex** for $\mathbf{T} \equiv (T_1, \ldots, T_n)$, and denoted $K(\mathbf{T}, \mathcal{X})$:

$$0 \stackrel{0}{\to} \mathcal{X} \otimes \wedge^0 \stackrel{D_{\mathbf{T}}^0}{\to} \mathcal{X} \otimes \wedge^1 \stackrel{D_{\mathbf{T}}^1}{\to} \cdots \stackrel{D_{\mathbf{T}}^{n-1}}{\to} \mathcal{X} \otimes \wedge^n \stackrel{D_{\mathbf{T}}^n \equiv 0}{\to} 0,$$

where $D_{\mathbf{T}}^i$ denotes the restriction of $D_{\mathbf{T}}$ to the subspace $\mathcal{X} \otimes \wedge^i$.

If $\mathrm{Ran} D_{\mathbf{T}}^i = \mathrm{Ker} D_{\mathbf{T}}^{i+1}$ (all $i \in \{0, 1, \cdots, n-1\}$), then the above cochain complex is said to **exact**.

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Taylor spectrum $\sigma_T(\mathbf{T})$:

Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of operators on a Banach space X. We define $\mathbf{T}$ to be invertible in case its associated Koszul complex $K(\mathbf{T}, \mathcal{X})$ is exact, that is,

$$\mathrm{Ran} D_{\mathbf{T}}^{i} = \mathrm{Ker} D_{\mathbf{T}}^{i+1} \ \ (\text{all } i \in \{0, 1\}) .$$

The commuting $\mathbf{T}$ is said to be **non-singular** on X, if $\mathrm{Ran} D_{\mathbf{T}}^{i} = \mathrm{Ker} D_{\mathbf{T}}^{i+1}$ (all $i \in \{0, 1\}$).

$$\sigma_T(\mathbf{T}) := \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : (T_1 - \lambda_1, T_2 - \lambda_2) \text{ is singular}\right\}$$

$$= \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : K\left((T_1 - \lambda_1, T_2 - \lambda_2), \mathcal{H}\right) \text{ is not exact}\right\}$$

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J. L. Taylor showed that, if $\mathcal{X}$ $(\neq \{0\})$ is a Banach space, then $\sigma_T(\mathbf{T})$ is a nonempty, compact subset of the polydisc of multiradius $r(\mathbf{T}) := (r(T_1), \ldots, r(T_n))$, where $r(T_i)$ is the spectral radius of T_i. ([Tay1], [Tay2]).

When $n = 1$, the Koszul complex is

$$0 \xrightarrow{0} \mathcal{X} \otimes \wedge^0 \xrightarrow{D_{\mathbf{T}}^0} \mathcal{X} \otimes \wedge^1 \xrightarrow{D_{\mathbf{T}}^1 \equiv 0} 0$$

and

$$D_{\mathbf{T}} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

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$$D_{\mathbf{T}} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

$(\because \quad D_{\mathbf{T}} = T \otimes E,$

$D_{\mathbf{T}}(x \otimes e_0) = D^0_{\mathbf{T}}(x \otimes e_0) = T \otimes E(x \otimes e_0) = Tx \otimes Ee_0$
$= Tx \otimes e_1 \wedge e_0 = Tx \otimes e_1$ and

$D_{\mathbf{T}}(x \otimes e_1) = D^1_{\mathbf{T}}(x \otimes e_1) = T \otimes E(x \otimes e_1)$
$= Tx \otimes Ee_1 = Tx \otimes e_1 \wedge e_1 = 0.)$

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Also

$$N(D_{\mathbf{T}}) = \left\{(x,y) : D_{\mathbf{T}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$$
$$= \left\{(x,y) : \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$$
$$= \{(x,y) : Tx = 0\} = N(T) \oplus \mathcal{X}.$$

and

$$R(D_{\mathbf{T}}) = \{0\} \oplus R(T)$$

If $N(D_{\mathbf{T}}) = R(D_{\mathbf{T}})$, then $\mathbf{T} \equiv T$ is invertible.
It follows that $\sigma_T = \sigma$.

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When $n = 2$, that is, $\mathbf{T} \equiv (T_1, T_2)$, the Koszul complex is

$$0 \stackrel{0}{\to} \mathcal{X} \otimes \wedge^0 \stackrel{D^0_{\mathbf{T}}}{\to} \mathcal{X} \otimes \wedge^1 \stackrel{D^1_{\mathbf{T}}}{\to} \mathcal{X} \otimes \wedge^2 \stackrel{D^2_{\mathbf{T}} \equiv 0}{\to} 0,$$

where $D^0_{\mathbf{T}}$ and $D^1_{\mathbf{T}}$ are defined by $D^0_{\mathbf{T}} x = T_1 x \oplus T_2 x$ ($x \in \mathcal{X}$) and $D^1_{\mathbf{T}}(x_1 \oplus x_2) = -T_2 x_1 + T_1 x_2$ ($x_1, x_2 \in \mathcal{X}$).

Then, we have

$N(D_{\mathbf{T}}) = \{N(T_1) \cap N(T_2)\} \oplus \{(x_1, x_2) : T_2 x_1 = T_1 x_2\} \oplus \mathcal{X}$

$R(D_{\mathbf{T}}) = 0 \oplus \{(T_1 x, T_2 x) : x \in X\} \oplus \{R(T_1) + R(T_2)\}$, where

$$D_{\mathbf{T}} = \sum_{i=1}^{2} T_i \otimes E_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ T_1 & 0 & 0 & 0 \\ T_2 & 0 & 0 & 0 \\ 0 & -T_2 & T_1 & 0 \end{pmatrix}.$$

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$$D_{\mathbf{T}} = \sum_{i=1}^{2} T_i \otimes E_i = \begin{pmatrix} 0 & 0 & 0 \\ \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} -T_2 & T_1 \end{pmatrix} & 0 \end{pmatrix}.$$

$(\because \quad D_{\mathbf{T}} = \sum_{i=1}^{2} T_i \otimes E_i,$
$D_{\mathbf{T}}(x \otimes e_0) = D_{\mathbf{T}}^0(x \otimes e_0) = T_1 x \otimes Ee_0 \oplus T_2 x \otimes Ee_0$
$= T_1 x \otimes e_1 \wedge e_0 \oplus T_2 x \otimes e_1 \wedge e_0 = T_1 x \otimes e_1 \oplus T_2 x \otimes e_1$ and
$D_{\mathbf{T}}(x_1 \otimes e_1 \oplus x_2 \otimes e_1) = D_{\mathbf{T}}^1(x_1 \otimes e_1 \oplus x_2 \otimes e_1)$
$= -T_2 x_1 \otimes e_2 \wedge e_1 + T_1 x_2 \otimes e_2 \wedge e_1.)$

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When $n = 3$,

$$D_{\mathbf{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} & 0 & 0 & 0 \\ 0 & \begin{pmatrix} 0 & -T_3 & T_2 \\ T_3 & 0 & -T_1 \\ -T_2 & T_1 & 0 \end{pmatrix} & 0 & 0 \\ 0 & 0 & \begin{pmatrix} T_1 & -T_2 & T_3 \end{pmatrix} & 0 \end{pmatrix}$$

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For an n-tuple $\mathbf{T} \equiv (T_1, \ldots, T_n)$, the first mapping $D^0_{\mathbf{T}}$ can be interpreted as $D^0_{\mathbf{T}} : \mathcal{X} \to \mathcal{X}^n$ defined by $D^0_{\mathbf{T}} x = \oplus^n_{i=1} T_i x$ $(x \in \mathcal{X})$. Similarly, $D^0_{\mathbf{T}} : \mathcal{X} \to \mathcal{X}^n$ is defined by $D^{n-1}_{\mathbf{T}} \left(\oplus^n_{i=1} x_i\right) = \sum^n_{i=1} (-1)^{i-1} \, T_i x_i$.

For hyponormal $W_\alpha \equiv \mathrm{shift}(\alpha_0, \alpha_1, \cdots)$,

$\sigma(W_\alpha) := \{\lambda \in \mathbb{C} : W_\alpha - \lambda$ is not invertible$\}$ is a closed disk with the radius $\|W_\alpha\|$

$S \in B(\mathcal{H})$ is called a Fredrolm operator if $S(\mathcal{H})$ is closed, $\dim(\ker S) < \infty$, and $\dim(\ker S^*) = \dim(\mathcal{H}/S(\mathcal{H})) < \infty$.

$\sigma_e(W_\alpha) := \{\lambda \in \mathbb{C} : W_\alpha - \lambda$ is not Fredrolm$\}$ is a circle with the radius $\|W_\alpha\|$

The Fredrolm index of the open disk is $\dim(\ker W_\alpha) - \dim(\ker W^*_\alpha) = -1$

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Recall: let $S \in B(\mathcal{H})$ and its range admits a closed complementary subspace. Then $S(\mathcal{H})$ is closed.

Proof: Let C be a closed complement for the range. We can assume that S is injective since $\ker S$ is a closed subspace and hence $\mathcal{H}/\ker S$ is a Banach space so we can replace S by the induced map from this quotient.

Consider $\mathcal{H} = S(\mathcal{H}) \oplus C$ and the map $W : \mathcal{H} \oplus C \to \mathcal{H}$ defined by $W(x, c) = S(x) + c$. Then, the space $\mathcal{H} \oplus C$ is Banach space with the norm $\|(x, c)\| = \|x\| + \|c\|$, W is bounded linear operator, and by the open mapping theorem,

$\mathrm{Ran}\,(W) = W(\mathcal{H} \oplus \{0\}) = S(\mathcal{H})$ is closed.

Thus, $\dim(\mathcal{H}/S(\mathcal{H})) < \infty \Longrightarrow S(\mathcal{H})$ is closed.

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Recall that the Taylor spectrum $\sigma_T(\mathbf{T})$ of $\mathbf{T} \equiv (T_1, T_2)$ is

$$\sigma_T(\mathbf{T}) := \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : K\left((T_1 - \lambda_1, T_2 - \lambda_2), \mathcal{H}\right) \text{ is not invertible}\right\}$$

$\mathbf{T}$ is called Fredholm if $\operatorname{ran} D_{\mathbf{T}}$ is closed and $\dim\left(\ker D_{\mathbf{T}}/\operatorname{ran} D_{\mathbf{T}}\right) < \infty$.

We can also define the Taylor essential spectrum $\sigma_{Te}(\mathbf{T})$ of $\mathbf{T} \equiv (T_1, T_2)$ as follows:

$$\sigma_{Te}(\mathbf{T}) := \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : (T_1 - \lambda_1, T_2 - \lambda_2) \text{ is not Fredholm}\}.$$

The Fredrolm index of (T_1, T_2) is $\sum_{i=0}^{1} (-1)^i \dim\left(\operatorname{Ker} D_{\mathbf{T}}^{i+1}/\operatorname{Ran} D_{\mathbf{T}}^{i}\right)$.

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Recall:
Assume that $(T_1, T_2) \equiv (V_1 P, V_2 P)$, where
$P = (T_1^* T_1 + T_2^* T_2)^{1/2}$, and let
$(\widehat{T_1, T_2}) \equiv (\widehat{T_1}, \widehat{T_2}) := (\sqrt{P} V_1 \sqrt{P}, \sqrt{P} V_2 \sqrt{P})$.
Assume also that (T_1, T_2) is commutative. Then we have:
(i) (V_1, V_2) is a (joint) partial isometry; more precisely,
$V_1^* V_1 + V_2^* V_2$ is the projection onto $\mathrm{ran}\ P$;
(ii) $(\widehat{T_1, T_2})$ is commutative.

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Theorem 1: Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair. Then, we have that $\mathbf{T}$ is (Taylor) invertible if and only if $\widehat{\mathbf{T}}$ is (Taylor) invertible.

Proof:

Consider a short Koszul complex $K(\mathbf{T}, \mathcal{H})$ associated to $\mathbf{T}$ on $\mathcal{H}$:

$$K(\mathbf{T}, \mathcal{H}) : 0 \longrightarrow \mathcal{H} \xrightarrow{T} \begin{matrix} \mathcal{H} \\ \bigoplus \\ \mathcal{H} \end{matrix} \xrightarrow{(-T_2, T_1)} \mathcal{H} \longrightarrow 0,$$

where $D^0_{\mathbf{T}} = T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ and $D^1_{\mathbf{T}} = (-T_2, T_1)$.

If $\mathbf{T}$ is invertible, that is, $K(\mathbf{T}, \mathcal{H})$ is exact, then T is injective, $(-T_2, T_1)$ is onto, and $\mathrm{ran}\ (T) = \ker(-T_2, T_1)$.

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(Continue Proof)

$$
\begin{array}{lccccccccc}
K(\mathbf{T},\mathcal{H}): & 0 & \longrightarrow & \mathcal{H} & \xrightarrow{T} & \begin{array}{c}\mathcal{H}\\ \bigoplus \\ \mathcal{H}\end{array} & \xrightarrow{(-T_2,T_1)} & \mathcal{H} & \longrightarrow & 0 \\
 & & & \downarrow \phi & & \downarrow \varphi & & \downarrow \psi & & \\
K(\widehat{\mathbf{T}},\mathcal{H}): & 0 & \longrightarrow & \mathcal{H} & \xrightarrow{\widetilde{T}} & \begin{array}{c}\mathcal{H}\\ \bigoplus \\ \mathcal{H}\end{array} & \xrightarrow{(\widehat{-T_2,T_1})} & \mathcal{H} & \longrightarrow & 0 \\
 & & & \downarrow \overline{\phi} & & \downarrow \overline{\varphi} & & \downarrow \overline{\psi} & & \\
K(\mathbf{T},\mathcal{H}): & 0 & \longrightarrow & \mathcal{H} & \xrightarrow{T} & \begin{array}{c}\mathcal{H}\\ \bigoplus \\ \mathcal{H}\end{array} & \xrightarrow{(-T_2,T_1)} & \mathcal{H} & \longrightarrow & 0.
\end{array}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (0)$$

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(Continue Proof)
Claim 1: If $\mathbf{T} = (T_1, T_2)$ is invertible, then $\sqrt{P}$ is invertible.

Proof of Claim 1:
Since $\mathbf{T}$ is invertible, $T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = VP = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P$ is injective.
Thus, $\ker(T) = \{0\}$. Since $\ker(T) = \ker(P) = \{0\}$, P is injective, that is, $\sqrt{P}$ is injective.
For any operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, we note

$$T \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \text{ is injective if and only if } R(T^*) \text{ is dense in } \mathcal{H} \quad \cdots (1)$$

33

(Continue Proof)

$$T \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \text{ is injective if and only if } R(T^*) \text{ is dense in } \mathcal{H} \quad \cdots (1)$$

(why?)

$$\begin{array}{ccc} \mathcal{H} & \overrightarrow{T = VP} & \mathcal{K} \\ \| & & \| \\ \ker T = \ker V = \ker P & & \ker T^* \\ \oplus & & \oplus \\ \overline{ranP} = (\ker T)^{\perp} = \overline{ranT^*} & \begin{matrix} \overrightarrow{V} \\ \overleftarrow{V^*} \end{matrix} & \overline{ranT} \end{array}$$

34
(Continue Proof)
Since T is injective, it follows from (1) that

$$\begin{aligned}&\overline{T_1^*(\mathcal{H})+T_2^*(\mathcal{H})}=\overline{P\left(V_1^*(\mathcal{H})+V_2^*(\mathcal{H})\right)}=\mathcal{H}\\ \Longrightarrow\ &\overline{P(\mathcal{H})}\supseteq\overline{P\left(V_1^*(\mathcal{H})+V_2^*(\mathcal{H})\right)}=\mathcal{H}.\end{aligned}\qquad\cdots\cdots(2)$$

Since P is continuous, by (2), P is onto, that is, $\sqrt{P}$ is onto.
Therefore, we have proved **Claim 1**.

35
(Continue Proof)
By **Claim 1**, we can see that $\phi = \psi =: \sqrt{P}$ and $\varphi := \sqrt{P} \oplus \sqrt{P}$ are all isomorphisms.
Since ϕ, φ, and ψ are all invertible, $\widetilde{T}$ is injective and $\widehat{\mathbf{T}}$ is onto, because T is injective and $\mathbf{T}$ is onto. Thus, we only need to show that

$$\mathrm{ran}\ \left(\widetilde{T}\right) = \ker\,(\widehat{-T_2, T_1}).$$

$(\subseteq)$:
If $y \in \mathrm{ran}\ \left(\widetilde{T}\right)$, then there exists $x \in \mathcal{H}$ such that $\widetilde{T}\,(x) = y = y_1 + y_2 \in \mathcal{H} \bigoplus \mathcal{H}$, that is,

$$\sqrt{P} V_1 \sqrt{P}\,(x) = y_1 \text{ and } \sqrt{P} V_2 \sqrt{P}\,(x) = y_2 \quad \cdots\cdots (3)$$

36
(Continue Proof)
Note that

$$
\begin{aligned}
&(-\sqrt{P}V_2\sqrt{P}, \sqrt{P}V_1\sqrt{P}) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\
&= \left(-\sqrt{P}V_2PV_1\sqrt{P} + \sqrt{P}V_1PV_2\sqrt{P}\right)(x) \\
&= \left(-\sqrt{P}V_2PV_1\sqrt{P} + \sqrt{P}V_1PV_2\sqrt{P}\right)\sqrt{P}(z) \\
&\qquad \left(\because \ \sqrt{P} \text{ is invertible}\right) \\
&= \sqrt{P}\left(-T_2T_1 + T_1T_2\right)(z) \\
&= \sqrt{P}\left(-T_1T_2 + T_1T_2\right)(z) = 0
\end{aligned}
$$

whenever $x \in \operatorname{ran} \sqrt{P}$ and $\sqrt{P}(z) = x$.

37

(Continue Proof)

Therefore, $y \in \ker(-\widehat{T_2, T_1})$. Thus, we have

$$\operatorname{ran}\left(\widetilde{T}\right) \subseteq \ker(-\widehat{T_2, T_1}) \cdots\cdots (4)$$

$(\supseteq)$:

Conversely, if $y \in \ker(-\widehat{T_2, T_1})$, then we can say $y = y_1 + y_2 \in \mathcal{H} \bigoplus \mathcal{H}$ and

$$\begin{aligned} &(-\sqrt{P}V_2\sqrt{P}, \sqrt{P}V_1\sqrt{P}) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \\ &\Longrightarrow \sqrt{P}\left(-V_2\sqrt{P}(y_1) + V_1\sqrt{P}(y_2)\right) = 0 \end{aligned} \qquad \cdots\cdots (5)$$

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(Continue Proof)

Now, by (5) and **Claim 1**, we have

$$-V_2\sqrt{P}(y_1) + V_1\sqrt{P}(y_2) = 0 \cdots\cdots (6)$$

If $y_1, y_2 \in \operatorname{ran} \sqrt{P}$ ($\because \sqrt{P}$ is invertible), then there exist $x_1, x_2 \in \mathcal{H}$ such that $y_1 = \sqrt{P}(x_1)$ and $y_2 = \sqrt{P}(x_2)$. Thus, (6) implies

$$-V_2P(x_1) + V_1P(x_2) = 0 \Longrightarrow (-T_2, T_1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \cdots (7)$$
$$\Longrightarrow x = x_1 + x_2 \in \ker(-T_2, T_1) = \operatorname{ran}(T)$$

Hence, there exists $z \in \mathcal{H}$ such that $T_1(z) = x_1$ and $T_2(z) = x_2$.

39
(Continue Proof)
Note that for $i = 1, 2$

$$\begin{aligned} & T_i(z) = x_i \\ \Longrightarrow\ & \sqrt{P} V_i \sqrt{P} \sqrt{P}(z) = \sqrt{P} x_i = y_i \\ \Longrightarrow\ & \sqrt{P} V_i \sqrt{P}(w) = y_i \\ \Longrightarrow\ & y = y_1 + y_2 \in \operatorname{ran}\left(\widetilde{T}\right), \end{aligned} \qquad \cdots\cdots (8)$$

where $w = \sqrt{P}(z) \in \mathcal{H}$.

40
(Continue Proof)
Therefore, we have

$$\ker(\widehat{-T_2, T_1}) \subseteq \operatorname{ran}\left(\widetilde{T}\right) \cdots\cdots (9)$$

Recall

$$\operatorname{ran}\left(\widetilde{T}\right) \subseteq \ker(\widehat{-T_2, T_1}) \cdots\cdots (4)$$

Therefore, by (4) and (9), we have

$$\ker(\widehat{-T_2, T_1}) = \operatorname{ran}\left(\widetilde{T}\right) \cdots\cdots (10)$$

that is, if $\mathbf{T}$ is invertible, then $\widehat{\mathbf{T}}$ is also invertible.

41
(Continue Proof)
($\Longleftarrow$) Let $\widehat{\mathbf{T}}$ be invertible. We first prove the following claim:
Claim 2: If $\widehat{\mathbf{T}}$ is invertible, then $\sqrt{P}$ is invertible.

Proof of Claim 2: Since $\widehat{\mathbf{T}}$ is invertible, $\widetilde{T}$ is injective and $\widehat{\mathbf{T}}$ is onto. Since $\widehat{\mathbf{T}}$ is onto, $\widehat{T}_1(\mathcal{H}) - \widehat{T}_2(\mathcal{H}) = \mathcal{H}$, that is

$$\begin{aligned}
&\sqrt{P}V_1\sqrt{P}(\mathcal{H}) - \sqrt{P}V_2\sqrt{P}(\mathcal{H}) = \mathcal{H} \\
&\Longleftrightarrow \sqrt{P}\left(V_1\sqrt{P}(\mathcal{H}) - V_2\sqrt{P}(\mathcal{H})\right) = \mathcal{H} \\
&\Longrightarrow \sqrt{P}\left(V_1\sqrt{P}(\mathcal{H}) - V_2\sqrt{P}(\mathcal{H})\right) \subseteq \sqrt{P}(\mathcal{H}) = \mathcal{H}.
\end{aligned}$$

Thus, $\sqrt{P}$ is onto. Since $\mathcal{H} = \left(\overline{\operatorname{ran}\sqrt{P}}\right) \oplus \ker\sqrt{P}$, $\ker\sqrt{P} = \{0\}$, which says that $\sqrt{P}$ is injective. Therefore, $\sqrt{P}$ is invertible and we have proved **Claim 2**.

42

(Continue Proof)

Since $\widehat{\mathbf{T}}$ is invertible, we let $\overline{\phi} = \overline{\psi} := \left(\sqrt{P}\right)^{-1}$ and $\overline{\varphi} = \left(\sqrt{P}\right)^{-1} \oplus \left(\sqrt{P}\right)^{-1}$ in (0).

By **Claim 2**, we can see that $\overline{\phi}$, $\overline{\varphi}$, and $\overline{\psi}$ are all isomorphisms. Since ϕ, φ, and ψ are all bijectives, $\widetilde{T}$ is injective and $\widehat{\mathbf{T}}$ is onto. Thus, we only need to show that

$$\mathrm{ran}\ (T) = \ker(\mathbf{T}).$$

$(\subseteq)$: If $y \in \mathrm{ran}\ (T)$, then there exists $x \in \mathcal{H}$ such that $T(x) = y = y_1 + y_2 \in \mathcal{H} \bigoplus \mathcal{H}$, that is, for $i = 1, 2$ $V_i P(x) = y_i$. Observe

$$(-V_2P, V_1P)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (-V_2PV_1P + PV_1PV_2P)(x) \quad \cdots (11)$$
$$= (-T_2T_1 + T_1T_2) = 0.$$

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(Continue Proof)
Thus, $y \in \ker(\mathbf{T})$. Therefore, we have

$$\text{ran }(T) \subseteq \ker(\mathbf{T}) \cdots\cdots (12)$$

$(\supseteq)$:
Conversly, if $y \in \ker(\mathbf{T})$, then $y = y_1 + y_2 \in \mathcal{H} \bigoplus \mathcal{H}$ and

$$\begin{aligned}&(-V_2 P(y_1) + V_1 P(y_2)) = 0 \Longrightarrow -\sqrt{P} V_2 P(y_1) + \sqrt{P} V_1 P(y_2) = 0\\ &\Longrightarrow -\sqrt{P} V_2 \sqrt{P}\left(\sqrt{P}(y_1)\right) + \sqrt{P} V_1 \sqrt{P}\left(\sqrt{P}(y_2)\right) = 0\\ &\Longrightarrow \begin{pmatrix} \sqrt{P}(y_1) \\ \sqrt{P}(y_2) \end{pmatrix} \in \ker\left(\widehat{\mathbf{T}}\right) = \text{ran }\left(\widetilde{T}\right).\end{aligned}$$

Observe that **Claim 2** implies that there exists $\sqrt{P}(z) \in \mathcal{H}$ for any $z \in \mathcal{H}$, because $\sqrt{P}$ is onto.

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(Continue Proof)
Thus, by (13), we have that for $i = 1, 2$

$$\sqrt{P} V_i \sqrt{P} \left(\sqrt{P} (z) \right) = \sqrt{P} (y_i)$$
$$\Longrightarrow T_i (z) = y_i \Longrightarrow y \in \operatorname{ran} \ (T).$$

Hence, we have

$$\ker (\mathbf{T}) \subseteq \operatorname{ran} \ (T) \cdots\cdots (14)$$

Therefore, by (12) and (14), we have

$$\ker (\mathbf{T}) = \operatorname{ran} \ (T),$$

that is, if $\widehat{\mathbf{T}}$ is invertible, then $\mathbf{T}$ is also invertible. Hence, we complete our proof.

45

Next consider whether $\mathbf{T}$ is Fredholm if and only if $\widehat{\mathbf{T}}$ is Fredholm.

Theorem 2: Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair.
Then, we have that $\mathbf{T}$ is Fredholm if and only if $\widehat{\mathbf{T}}$ is Fredholm.
Proof:
Claim 1: If $\mathbf{T} = (T_1, T_2)$ is Fredholm, then $\sqrt{P}$ is Fredholm, that is, $\sqrt{P}(\mathcal{H})$ is closed, dim $\left(\ker \sqrt{P}\right) < \infty$, and dim $\left(\mathcal{H}/\sqrt{P}(\mathcal{H})\right) < \infty$.

Proof of Claim 1: Since $\mathbf{T}$ is Fredholm, $\mathrm{ran}\ (T)$ is closed in $\mathcal{H} \bigoplus \mathcal{H}$ and $\mathrm{ran}\ (-T_2, T_1)$ is closed in $\mathcal{H}$, and

$$\dim\left(\ker\left(T\right)\right), \dim\left(\ker\left(-T_2, T_1\right)/\mathrm{ran}\left(T\right)\right), \dim\left(\mathcal{H}/\mathrm{ran}(-T_2, T_1)\right) < \infty.$$

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(Continue Proof)
Since $\dim(\ker(T)) < \infty$ and $\ker(T) = \ker(P)$, we have

$$\dim(\ker(P)) = \dim\left(\ker(\sqrt{P})\right) < \infty.$$

Since $\dim\left(\ker(\sqrt{P})\right) < \infty$ and $\sqrt{P}$ is continuou, we have
$\mathcal{H} = \ker\sqrt{P} \oplus \left(\overline{\text{ran }\sqrt{P}}\right) = \ker\sqrt{P} \oplus \left(\text{ran }\sqrt{P}\right)$ and
$\dim\left(\mathcal{H}/\sqrt{P}(\mathcal{H})\right) < \infty$.
Thus, $\sqrt{P}(\mathcal{H})$ is closed and $\sqrt{P}$ is Fredholm.
Therefore, we have proved **Claim 1**.

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(Continue Proof)
Claim 2: If $\widehat{\mathbf{T}}$ is is Fredholm, then $\sqrt{P}$ is Fredholm.
Proof of Claim 2: Since $\widehat{\mathbf{T}}$ is Fredholm, $\operatorname{ran}(\widetilde{T})$ is closed in $\mathcal{H}\bigoplus\mathcal{H}$ and $\operatorname{ran}(-\widehat{T}_2, \widehat{T}_1)$ is closed in $\mathcal{H}$, and

$\dim(\ker(\widetilde{T})), \dim(\ker(-\widehat{T}_2, \widehat{T}_1)/\operatorname{ran}(\widetilde{T})), \dim(\mathcal{H}/\operatorname{ran}(-\widehat{T}_2, \widehat{T}_1)) < \infty$.

Since $\dim\left(\mathcal{H}/\operatorname{ran}(-\widehat{T}_2, \widehat{T}_1)\right) < \infty$, we have
$\mathcal{H} = \operatorname{ran}(-\widehat{T}_2, \widehat{T}_1) \oplus N$, where $\dim(N) < \infty$, that is,

$$\begin{aligned}
&\sqrt{P}V_1\sqrt{P}(\mathcal{H}) - \sqrt{P}V_2\sqrt{P}(\mathcal{H}) + N = \mathcal{H} \\
&\Longleftrightarrow \sqrt{P}\left(V_1\sqrt{P}(\mathcal{H}) - V_2\sqrt{P}(\mathcal{H})\right) + N = \mathcal{H} \\
&\Longrightarrow \sqrt{P}(\mathcal{H}) + N \supseteq \mathcal{H} \Longrightarrow \sqrt{P}(\mathcal{H}) + N = \mathcal{H}.
\end{aligned}$$

Thus, $\dim\left(\mathcal{H}/\sqrt{P}(\mathcal{H})\right) < \infty$. Since $\mathcal{H} = \ker\sqrt{P} \oplus \left(\operatorname{ran}\sqrt{P}\right)$, we have $\dim\left(\ker(\sqrt{P})\right) < \infty$.

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(Continue Proof)

($\Longrightarrow$) Let $\mathbf{T}$ be Fredholm. Then, by **Claim 1**, $\sqrt{P}$ is invertible in the Calkin algebra $\mathcal{E} \equiv \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ is a maximal norm-closed ideal of compact operators in $\mathcal{B}(\mathcal{H})$. Consider the following Koszul complexes:

Let $K(\mathbf{T}) := K(\mathbf{T}, \mathcal{E})$ and $K(\widehat{\mathbf{T}}) := K(\widehat{\mathbf{T}}, \mathcal{E})$.

$$
\begin{array}{cccccccccc}
K(\mathbf{T}) & 0 & \longrightarrow & \mathcal{E} & \xrightarrow{T} & \begin{array}{c}\mathcal{E}\\ \bigoplus \\ \mathcal{E}\end{array} & \xrightarrow{(-T_2, T_1)} & \mathcal{E} & \longrightarrow & 0 \\
 & & & \overline{\phi}\uparrow\downarrow\phi & & \overline{\varphi}\uparrow\downarrow\varphi & & \overline{\psi}\uparrow\downarrow\psi & & \\
K(\widehat{\mathbf{T}}) & 0 & \longrightarrow & \mathcal{E} & \xrightarrow{\widetilde{T}} & \begin{array}{c}\mathcal{E}\\ \bigoplus \\ \mathcal{E}\end{array} & \xrightarrow{\widehat{(-T_2, T_1)}} & \mathcal{E} & \longrightarrow & 0.
\end{array}
$$

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(Continue Proof)

Let $\phi = \psi = \sqrt{P}$ and $\varphi = \sqrt{P} \oplus \sqrt{P}$. Then, $\widetilde{T} \circ \phi = \varphi \circ T$. Hence, by the similar argument of **Claim 1** in the proof of Theorem 1, we can see that $\widehat{\mathbf{T}}$ is Fredholm.

($\Longleftarrow$) Let $\widehat{\mathbf{T}}$ be Fredholm. By **Claim 2**, we have that $\sqrt{P}$ is invertible in the Calkin algebra $\mathcal{E}$. Let $\overline{\phi} = \overline{\psi} = \left(\sqrt{P}\right)^{-1}$ and $\overline{\varphi} = \left(\sqrt{P}\right)^{-1} \oplus \left(\sqrt{P}\right)^{-1}$. Then $\overline{\varphi} \circ \widetilde{T} = T \circ \overline{\phi}$. By the similar argument of **Claim 2** in the proof of Theorem 1, we have that $\mathbf{T}$ is Fredholm, as desired.

50

We now consider whether $\mathbf{T}-\lambda$ is invertible if and only if $\widehat{\mathbf{T}-\lambda}$ is invertible, where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}$.

For this, we recall the criss-cross commutativity of pair of operators.

Let $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2)$ be pairs and consider $\mathbf{AB} := (A_1B_1, A_2B_2)$.

If $\mathbf{A}$ and $\mathbf{B}$ are commuting pairs, there is no reason that $\mathbf{AB}$ remains a commuting.

To ensure that $\mathbf{AB}$ remains a commuting pair, suitable extra conditions are needed.

One of conditions is the so-called "criss-cross commutativity".

The pairs $\mathbf{A}$ and $\mathbf{B}$ are said to be criss-cross commuting provided that for every $1 \leq i, j, k \leq 2$

$$A_iB_jA_k = A_kB_jA_i \text{ and } B_iA_jB_k = B_kA_jB_i.$$

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The pairs **A** and **B** are said nearly commuting provided that $A_i B_j = B_j A_i$ for every $i \neq j$.

Example of criss-cross commuting tuples of operators:

Let $\mathbf{A} = (A_1, A_2) = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)$

and $\mathbf{B} = (B_1, B_2) = \left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right)$.

Then **A** and **B** are commuting pairs. Furthermore. they are criss-cross commuting.

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Recently, C. Benhida and R. Curto have proved the following result:

Lemma 3: Let $\mathbf{S} \equiv (S_1, S_2)$ and $\mathbf{T} \equiv (T_1, T_2)$ be pairs of operators satisfying the criss-cross commutativity condition.

If **ST** and **TS** are both commuting, then

$$\sigma_T(\mathbf{ST}) \setminus \{(0,0)\} = \sigma_T(\mathbf{TS}) \setminus \{(0,0)\}$$
$$\text{and } \sigma_{Te}(\mathbf{ST}) \setminus \{(0,0)\} = \sigma_{Te}(\mathbf{TS}) \setminus \{(0,0)\},$$

where $\sigma_{Te}(\mathbf{T})$ means the Taylor essential spectrum of **T**.

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Corollary 4: Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair. Then, we have

$$\sigma_T\left(\widehat{\mathbf{T}}\right) = \sigma_T(\mathbf{T}) \cdots\cdots (16)$$

Proof: We put $\mathbf{A} = (V_1\sqrt{P}, V_2\sqrt{P})$ and $\mathbf{B} = (\sqrt{P}, \sqrt{P})$. If $\lambda = (0,0)$, then we use Theorem 1 for (16).
If $\lambda \neq (0,0)$, then we use Lemma 4 for (16) and our proof is completed.

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Corollary 5: Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair. Then, we have

$$\sigma_{Te}\left(\widehat{\mathbf{T}}\right) = \sigma_{Te}(\mathbf{T}) \cdots\cdots (17)$$

Proof: We put $\mathbf{A} = (V_1\sqrt{P}, V_2\sqrt{P})$ and $\mathbf{B} = (\sqrt{P}, \sqrt{P})$. If $\lambda = (0, 0)$, then we use Theorem 3 for (17).
If $\lambda \neq (0, 0)$, then we use Lemma 4 for (17) and our proof is completed.

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Theorem 6: Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair. Then, for $0 \leq \epsilon \leq 1$, we have

$$\sigma_T\left(\widehat{\mathbf{T}}^{\epsilon}\right) = \sigma_T(\mathbf{T})$$

and

$$\sigma_{Te}\left(\widehat{\mathbf{T}}^{\epsilon}\right) = \sigma_{Te}(\mathbf{T})$$

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We next study the Fredrolm index of (T_1, T_2). We recall that the Fredrolm index of $\mathbf{T} \equiv (T_1, T_2)$ is

$$\operatorname{ind}(\mathbf{T}) := \sum_{i=0}^{1} (-1)^i \dim \left(\ker D_{\mathbf{T}}^{i+1} / \operatorname{ran} D_{\mathbf{T}}^{i}\right).$$

Theorem 7: Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair. Then, we have that $\mathbf{T}$ is Fredholm if and only if $\widehat{\mathbf{T}}$ is Fredholm. Furthermore,

$$\operatorname{ind}\left(\widehat{\mathbf{T}}\right) = \operatorname{ind}(\mathbf{T}).$$

Proof: We note that if $\mathbf{T}$ (resp $\widehat{\mathbf{T}}$) is Fredholm, then $\phi = \psi = \sqrt{P}$ is invertible in Calkin algebra $\mathcal{E} \equiv \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.
Since $\sqrt{P}$ is invertible in Calkin algebra $\mathcal{E}$, by the similar proof in Theorem 1, we have that $\operatorname{ind}\left(\widehat{\mathbf{T}}\right) = \operatorname{ind}(\mathbf{T})$, that is,

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(Continue Proof)

$$\begin{aligned}
&\operatorname{ind}(\mathbf{T}) \\
&= \dim(\ker T_1 \cap \ker T_2) - \dim(\ker(-T_2, T_1)/\operatorname{ran} T) + \mathcal{H}/\operatorname{ran}(-T_2, T_1) \\
&= \dim(\ker \widehat{T}_1 \cap \ker \widehat{T}_2) - \dim(\ker(\widehat{-T_2, T_1})/\operatorname{ran} \widetilde{T}) + \mathcal{H}/\operatorname{ran}(\widehat{-T_2, T_1}) \\
&= \operatorname{ind}\left(\widehat{\mathbf{T}}\right).
\end{aligned}$$

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Lemma 8: Let $\mathbf{A} \equiv (A_1, A_2)$ and $\mathbf{B} \equiv (B_1, B_2)$ be criss-cross commuting pairs
Let $\mathbf{C} \equiv (A_1B_1 - \lambda_1, A_2B_2 - \lambda_2)$ and $\mathbf{D} \equiv (B_1A_1 - \lambda_1, B_2A_2 - \lambda_2)$, where there exists at least one k such that $\lambda_k \neq 0$ $(k = 1, 2)$.
Then, $\mathbf{C}$ is Fredholm if and only if $\mathbf{D}$ is Fredholm.
In this case, we have

$$\mathrm{ind}\,(\mathbf{C}) = \mathrm{ind}\,(\mathbf{D}) \text{ and}$$
$$\dim \left(\ker D_{\mathbf{C}}^{i+1} / \mathrm{ran}\, D_{\mathbf{C}}^{i}\right) = \dim \left(\ker D_{\mathbf{D}}^{i+1} / \mathrm{ran}\, D_{\mathbf{D}}^{i}\right) \; (i = 0, 1),$$

where $\mathrm{ind}\,(\mathbf{C})$ is the Fredrolm index of $\mathbf{C}$.

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Theorem 9: For $0 \leq \epsilon \leq 1$ and $\mathbf{T} \equiv (T_1 - \lambda_1, T_2 - \lambda_2)$, we have

$$\mathrm{ind}\left(\widehat{\mathbf{T}}^{\epsilon}\right) = \mathrm{ind}\left(\mathbf{T}\right),$$

where $\lambda_k \neq 0$ $(k = 1, 2)$.

Proof: Clear from Theorem 7 and Lemma 8.

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Open problems:

Problem 1. ([Exn], [LY2])
If W_α is a subnormal weighted shift with Berger measure μ, are the following statements equivalent?

(i) μ has a square root; (ii) The Aluthge transform $\widetilde{W}_\alpha$ is subnormal.

Problem 2. [CuYo5] Let S be an operator and let $k \geq 2$.
Do the k-hyponormality of S imply the k-hyponormality of S^2?
Do the k-hyponormality of S and $\widetilde{S}$ imply the k-hyponormality of S^2?
Concretely, the k-hyponormality of W_α and $\widetilde{W}_\alpha$ imply the k-hyponormality of W_α^2?

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A quasinormal operator is said to be purely quasinormal if there exists no subspace M of H which is invariant under T such that $T|_{\mathcal{M}}$ is normal, where $T|_{\mathcal{M}}$ means the restriction of T to the invariant subspace M.

We recall that the Hilbert space dimension of the subspace $(U_+(\mathcal{H}))^\perp$ is called the multiplicity of a unilateral shift U_+.

We let $\text{multi}\,(U_+)$ be the multiplicity of a unilateral shift U_+.

Theorem 8: ([Bro], [Con]) $S \in B(H)$ with a polar decomposition $S = U|S|$ is a (purely) quasinormal operator if and only if there exists a positive operator $A \in B(H)$ with $\ker A = \{0\}$ such that $S \cong U_+ \otimes A$, where U_+ is a unilateral shift with $\text{multi}\,(U_+) = n \in N$, $U \cong U_+ \otimes I_{\mathcal{N}}$, and $|S| \cong I_{\mathcal{M}} \otimes A$ with $H = M \otimes N$. Furthermore, if the polar decomposition $S = U|S|$ is unique, then, up to a unitary equivalence, U_+ and A in $S \cong U_+ \otimes A$ are uniquely determined.

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Hence for $n \geq 1$, $S^n = U_+^n \otimes A^n$ and $(S^*)^n = \left(U_+^*\right)^n \otimes A^n$, so that

$$(S^*)^n S^n = I \otimes A^{2n},\ S^n (S^*)^n S^n = U_+^n \otimes A^{3n},$$
and $S^n (S^*)^n S^n = U_+^n \otimes A^{3n}$.

Therefore, we have $\left[S^n, (S^*)^n S^n\right] = 0$, that is, S^n is quasinormal. Thus, we can ask:

Problem 3. If S^2 is quasinormal, then is S quasinormal?

Problem 4. If S^2 and $\left(\widetilde{S}\right)^2$ are both quasinormal, then is S quasinormal?

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We can also give an answer to **Problem 4.**
For this, we let $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+) = \bigvee_{i=0}^{\infty}\{e_i\}$.
Given integers m and h ($h \geq 1$, $0 \leq m \leq h-1$), define
$\mathcal{H}_m := \bigvee_{i=0}^{\infty}\{e_{hi+m}\}$; clearly,

$$\mathcal{H} = \bigoplus_{m=0}^{h-1} \mathcal{H}_m \cdots\cdots (15)$$

For a weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$, we let

$$W_{\alpha(h:m)} := \text{shift}\left(\Pi_{n=0}^{h-1}\alpha_{hi+m+n}\right)_{i=0}^{\infty};$$

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For a weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$, we let

$$W_{\alpha(h:m)} := \text{shift}\left(\Pi_{n=0}^{h-1}\alpha_{hi+m+n}\right)_{i=0}^{\infty};$$

that is, $W_{\alpha(h:m)}$ denotes the sequence of products of weights in adjacent packets of size h, beginning with $\alpha_m \cdots \alpha_{m+h-1}$.
For example, given a weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$, we have
$W_{\alpha(2:0)} = \text{shift}(\alpha_0\alpha_1, \alpha_2\alpha_3, \cdots)$ and
$W_{\alpha(3:2)} = \text{shift}(\alpha_2\alpha_3\alpha_4, \alpha_5\alpha_6\alpha_7, \cdots)$.
For $h \geq 1$, and $0 \leq m \leq h-1$, we note that $W_{\alpha(h:m)}$ is unitarily equivalent to $W_\alpha^h|_{\mathcal{H}_m}$. Therefore, W_α^h is unitarily equivalent to $\bigoplus_{i=0}^{h-1} W_{\alpha(h:m)}$ [CuP].

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Problem 5. [CuYo5] When does the subnormality of S^2 imply the subnormality of S ?

Problem 6. Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair and spherically quasinormal with purely quasinormals T_1 and T_2. Can we say that there exist a (joint) isometry $\mathbf{U} = (U_1, U_2)$ and $P \geq 0$ such that $\mathbf{T} = \mathbf{U} \otimes P$?

Problem 7. If $W_{(\alpha,\beta)}$ is a subnormal with Berger measure μ, are the following statements equivalent?
(i) μ has a square root; (ii) The spherical Aluthge transform $\widehat{W}_{(\alpha,\beta)}$ of $W_{(\alpha,\beta)}$ is subnormal.

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We use $\mathfrak{H}_0$ (resp. $\mathfrak{H}_\infty$) to denote the set of commuting pairs of subnormal operators (resp. subnormal pairs) on Hilbert space. For $k \geq 1$, we let $\mathfrak{H}_k$ denote the class of k-hyponormal pairs in $\mathfrak{H}_0$.

Clearly, $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$. The main results in ([CLY1], [CuYo1]) show that these inclusions are all proper.

Recently, in [LLY3] we gave a negative answer to the Lubin's question **(iii):**

If (T_1, T_2) is a pair of commuting subnormal operators on $\mathcal{H}$, do they admit commuting normal extensions **(i)** when $p(T_1, T_2)$ is subnormal for every 2-variable polynomial p, **(ii)** when $T_1 + sT_2$ (all $s \in \mathbb{C}$) is subnormal, or more weakly, and **(iii)** when $T_1 + T_2$ is subnormal?

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Problem 8. [Lu1] If $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$, do $\mathbf{T}$ admit commuting normal extensions when $T_1 + sT_2$ (all $s \in \mathbb{C}$) is subnormal?

Problem 9. [CLY1] If $\mathbf{T}^{(2,1)} \equiv (T_1^2, T_2), \mathbf{T}^{(1,2)} \equiv (T_1, T_2^2) \in \mathfrak{H}_\infty$, then do $\mathbf{T}$ admit commuting normal extensions?

For **Problem 9**, we split the ambient space $\ell^2(\mathbb{Z}_+^2)$ into an orthogonal direct sum $\oplus_{p=0}^{m-1} \oplus_{q=0}^{n-1} \mathcal{H}_{(p,q)}^{(m,n)}$, where

$$\mathcal{H}_{(p,q)}^{(m,n)} := \vee\{e_{(m\ell+p,nk+q)} : k = 0, 1, 2, \cdots, \ell = 0, 1, 2, \cdots\}.$$

Let $W_{(\alpha,\beta)}^{(m,n)}|_{\mathcal{H}_{(p,q)}^{(m,n)}}$ be the restriction of $W_{(\alpha,\beta)}^{(m,n)}$ to the space $\mathcal{H}_{(p,q)}^{(m,n)}$. Each of $\mathcal{H}_{(p,q)}^{(m,n)}$ reduces T_1^m and T_2^n, and $W_{(\alpha,\beta)}^{(m,n)}$ is subnormal if and only if each $W_{(\alpha,\beta)}^{(m,n)}|_{\mathcal{H}_{(p,q)}^{(m,n)}}$ is subnormal.

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We let $\alpha^{(m,n)}_{(k_1,k_2)}|_{\mathcal{H}^{(m,n)}_{(p,q)}}$ and $\beta^{(m,n)}_{(k_1,k_2)}|_{\mathcal{H}^{(m,n)}_{(p,q)}}$ be the weights of $W^{(m,n)}_{(\alpha,\beta)}|_{\mathcal{H}^{(m,n)}_{(p,q)}}$.

Problem 10. [CLY11] If one of $\mathbf{T}^{(2,1)}$ and $\mathbf{T}^{(1,2)}$ is spherically quasinormal, then do $\mathbf{T}$ admit commuting normal extensions?

Problem 11. Let $\mathbf{S} = (S_1, S_2)$ and $\mathbf{T} = (T_1, T_2)$ be doubly commutative.

If $(S_1, S_2) = (W_1 Q, W_2 Q)$ (resp. $(T_1, T_2) = (V_1 P, V_2 P)$) is the spherical polar decomposition of $\mathbf{S}$ (resp. $\mathbf{T}$),

is it true that $(S_1 T_1, S_2 T_2) = (W_1 V_1 QP, W_2 V_2 QP)$ is the spherical polar decomposition of $\mathbf{ST}$ $=(S_1 T_1, S_2 T_2)$?

Problem 12: If $\widehat{\mathbf{T}} = \mathbf{T} = \widetilde{\mathbf{T}}$, then is $\mathbf{T}$ (jointly) quasinormal?

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Thank you for your attention!