Complex symmetric operators and their applications

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Plan of Lectures

Lecture 1. Preliminaries

Lecture 2. Complex symmetric operators
- On the generalized mean transform of complex symmetric operators
- m-Complex symmetric operators
- ∞-Complex symmetric operators

Lecture 3. Complex symmetric operators and their applications
- Complex symmetric Toeplitz operators on the Hardy space
- Complex symmetric Toeplitz operators on the weighted Bergman space

Motivation

- $\mathcal{H}$: a complex (separable) Hilbert space
- $\mathcal{L}(\mathcal{H})$: the algebra of all bounded linear operators on $\mathcal{H}$.

Motivation

- In 1925, Takagi observed the relevance of antilinear eigenvalue problem:

$$Tx = \lambda \bar{x}, \ (x \neq 0)$$

where T is an $n \times n$ complex symmetric matrix and $\bar{x}$ denotes complex conjugation of a vector x in $\mathbb{C}^n$.

Ta T. Takagi, *On an algebraic problem related to an analytic theorem of Caratheodory and Fejer and on an allied theorem of Landau*, Japan J. Math. **1**(1925), 83-93.

Study

GP1 S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), 1285-1315.

GP2 __________, *Complex symmetric operators and applications II*, Trans. Amer. Math. Soc. **359**(2007), 3913-3931.

 

- M. Putinar, University of California at Santa Barbara, USA
- S. R. Garcia, Pomona college, USA

Conjugation

1.1 Conjugation

Conjugation

Definition 1.1.1

$C : \mathcal{H} \to \mathcal{H}$ is a conjugation operator on $\mathcal{H}$
if the following conditions hold:
(i) C is antilinear; $C(ax + by) = \bar{a}Cx + \bar{b}Cy$ for all $a, b \in \mathbb{C}$ and $x, y \in \mathcal{H}$.
(ii) C is isometric; $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$
(iii) C is involutive; $C^2 = I$.

- By the polarization identity, the second condition (ii) is equivalent to $\|Cx\| = \|x\|$ for all $x \in \mathcal{H}$.
- Note that $(CTC)^k = CT^kC$ and $(CTC)^* = CT^*C$ for every positive integer k, and $\|C\| = 1$.

Properties of Conjugations

Lemma 1.1.2

For a conjugation C on $\mathcal{H}$, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (Such a basis is C-real).

Proof.

Consider the set $\mathcal{K} = (I + C)\mathcal{H}$. Note that each vector in $\mathcal{K}$ is fixed by C. Since

$$\langle x, y\rangle = \langle Cy, Cx\rangle = \langle y, x\rangle = \overline{\langle x, y\rangle},\ x, y \in \mathcal{K},$$

we conclude that $\mathcal{K}$ is a real Hilbert space. Let $\{e_n\}$ be an orthonormal basis for $\mathcal{K}$. Since $\mathcal{H} = \mathcal{K} + i\mathcal{K}$, it follows that $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$. □

[Ref.] S. R. Garcia, *2011-RENNES01*, PPT.

Properties of Conjugations

Lemma 1.1.3

Any conjugation operator is unitarily equivalent to complex conjugation on an ℓ^2 space of the appropriate dimension.

Proof.

If $\{e_n\}$ is a C-real basis for $\mathcal{H}$, then

$$C(\sum_n \alpha_n e_n) = \sum_n \overline{\alpha_n} e_n.$$

The coordinate map $U : \mathcal{H} \to \ell^2$ defined by $Uf = \{\langle f, e_n \rangle\}$ is unitary and satisfies $JU = UC$ where $J : \ell^2 \to \ell^2$ is the canonical conjugation $J(z_1, z_2, \cdots) = (\overline{z_1}, \overline{z_2}, \cdots)$. □

[Ref.] S. R. Garcia, *2011-RENNES01*, PPT.

Examples of conjugation

- The most trivial example of a conjugation operator is simply complex conjugation on $\mathbb{C}$.
- (Canonical conjugation) $C(x_1, x_2, x_3, \cdots, x_n) = (\overline{x_1}, \overline{x_2}, \overline{x_3}, \cdots, \overline{x_n})$ on $\mathbb{C}^n$.
- (Toeplitz conjugation) $C(x_1, x_2, x_3, \cdots, x_n) = (\overline{x_n}, \overline{x_{n-1}}, \overline{x_{n-2}}, \cdots, \overline{x_1})$ on $\mathbb{C}^n$.

Let's define an operator C as follows:

- (pointwise conjugation) $[Cf](x) = \overline{f(x)}$ on a Lebesgue space $\mathcal{L}^2(\mathcal{X}, \mu)$.
- $[Cf](x) = \overline{f(1-x)}$ on $L^2([0,1])$.
- $[Cf](x) = \overline{f(-x)}$ on $L^2(\mathbb{R}^n)$.
- $[Cf](z) = \overline{zf(z)}u(z) \in \mathcal{K}_u$ for all $f \in \mathcal{K}_u$ where u is an inner function and $\mathcal{K}_u = H^2 \ominus uH^2$ is a Model space.

Conjugation on $\mathbb{C}^2$.

Example 1.1.4

Let the operator C be given by

$$C\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & \frac{a}{\sqrt{1+a^2}} \\ \frac{a}{\sqrt{1+a^2}} & -\frac{1}{\sqrt{1+a^2}} \end{pmatrix} \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}.$$

Then C is the conjugation on $\mathbb{C}^2$.

Proof.

We may assume that a is real. (1) C is involutive;

$$\begin{aligned} C^2\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= C\begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & \frac{a}{\sqrt{1+a^2}} \\ \frac{a}{\sqrt{1+a^2}} & -\frac{1}{\sqrt{1+a^2}} \end{pmatrix} \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix} \\ &= \frac{1}{1+a^2}\begin{pmatrix} 1+a^2 & 0 \\ 0 & 1+a^2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \end{aligned}$$

Conjugation on $\mathbb{C}^2$.

Proof.

(2) C is isometric;

$$
\begin{aligned}
& \left\langle C\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, C\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle \\
= \ & \left\langle \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & \frac{a}{\sqrt{1+a^2}} \\ \frac{a}{\sqrt{1+a^2}} & -\frac{1}{\sqrt{1+a^2}} \end{pmatrix} \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & \frac{a}{\sqrt{1+a^2}} \\ \frac{a}{\sqrt{1+a^2}} & -\frac{1}{\sqrt{1+a^2}} \end{pmatrix} \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix} \right\rangle \\
= \ & \frac{1}{1+a^2} \begin{pmatrix} 1+a^2 & 0 \\ 0 & 1+a^2 \end{pmatrix} \left\langle \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}, \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix} \right\rangle \\
= \ & \left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle.
\end{aligned}
$$

Since C is clearly an antilinear, it follows that C is the conjugation on $\mathbb{C}^2$. □

Hardy space

- $L^2 := L^2(\partial\mathbb{D})$ is the usual Lebesque space on the unit circle $\partial\mathbb{D}$.
- L^∞ is the Banach space consisting of all essentially bounded functions on $\partial\mathbb{D}$.
- $\{z^n : n = 0, \pm 1, \pm 2, \pm 3, \cdots\}$ is an orthonormal basis for L^2.
- The *Hilbert Hardy space*, denoted by H^2, consists of all analytic functions f on $\mathbb{D}$ with power series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, or equivalently, with

$$\sup_{0<r<1} \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\Big) < \infty.$$

- $H^2 = \overline{\text{span}\{z^n : n = 0, 1, 2, 3, \cdots\}}$.
- H^∞ is the space of bounded analytic functions on $\mathbb{D}$.

Conjugation on a Model space $\mathcal{K}_u$

- A function $u \in H^2$ is called *inner* if $|u| = 1$ a.e. on $\partial\mathbb{D}$.

Example 1.1.5

Let u be nonconstant inner function and let $\mathcal{K}_u = H^2 \ominus uH^2$ be Model space. If the operator C is defined by

$$[Cf](z) = \overline{zf(z)}u(z) \in \mathcal{K}_u \text{ for all } f \in \mathcal{K}_u,$$

then C is a conjugation on $\mathcal{K}_u$.

Proof.

Let f be an arbitrary function in $\mathcal{K}_u$ and consider the function $\overline{fz}u$ in $L^2(\mathbb{T})$. Since $\langle \overline{fz}u, \overline{zh}\rangle = \langle uh, f\rangle = 0$ and $\langle \overline{fz}u, uh\rangle = \langle \overline{zh}, f\rangle = 0$ for each $f \in \mathcal{K}_u$ and $h \in H^2$, the antilinear C maps $\mathcal{K}_u$ to itself.

Example

Proof.
On the other hand, since $|u| = 1$ a.e. on $\mathbb{T}$, it follows that

$$\langle Cf, Cg \rangle = \langle \overline{fz}u, \overline{gz}u \rangle = \langle g, f \rangle$$

for each $f, g \in \mathcal{K}_u$. Thus C is isometric. Finally, since

$$C^2 f = C(\overline{fz}u) = \overline{\overline{fz}uz}u = f|z|^2|u|^2 = f$$

for each $f \in \mathcal{K}_u$, we have $C^2 = I$. Thus C is involutive. Hence C is a conjugation on $\mathcal{K}_u$. □

Conjugation on $L^2([-1,1])$

Consider a bounded, positive continuous weight ρ on the interval [-1,1], symmetric with respect to the midpoint of the interval, i.e., $\rho(t) = \rho(-t)$ for $t \in [0,1]$. Let P_n the associated orthogonal polynomials, normalized by the conditions

$$\int_{-1}^{1} P_n(t)^2 \rho(t)dt = 1, \lim_{x\to\infty} P_n(x)/x^n = 1.$$

Due to their uniqueness, these polynomials have real coefficients and satisfy

$$P_n(-t) = (-1)^n P_n(t)$$

for all t. Thus, $e_n(t) = i^n P_n(t)$ for all $n \geq 0$ is a C-real basis for $L^2([-1,1], \rho dt)$ with respect to the symmetry $Cf(t) := \overline{f(-t)}$.

CSO

1.2 Complex symmetric operators

Complex symmetric operators

Definition 1.2.1

- An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be complex symmetric if there exists a conjugation C on $\mathcal{H}$ such that

$$T = CT^*C \tag{1}$$

where T^* is the adjoint of T.

- In this case, we say that T is a complex symmetric operator (CSO) with a conjugation C.

CSOs on finite dimensional spaces

Example 1.2.2

- All 2×2 complex matrix on $\mathbb{C}^2$.
- For distinct complex numbers a and b, let

$$T = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix} : \mathbb{C}^3 \to \mathbb{C}^3.$$

If $\mid a \mid = \mid b \mid$, then T is complex symmetric with respect to conjugation $C(x_1, x_2, x_3) = (\overline{x_3}, \overline{x_2}, \overline{x_1})$ (see [GP1]).
If $\mid a \mid \neq \mid b \mid$, then T is not complex symmetric.

- Finite Toeplitz matrix (e.g., finite Jordan blocks)
- Finite Hankel matrix (any size)
- Complex symmetric matrices

CSMs occur in the study of

(Sorry, we will not discuss the following here)

- thermoelastic waves
- electric power modeling
- quantum reaction dynamics
- multicomponent transport
- vertical cavity surface emitting lasers(VCSELs)
- numerical simulation of high-voltage insulators

[Ref.] S. R. Garcia, *2011-RENNES01*, PPT.

Finite dimensional case

Example 1.2.3

- A 3×3 Toeplitz matrix $T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} \\ a_1 & a_0 & a_{-1} \\ a_2 & a_1 & a_0 \end{pmatrix}$ satisfies $T = CT^*C$ where $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$ for all $z_1, z_2, z_3 \in \mathbb{C}^3$.
- A 3×3 Hankel matrix $T = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}$ satisfies $T = CT^*C$ where $C(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$ for all $z_1, z_2, z_3 \in \mathbb{C}^3$.

Finite Toeplitz matrix is CSO

For a 3×3 Toeplitz matrix, we show that $CT^*C = T$;

$$\begin{aligned}
CT^*C(z_1,z_2,z_3) &= C\begin{pmatrix} \overline{a_0} & \overline{a_1} & \overline{a_2} \\ \overline{a_{-1}} & \overline{a_0} & \overline{a_1} \\ \overline{a_{-2}} & \overline{a_{-1}} & \overline{a_0} \end{pmatrix}\begin{pmatrix} \overline{z_3} \\ \overline{z_2} \\ \overline{z_1} \end{pmatrix} \\
&= C\begin{pmatrix} \overline{a_0 z_3 + a_1 z_2 + a_2 z_1} \\ \overline{a_{-1} z_3 + a_0 z_2 + a_1 z_1} \\ \overline{a_{-2} z_3 + a_{-1} z_2 + a_0 z_1} \end{pmatrix} \\
&= \begin{pmatrix} a_{-2} z_3 + a_{-1} z_2 + a_0 z_1 \\ a_{-1} z_3 + a_0 z_2 + a_1 z_1 \\ a_0 z_3 + a_1 z_2 + a_2 z_1 \end{pmatrix} \\
&= \begin{pmatrix} a_0 & a_{-1} & a_{-2} \\ a_1 & a_0 & a_{-1} \\ a_2 & a_1 & a_0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \\
&= T(z_1,z_2,z_3).
\end{aligned}$$

CSOs on infinite dimensional spaces

Example 1.2.4

- Normal operators (i.e., $T^*T = TT^*$)
- Aluthge transforms of CSOs
- Algebraic operator of order 2 (i.e., $T^2 + aT + b = 0$)
- Truncated Toeplitz operators (i.e., $A^u_\varphi f = P_u(\varphi f), P_u : H^2 \to \mathcal{K}_u := H^2 \ominus uH^2$)
- The Volterra integration operator $Tf(x) = \int_0^x f(y)dy$ on $L^2([0,1])$ satisfies $T = CT^*C$ where $Cf(x) = \overline{f(1-x)}$ on $L^2([0,1])$.

Infinite dimensional case

Example 1.2.5

- Foguel-Hankel operators, i.e., $T = \begin{pmatrix} S^* & H \\ 0 & S \end{pmatrix}$ where S is the unilateral shift on ℓ^2 and H is a Hankel matrix, is CSO w.r.t. $\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$.
- $S^* \oplus S$ where S is the unilateral shift on ℓ^2.
- Binormal operators, that is, 2×2 block operators whose entries are commuting normal operators
- Rank one perturbation of normal operators

Normal is CSO

Theorem 1.2.6

Every normal operator is complex symmetric.

Proof.

We may assume that

$$Tf = \varphi f \quad (*)$$

on $L^2(X, \mu)$ where $\varphi \in L^\infty(X, \mu)$. Let $Cf = \overline{f}$. Then C is a conjugation on $L^2(X, \mu)$. Since

$$T^* f = \overline{\varphi} f,$$

we verify that $CT^*Cf = CT^*\overline{f} = C\overline{\varphi f} = \varphi f = Tf$. Hence $T = CT^*C$ and so T is complex symmetric. □

Normal is CSO

- $(*)$ The building blocks, that is orthogonal summands, of any normal operator are the multiplication operators M_z on a Lebesgue space $L^2(\mu)$ of a planar, positive Borel measure μ with compact support.
- In general, subnormal operators, i.e., it has normal extension, are not complex symmetric.

Rank one operator

- $u \otimes v$ denotes the *rank one* operator given by

$$(u \otimes v)f := \langle f, v \rangle u.$$

Lemma 1.2.7

Let $T = u \otimes v$. Then T is complex symmetric with a conjugation C if and only if T is a constant multiple of $u \otimes Cu$.

Proof.

Since $C\langle f, v\rangle u = \overline{\langle f, v\rangle} Cu = \langle v, f\rangle Cu = \langle Cf, Cv\rangle Cu$, it follows that

$$C(u \otimes v)f = (Cu \otimes Cv)Cf$$

for all $f, u, v \in \mathcal{H}$. Note that $(u \otimes v)^* = (v \otimes u)$. From this, $(u \otimes v)^* = C(u \otimes v)C$ if and only if $(v \otimes u) = (Cu \otimes Cv)$ if and only if $v = \lambda Cu$ for some $\lambda \in \mathbb{C}$. Hence the proof is completed. $\square$

Compact complex symmetric operators

Definition 1.2.8

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *compact* if it can be written in the form $T = \sum_{n=1}^{\infty} \lambda_n \langle f_n, \cdot \rangle g_n$ where $f_1, f_2, \cdots$ and $g_1, g_2, \cdots$ are orthonormal sets (not necessarily complete), and $\lambda_1, \lambda_2, \cdots$ is a sequence of positive numbers with limit zero.

Theorem 1.2.9

Every compact complex symmetric operator T is of the form

$$T = \sum_{n=1}^{\infty} a_n (e_n \otimes Ce_n)$$

where the e_n are certain orthonormal eigenvectors of $|T| = \sqrt{T^*T}$ and $\{a_n\}$ are the nonzero eigenvalues of T, repeated according to multiplicity.

Truncated Toeplitz operators (TTO)

- For any $\varphi \in L^\infty$, the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by the formula

$$T_\varphi f = P(\varphi f), f \in H^2$$

where P denotes the orthogonal projection of L^2 onto H^2.

- For an inner function u and $\varphi \in L^2$, the *truncated Toeplitz operator* $A_\varphi^u : \mathcal{K}_u \to \mathcal{K}_u$ (possibly unbounded) is the compressed operator of T_φ to the space $\mathcal{K}_u$ defined by

$$A_\varphi^u f := P_u T_\varphi P_u f = P_u(\varphi f), f \in \mathcal{K}_u \cap L^\infty(\partial\mathbb{D})$$

where P_u denotes the orthogonal projection of L^2 onto $\mathcal{K}_u$.

Truncated Toeplitz operators (TTO)

Proposition 1.2.10

If u is a nonconstant inner function, then A_φ^u is complex symmetric with the conjugation C.

Proof.

For $f, g \in \mathcal{K}_u$ we have

$$
\begin{aligned}
\langle CA_\varphi^u f, g\rangle &= \langle Cg, A_\varphi^u f\rangle = \langle Cg, P_u T_\varphi P_u f\rangle \\
&= \langle P_u Cg, T_\varphi f\rangle = \langle Cg, P(\varphi f)\rangle \\
&= \langle PCg, \varphi f\rangle = \langle Cg, \varphi f\rangle \\
&= \langle \overline{gz}u, \varphi f\rangle = \langle \overline{fz}u, \varphi g\rangle \\
&= \langle Cf, \varphi g\rangle = \langle PP_u Cf, \varphi g\rangle \\
&= \langle P_u Cf, T_\varphi g\rangle = \langle Cf, P_u T_\varphi P_u g\rangle \\
&= \langle Cf, A_\varphi^u g\rangle = \langle A_\varphi^{u*} Cf, g\rangle.
\end{aligned}
$$

Hence A_φ^u is complex symmetric with the conjugation C. □

Volterra integration operator

Proposition 1.2.11

Let the Volterra integration operator $V : L^2[0,1] \to L^2[0,1]$ be defined by

$$[Vf](x) = \int_0^x f(t)dt.$$

Then V is complex symmetric with the conjugation C given by $[Cf](x) = \overline{f(1-x)}$ on $L^2[0,1]$.

Proof.

Note that $[V^*f](x) = \int_x^1 f(t)dt$. Then

$$\begin{aligned} CV^*f(x) &= C\int_x^1 f(t)dt = CF(x) \text{ where } F(x) := \int_x^1 f(t)dt \\ &= \overline{F(1-x)} = \overline{\int_{1-x}^1 f(t)dt} \qquad (2) \end{aligned}$$

Volterra integration operator

and

$$\begin{aligned} VCf(x) &= V\overline{f(1-x)} \text{ where } g(x) := \overline{f(1-x)} \\ &= Vg(x) = \int_0^x g(y)dy = \int_0^x \overline{f(1-y)}dy \\ &= \int_1^{1-x} \overline{f(t)}(-dt) = \int_{1-x}^1 \overline{f(t)}dt. \end{aligned} \tag{3}$$

Hence $CV^* = VC$ holds.

Basic properties of CSO

In 2006, [S.R. Garcia and M. Putinar, TAMS]

Let $T \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator with a conjugation C.

- T is left invertible if and only if T is right invertible.
- If T^{-1} exists, then T^{-1} is also complex symmetric.
- $\ker T$ is trivial if and only if $\operatorname{ran} T$ is dense in $\mathcal{H}$.
- $p(T)$ is complex symmetric for any polynomial $p(z)$.
- For each λ and $n \geq 0$, the map C establishes an antiliner isometric isomorphisim between $\ker(T - \lambda I)^n$ and $\ker(T^* - \overline{\lambda} I)^n$.
- If T_1 is complex symmetric with a conjugation C_1, then UT_1U^* is complex symmetric with the conjugation UC_1U^* where U is unitary.

Example

Example 1.2.12

The unilateral shift S on H^2 is not complex symmetric.

Indeed, S has no eigenvalues while the backward shift S^* have many eigenvalues.

Refined polar decomposition

Theorem 1.2.13 (Godic-Lucenko [GL])

If U is a unitary operator on $\mathcal{H}$, then there exist conjugations C and J on $\mathcal{H}$ such that $U = CJ$.

Lemma 1.2.14

If both C and J are conjugations on $\mathcal{H}$, then $U = CJ$ is a unitary operator. Moreover, U is complex symmetric with both conjugations C and J.

Example 1.2.15

If U denotes the unitary operator $[Uf](e^{i\theta}) = e^{i\theta} f(e^{i\theta})$ on $L^2(\partial\mathbb{D}, \mu)$, then $U = CJ$ where

$$[Cf](e^{i\theta}) = e^{\frac{i}{2}\theta}\overline{f(e^{i\theta})} \text{ and } [Jf](e^{i\theta}) = e^{-\frac{i}{2}\theta}\overline{f(e^{i\theta})}$$

for all $f \in L^2(\partial\mathbb{D}, \mu)$.

Refined polar decomposition

- Recall that for $T \in \mathcal{L}(\mathcal{H})$, the polar decomposition of T expresses $T = U|T|$ uniquely where $|T| = \sqrt{T^*T}$ and U is a partial isometry with $\ker T = \ker U = \ker |T|$ and that map the initial space $(\ker |T|)^{\perp}$ on the final space $\overline{ran T}$.

Theorem 1.2.16

If $T = U|T|$ is the polar decomposition of a complex symmetric operator T, then $T = CJ|T|$ where J is a partial conjugation, supported on $\overline{ran|T|}$, which commutes with $|T| = \sqrt{T^*T}$. In particular, the partial isometry U is complex symmetric with the conjugation C and $U = CJ$.

Refined polar decomposition

Proof.

Let $T = U|T|$ be the polar decomposition of a complex symmetric operator T. Since U^*U is the orthogonal projection onto $\overline{ran|T|}$, it follows that

$$T = CT^*C = C|T|U^*C = C(U^*U)|T|U^*C = (CU^*C)(CU|T|U^*C) \tag{4}$$

Setting $W = CU^*C$, it follows that $W^* = CUC$ and hence $WW^*W = W$ since $U^*UU^* = U^*$. Thus W is a partial isometry and $A = CU|T|U^*C = CU|T|(CU)^*$ is clearly positive.
If we can show that $\ker A = \ker W = \ker T$, then the uniqueness of the polar decomposition of T gives that $W = U$ and $A = |T|$.

Refined polar decomposition

Proof.

Since U and U^* have $\overline{ran|T|}$ as their initial and final spaces, respectively, it follows that

$$\ker W = \ker A = \ker U^* C.$$

We claim that $\ker T = \ker U^* C$. By (4), $\ker U^* C \subseteq \ker T$. Conversely, if $Tf = 0$, then (4) implies that $|T| U^* Cf = 0$. Since $\overline{ran(U^*)} = \overline{ran|T|}$, we have $U^* Cf = 0$ and so $\ker T = \ker U^* C$. Hence $W = U$ and $A = |T|$.

Since $U = CU^* C$, it follows that U is complex symmetric with C. Writing $J = CU = U^* C$, we have $J^2 = (U^* C)(CU) = U^* U$ and it is the orthogonal projection onto $\overline{ran|T|}$. Since $CU|T|U^* C = |T|$, it follows that $J|T|J = |T|$ and so $J|T| = |T|J$.

Refined polar decomposition

Proof.

Since $J = CU$, we have $\ker J = \ker U = \ker |T| = (\overline{ran|T|})^{\perp}$. Moreover, since $J = U^*C$, it follows that $\overline{ran(J)} = ran(U^*) = \overline{ran|T|}$. Finally, J is clearly isometric on $\overline{ran|T|}$ since CU is isometric there. Thus J is a partial conjugation supported on $\overline{ran|T|}$ which commutes with $|T|$. The proof is completed.

Refined polar decomposition

Corollary 1.2.17

Let T be complex symmetric with the conjugation C. Then the following properties hold.
(i) $T = W|T|$ where W is a unitary complex symmetric operator with the conjugation C.
(ii) T^*T and TT^* are unitarily equivalent.
(iii) T is invertible if and only if $|T|$ is invertilble.

Refined polar decomposition

Example 1.2.18

Let S be the unilateral shift given by $Se_n = e_{n+1}$ and let C be the canonical conjugation on $\ell^2(\mathbb{N})$. Then $SC = CS$ and $S^*C = CS^*$. Hence $S^* \oplus S$ is complex symmetric with respect to the conjugation $\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$. A direct computation shows that

$$\underbrace{\begin{pmatrix} S^* & 0 \\ 0 & S \end{pmatrix}}_{\mathcal{T}} = \underbrace{\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}}_{\mathcal{C}} \underbrace{\begin{pmatrix} 0 & CS \\ CS^* & 0 \end{pmatrix}}_{\mathcal{J}} \underbrace{\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}}_{|\mathcal{T}|}$$

where P is the orthogonal projection

$$P(a_0, a_1, a_2, \cdots) = (0, a_1, a_2, \cdots).$$

Invariant subspaces of CSO

Proposition 1.2.19

Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric with a conjugation C.

- $\mathcal{M}$ is C-invariant if and only if $\mathcal{M}^{\perp}$ is C-invariant
- If $\mathcal{M}$ is a subspace of $\mathcal{H}$ and it is invariant under C and T, then $\mathcal{M}$ reduces T.
- $\mathcal{M}$ reduces T if and only if $C\mathcal{M}$ reduces T.
- If $\mathcal{M}$ is a C-invariant subspace of $\mathcal{H}$ and P is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$, then the compression $A = PTP$ of T to $\mathcal{M}$ satisfies $CA = A^*C$.

Invariant subspaces of CSO

Example 1.2.20

- There are no proper nontrivial subspaces of $\mathbb{C}^n$ which are simultaneously invariant for both the Jordan block $J := J_n(\lambda)$ and w.r.t. the conjugation C.
- If $\mathcal{M}$ is a nontrivial subspace of $\mathbb{C}^n$ which is J-invariant, then it must contain the vector $(1, 0, \cdots, 0)$. But, $C(1, 0, 0, \cdots) = (0, \cdots, 0, 1)$. So if $\mathcal{M}$ is also C-invariant, then $\mathcal{M}$ must be all of $\mathbb{C}^n$.

Invariant subspaces of CSO

Example 1.2.21

Let $\mathcal{X}_{[0,a)}$ be the characteristic function of the interval $[0, a)$ and $a \in [0, 1]$. Then the subspace $\mathcal{X}_{[0,a)}L^2([0, 1])$ is the only invariant subspaces for the Volterra integration operator. There are no proper nontrivial subspaces of V-invariant subspaces of $L^2[0, 1]$ which are also C-invariant.

Aluthge transforms of CSO

1.3 Aluthge transforms of complex symmetric operators

Ga S. R. Garcia, *Aluthge transforms of complex symmetric operators and applications*, Int. Eq. Op. Th. **60**(2008), 357-367.

WG X. Wang and Z. Gao, *A note on Aluthge transforms of complex symmetric operators and applications*, Int. Eq. Op. Th. **65**(2009), 573-580.

Aluthge transforms

- A. Aluthge (1990) introduced the Aluthge transform $\widetilde{T}$ which given by
$$\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$$
for an operator $T = U|T| \in \mathcal{L}(\mathcal{H})$.
- A. Aluthge showed that if T is p-hyponormal with $0 < p < \frac{1}{2}$, then $\widetilde{(\widetilde{T})}$ is hyponormal.
- I. B. Jung, E. Ko, and C. Pearcy proved that if T is a quasiaffinity, then Lat(T) is nontrivial if and only if Lat($\widetilde{T}$) is nontrivial.

Aluthge transform of CSOs

Lemma 1.3.1

If $T \in \mathcal{L}(\mathcal{H})$ is complex symmetric with the conjugation C, then $T = CJ|T|$ where J is a partial conjugation, supported on $\overline{(ran|T|)}$ which commutes with $|T|$.

Remark

We may write $T = CJ|T|$ where J is a conjugation on all of $\mathcal{H}$.

Theorem 1.3.2

The Aluthge transform of a complex symmetric operator is complex symmetric. In other words, if $T = CT^*C$ for some conjugation C, then there exists a conjugation J such that $\widetilde{T} = J(\widetilde{T})^*J$.

Aluthge transform of CSOs

Proof.

By Lemma 1.3.1 and Remark, we may write

$$T = CJ|T|$$

where J is a conjugation on all of $\mathcal{H}$ which commutes with $|T|$. Since $\widetilde{T} = |T|^{\frac{1}{2}}CJ|T|^{\frac{1}{2}}$ and $(CJ)^* = JC$, we have

$$J(\widetilde{T})^*J = J|T|^{\frac{1}{2}}JC|T|^{\frac{1}{2}}J = |T|^{\frac{1}{2}}CJ|T|^{\frac{1}{2}} = \widetilde{T}. \tag{5}$$

□

Aluthge transform of CSOs

Theorem 1.3.3

If T is complex symmetric, then $(\widetilde{T})^* \cong (\widetilde{T^*})$ where $\cong$ denotes unitary equivalence.

Proof.

Since T is complex symmetric, there exist conjugations C and J such that $T = CJ|T|$ and $J|T| = |T|J$. It suffices to establish that

$$\widetilde{T} = J(\widetilde{T})^* J \text{ and } \widetilde{T} = C(\widetilde{T^*})C.$$

Since (5) holds, we only show $\widetilde{T} = C(\widetilde{T^*})C$.

Aluthge transform of CSOs

Proof.

Since T is complex symmetric, it follows that $C(TT^*)C = T^*T$ and so $C(TT^*)^pC = (T^*T)^p$ for all $p \geq 0$. In particular, we have

$$T^* = CTC = C(CJ|T|)C = J|T|C = JC|T^*|.$$

Hence

$$\begin{aligned} C(\widetilde{T^*})C &= C[|T^*|^{\frac{1}{2}}JC|T^*|^{\frac{1}{2}}]C \\ &= |T|^{\frac{1}{2}}CJ|T|^{\frac{1}{2}} = \widetilde{T}. \end{aligned}$$

□

Aluthge transform of CSOs

Example 1.3.4

If S is the unilateral shift on $\mathcal{H}$, then $(\widetilde{S})^*$ and $(\widetilde{S^*})$ are not unitarily equivalent. Indeed, since $S = SI$ and $S^* = S^*(SS^*)$ are the polar decompositions of S and S^*, respectively, we have $(\widetilde{S})^* = (ISI)^* = S^*$ and

$$(\widetilde{S^*}) = (SS^*)S^*(SS^*) = (SS^*)(S^*S)S^* = S(S^*)^2.$$

Hence $(\widetilde{S})^*$ and $(\widetilde{S^*})$ are not unitarily equivalent.

Aluthge transform of CSOs

Lemma 1.3.5

If T is complex symmetric with the conjugation C, then the following are equivalent;
(i) T is quasinormal, i.e., $[T, T^*T] = 0$,
(ii) C and $|T|$ commute,
(iii) T is normal.

Proof.

(i)⇒(ii): If T is quasinormal, then $U|T| = |T|U$. Since $U = CJ$ and $J|T| = |T|J$, it follows that $C|T|J = CJ|T| = |T|CJ$. Thus $C|T| = |T|C$.
(ii)⇒(iii): If C and $|T|$ commute, then

$$TT^* = (CJ|T|)(|T|JC) = C|T|^2C = |T|^2 = T^*T.$$

Hence T is normal. □

Aluthge transform of CSOs

Theorem 1.3.6
If T is complex symmetric, then $\widetilde{T} = T$ if and only if T is normal.

Proof.
Since $\widetilde{T} = T$ if and only if T is quasinormal by [JKP], it follows from Lemma 1.3.5 that $\widetilde{T} = T$ if and only if T is normal. □

JKP I. Jung, E. Ko and C. Pearcy, *Aluthge transform of operators*, Int. Eq. Op. Th., **37**(2000), 437-448.

Aluthge transform of CSOs

Theorem 1.3.7

Let $T \in \mathcal{L}(\mathcal{H})$. Then $\widetilde{T} = 0$ if and only if $T^2 = 0$.

Proof.

Let $T = U|T|$ be the polar decomposition of T. ($\Rightarrow$) If $\widetilde{T} = 0$, then

$$T^2 = U|T|U|T| = U|T|^{\frac{1}{2}}\widetilde{T}|T|^{\frac{1}{2}} = 0.$$

($\Leftarrow$) If $T^2 = 0$, then $U|T|U|T| = 0$. Since U^*U is the orthogonal projection onto $\overline{ran(|T|)}$, $|T|^{\frac{1}{2}}\widetilde{T}|T|^{\frac{1}{2}} = 0$. Moreover, since $\widetilde{T}$ vanishes on $\ker|T|$, it suffices to show that $\widetilde{T}$ vanishes on $\overline{ran|T|}$.

Aluthge transform of CSOs

Proof.

Assume that $y \in ran|T|$ and $z = \widetilde{T}y \neq 0$. Then $y = |T|^{\frac{1}{2}}x$ for some x and so

$$0 = |T|^{\frac{1}{2}}\widetilde{T}|T|^{\frac{1}{2}}x = |T|^{\frac{1}{2}}\widetilde{T}y = |T|^{\frac{1}{2}}z \neq 0$$

since z is a nonzero vector in ran$|T|$. This contradiction shows that $\widetilde{T}$ vanishes on ran$|T|$ and hence on $\overline{ran|T|}$ as well. Hence $\widetilde{T} = 0$. □

Aluthge transform of CSOs

Theorem 1.3.8

If T is nilpotent of order two, then T is complex symmetric.

Example 1.3.9

Let $A \in \mathcal{L}(\mathcal{H})$ be any operator. If $T = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, then T is nilpotent of order two. Hence T is complex symmetric by Theorem 1.3.8.

Aluthge transform of CSOs and applications

- Let $T = U|T| \in \mathcal{L}(\mathcal{H})$ be the polar decomposition of T. Then the Aluthge transform $\widetilde{T}_{s,t}$ is given by

$$\widetilde{T}_{s,t} = |T|^s U |T|^t \text{ for } s, t > 0.$$

Theorem 1.3.10

If T is complex symmetric, then

$$\widetilde{T}_{s,t} = J(\widetilde{T}_{t,s})^* J \text{ and } \widetilde{T}_{s,t} = C(\widetilde{T^*_{s,t}})C.$$

Hence $(\widetilde{T}_{t,s})^* \cong (\widetilde{T^*_{s,t}})$ where $\cong$ denotes unitary equivalence.

Aluthge transform of CSOs and applications

Corollary 1.3.11

If T is complex symmetric, then

$$\widetilde{T}_{t,t} = |T|^t U |T|^t$$

is complex symmetric. In other words, if $T = CT^*C$ for some conjugation C, then there exists a conjugation J such that

$$\widetilde{T}_{t,t} = J(\widetilde{T}_{t,t})^* J.$$

Aluthge transform of CSOs and applications

Example 1.3.12

Consider $T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$. Let $T = U|T|$ be the polar decomposition of T. Then

$$U = \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 & 0 \end{pmatrix} \text{ and } |T| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{3}{\sqrt{5}} \end{pmatrix}.$$

Then T is complex symmetric with respect to $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$.

Aluthge transform of CSOs and applications

(i) We consider the case $s = t$. If $s = t = 1$, then

$$\widetilde{T}_{s,t} = |T|U|T| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{5}} \\ 0 & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

and

$$\begin{aligned} J(\widetilde{T_{s,t}})^* J &= J|T|U^*|T|J \\ &= CU|T|U^*|T|U^*C = CU(T^*)^2C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{5}} \\ 0 & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}. \end{aligned}$$

Hence $\widetilde{T}_{s,t} = J(\widetilde{T}_{s,t})^* J$ and $\widetilde{T}_{s,t}$ is complex symmetric with the conjugation J.

Aluthge transform of CSOs and applications

(ii) We consider the case $s \neq t$. If $s = 2, t = 1$, then

$$\widetilde{T}_{s,t} = |T|^2 U |T| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$J(\widetilde{T_{s,t}})^* J = CU|T|U^*|T|^2 U^* C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{5} & \frac{6}{5} \\ 0 & \frac{1}{5} & \frac{3}{5} \end{pmatrix}.$$

Hence $\widetilde{T_{s,t}} \neq J(\widetilde{T_{s,t}})^* J$.

Aluthge transform of CSOs and applications

- For an operator $T \in \mathcal{L}(\mathcal{H})$, T^* denotes the adjoint of T.
- T is said to be *normal* if $T^*T = TT^*$
- T is said to be *quasinormal* if T^*T and T commute
- T is said to be *hyponormal* if $T^*T - TT^* \geq 0$.
- For $0 < p \leq 1$, we say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$.

Aluthge transform of CSOs and applications

- ([Aluthge and Wang]) T is *w-hyponormal* if and only if

$$|T| \geq (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}} \text{ and } (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*|,$$

- if and only if

$$|\widetilde{T}| \geq |T| \geq |(\widetilde{T})^*|.$$

- ([Ito]) T belongs to *class* $wA(s,t)$ for $s,t > 0$ if and only if

$$|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}} \text{ and } (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t},$$

- if and only if

$$|\widetilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t} \text{ and } |T|^{2s} \geq |(\widetilde{T}_{s,t})^*|^{\frac{2s}{s+t}}.$$

Aluthge transform of CSOs and applications

Lemma 1.3.13

If T is complex symmetric and p-hyponormal, then T is normal.

Proof.

Let T be complex symmetric with the conjugation C. Since T is p-hyponormal, it follows that

$$\langle Cx, C((T^*T)^p - (TT^*)^p)x\rangle = \langle((T^*T)^p - (TT^*)^p)x, x\rangle \geq 0 \quad (6)$$

for any $x \in \mathcal{H}$. Moreover, since $C(T^*T)^p = (TT^*)^p C$, we have

$$\langle Cx, ((TT^*)^p - (T^*T)^p)Cx\rangle \geq 0$$

for any $x \in \mathcal{H}$, i.e., $\langle y, ((TT^*)^p - (T^*T)^p)y\rangle \geq 0$ for any $y \in \mathcal{H}$.

Aluthge transform of CSOs and applications

Proof.

Thus

$$\langle ((TT^*)^p - (T^*T)^p)y, y\rangle \geq 0 \tag{7}$$

for any $y \in \mathcal{H}$. From (6) ad (7), we get $(T^*T)^p = (TT^*)^p$ and so $T^*T = TT^*$. Hence T is normal. □

Theorem 1.3.14

If T is complex symmetric with the conjugation C, then the following are equivalent;
(i) T is w-hyponormal,
(ii) T belongs to class $wA(t,t)$,
(iii) C and $|T|$ commute,
(iv) T is normal.

Aluthge transform of CSOs and applications

Proof.

Since note that T is w-hyponormal if and only if T belongs to class $wA(\frac{1}{2},\frac{1}{2})$, it suffices to prove $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.

$(ii) \Rightarrow (iii)$; If T belongs to class $wA(t,t)$, then $\widetilde{T_{t,t}}$ is semi-hyponormal. Since T is complex symmetric, it follows from Corollary 1.3.11 that $\widetilde{T_{t,t}}$ is complex symmetric. By Lemma 1.3.13, $\widetilde{T_{t,t}}$ is normal and so T is normal.

$(iii) \Rightarrow (iv)$; The proof follows from Lemma 1.3.5.

$(iv) \Rightarrow (i)$; Since $C|T| = |T|C$, it follows that $|T| = |T^*|$. Hence T is w-hyponormal. □

Complex symmetric operators

2. Complex symmetric operators

m-Complex symmetric operators

2.1. The generalized mean transform of complex symmetric operators

BCKL C. Benhida, M. Cho, E. Ko, and J. E. Lee, *On the generalized mean transform of complex symmetric operators*, Banach J. Math. Anal., in press.

Generalized mean transforms

- If $T = U|T|$ is the polar decomposition of an operator $T \in \mathcal{L}(\mathcal{H})$, then the *generalized Aluthge transform* $\widetilde{T}(t)$ of T is defined by $\widetilde{T}(t) := |T|^t U|T|^{1-t}$ for some $t \in [0, \frac{1}{2}]$ where $\widetilde{T}(0) = T$.
- In particular, $\widetilde{T}(\frac{1}{2}) := |T|^{\frac{1}{2}} U|T|^{\frac{1}{2}}$ is called the *Aluthge transform* of T (see [JKP]).
- The *Duggal transform* $\widetilde{T}^D$ of T is given by $\widetilde{T}^D := |T|U$ (see [JKP2]).
- The *mean transform* $\widehat{T}$ of T is defined by $\widehat{T} := \frac{1}{2}(T + \widetilde{T}^D)$ (see [LLY]).

Generalized mean transforms

Definition 2.1.1

The generalized mean transform $\widehat{T}(t)$ *of* $T \in \mathcal{L}(\mathcal{H})$ *is defined by*

$$\widehat{T}(t) := \frac{1}{2}[\widetilde{T}(t) + \widetilde{T}(1-t)]$$

where $\widetilde{T}(t) = |T|^t U |T|^{1-t}$ *denotes the generalized Aluthge transform of* T *for some* $t \in [0, \frac{1}{2}]$*. In particular,* $\widehat{T}(0) = \widehat{T}$ *is the mean transform of* T *and* $\widehat{T}(\frac{1}{2}) = \widetilde{T}$ *is the Aluthge transform of* T*.*

Generalized mean transforms

Recall that the *numerical range* $W(T)$ of T is defined as

$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

and the *numerical radius* $w(T)$ of T is defined by

$$w(T) := \sup\{|\lambda| : \lambda \in W(T)\}.$$

Generalized mean transforms

Theorem 2.1.2

Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{L}(\mathcal{H})$ and let $t \in [0, \frac{1}{2}]$. Then the following properties hold;

(i) $\widehat{kT}(t) = k\widehat{T}(t)$ *for every complex number* k.

(ii) $\widehat{VTV^*}(t) = V\widehat{T}(t)V^*$ *for every unitary operator* V.

(iii) T *is quasinormal if and only if* $\widehat{T}(t) = T$.

(iv) $\|\widehat{T}(t)\| \leq \|\widehat{T}\| \leq \|T\|$ *and* $w(\widetilde{T}) \leq w(\widehat{T}(t))$.

(v) *If* $\ker(T) \subset \ker(T^*)$, *then*

$$\ker(\widehat{T}(t)) \subset \ker(T) \cap \ker(\widetilde{T}(t))$$

holds for $t \neq 0$. *In particular, if* $\widehat{T}(t) = 0$, *then* $T = 0$.

(vi) *If* T *is invertible, then* $\widehat{T}(t)$ *is invertible.*

Generalized mean transforms

Corollary 2.1.3

Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric and let $t \in (0, \frac{1}{2})$. If T is p-hyponormal, then $\widehat{T}(t)$ is normal.

Proof.

Since T is complex symmetric and p-hyponormal, it follows from Lemma 1.3.13 that T is normal. By Theorem 2.1.2, $\widehat{T}(t) = T$ and it is normal. □

Generalized mean transforms

Let W_α be the weighted shift on $\mathcal{H}$ with positive weights $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. By a direct computation, the mean transform of W_α is the following weighted shift operator (see [LLY]);

$$\widehat{W_\alpha} = (\frac{\alpha_0 + \alpha_1}{2}, \frac{\alpha_1 + \alpha_2}{2}, \cdots, \cdots) \tag{8}$$

and its generalized mean transforms are given by

$$\widehat{W_\alpha}(t) = (\frac{{\alpha_0}^t {\alpha_1}^{1-t} + {\alpha_0}^{1-t} {\alpha_1}^t}{2}, \frac{{\alpha_1}^t {\alpha_2}^{1-t} + {\alpha_1}^{1-t} {\alpha_2}^t}{2}, \cdots, \cdots). \tag{9}$$

LLY S. Lee, W. Lee and J. Yoon, *The mean transform of bounded linear operators*, J. Math. Anal. Appl. **410**(2014), 70-81.

Generalized mean transforms

Proposition 2.1.3

Let W_α be the weighted shift on $\mathcal{H}$ with positive weights $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ and let $t \in (0, \frac{1}{2}]$. Then the following statements hold.

(i) If W_α is hyponormal, then $\widehat{W_\alpha}(t)$ is hyponormal.

(ii) If $\widehat{W_\alpha}$ is hyponormal, then $\widehat{W_\alpha}(t)$ is hyponormal.

(iii) $\widehat{W_\alpha}(t)$ is hyponormal if and only if

$$\gamma_n(\gamma_{n+1}^t + \gamma_{n+1}^{1-t}) \geq \gamma_n^t + \gamma_n^{1-t} \tag{10}$$

where $\gamma_n = \frac{\alpha_{n+1}}{\alpha_n}$ for all $n \geq 0$.

Generalized mean transforms

Example 2.1.4

With the same notations as in Proposition 2.1.3, if

$$\alpha = \{1, 2, 1, 2, \cdots, \cdots\}.$$

then

$$\gamma = \{2, \frac{1}{2}, 2, \frac{1}{2}, \cdots, \cdots\}.$$

If $\gamma_{n+1} = 2$, then $\gamma_n = \frac{1}{2}$ and so $\frac{1}{2}(2^t + 2^{1-t}) = \frac{1}{2^{1-t}} + \frac{1}{2^t}$ holds.
If $\gamma_{n+1} = \frac{1}{2}$, then $\gamma_n = 2$ and so $2(\frac{1}{2^t} + \frac{1}{2^{1-t}}) = 2^t + 2^{1-t}$ holds.
Hence, by Proposition 2.1.3(iii), $\widehat{W_\alpha}(t)$ is hyponormal.

Generalized mean transforms

Remark

If $t = 0$ in (10), then the inequality

$$\gamma_n(\gamma_{n+1}^0 + \gamma_{n+1}^1) \geq \gamma_n^0 + \gamma_n^1$$

implies $\gamma_n \gamma_{n+1} \geq 1$, which means that $\widehat{W_\alpha}$ is hyponormal. The converse of Proposition 2.1.3(ii) does not hold, in general.

Generalized mean transforms

Example 2.1.5

With the same notations as in Proposition 2.1.3, if

$$\alpha = \{1, 1, 1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \cdots, \cdots\},$$

then

$$\gamma = \{1, 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \cdots, \cdots\}.$$

If $\gamma_{n+1} = k$ for $k \in \mathbb{N}$, then $\gamma_n = \frac{1}{k}$ and so $\frac{1}{k}(k^t + k^{1-t}) = \frac{1}{k^{1-t}} + \frac{1}{k^t}$ holds. If $\gamma_{n+1} = \frac{1}{k}$ for $k \in \mathbb{N}$, then $\gamma_n = k$ and so $k(\frac{1}{k^t} + \frac{1}{k^{1-t}}) = k^t + k^{1-t}$ holds. Hence, by Proposition 2.1.3(iii), $\widehat{W_\alpha}(t)$ is hyponormal. However, $\widehat{W_\alpha}$ is not hyponormal since $\alpha_n \leq \alpha_{n+2}$ does not hold.

Generalized mean transforms

We now consider the property of an operator T in the class $\delta(\mathcal{H})$ (see [LLY]). Put

$$\delta(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) : U^2|T| = |T|U^2\}.$$

Theorem 2.1.6

If $T \in \mathcal{L}(\mathcal{H})$ is hyponormal which belongs in class $\delta(\mathcal{H})$, then the generalized mean transform $\widehat{T}(t)$ of T is normal all $t \in (0, \frac{1}{2}]$.

Generalized mean transforms

The converse of Theorem 2.1.6 does not hold as it is shown by the following example.

Example 2.1.7

Consider $T = \begin{pmatrix} 0 & P \\ I & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ where P is a positive semidefinite compact operator with a nontrivial kernel. Then $T = U_T|T| = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix}$ and so $T \in \delta(\mathcal{H} \oplus \mathcal{H})$. Since $|T| = \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix}$, it follows that $|T|^t = (I \oplus P)^t = I \oplus P^t$.

Generalized mean transforms

Example

Moreover, since $U_T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, we have

$$\widetilde{T}(t) = |T|^t U_T |T|^{1-t} = \begin{pmatrix} 0 & P^{1-t} \\ P^t & 0 \end{pmatrix} = \widetilde{T}(1-t)^*$$

and

$$\widetilde{T}(t)^* = |T|^{1-t} U_T^* |T|^t = \begin{pmatrix} 0 & P^t \\ P^{1-t} & 0 \end{pmatrix} = \widetilde{T}(1-t).$$

Therefore $\widehat{T}(t) = \widehat{T}(t)^* = \begin{pmatrix} 0 & \frac{P^t + P^{1-t}}{2} \\ \frac{P^{1-t}+P^t}{2} & 0 \end{pmatrix}$. Hence $\widehat{T}(t)$ is normal for $t \in (0, \frac{1}{2}]$. However, T is not hyponormal.

Generalized mean transform of CSOs

Lemma 2.1.8

Let $T = U|T|$ be the polar decomposition $T \in \mathcal{L}(\mathcal{H})$. If $T = V|T|$ is another decomposition of T, then

$$\widetilde{T}(t) = |T|^t U|T|^{1-t} = |T|^t V|T|^{1-t} \quad \text{for every } 0 \leq t < 1.$$

Remark

It is important to notice that this may not be true for $t = 1$.

Generalized mean transform of CSOs

Lemma 2.1.9

Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{L}(\mathcal{H})$. Suppose that T is complex symmetric with a conjugation C. Then $T = U|T| = CJ|T| = C\tilde{J}|T|$, where $\tilde{J}$ is any conjugation that extends J and we have

1. $\tilde{J}|T|^t = |T|^t\tilde{J}$ for all $0 \leq t \leq 1$.
2. $\tilde{J}\widetilde{T}(t)\tilde{J} = \big(\widetilde{T}(1-t)\big)^*$ for all $0 < t < 1$.

Generalized mean transform of CSOs

Example 2.1.10

If $T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^4$, then T is complex symmetric and the generalized Aluthge transforms $\widetilde{T}(t)$ are not complex symmetric for $t \in (0, \frac{1}{2})$ by [Lemma 2.8 and Proposition 5.3 in [LZ2]. However, the generalized mean transform $\widehat{T}(t)$ is complex symmetric.

LZ2 S. Zhu and C. G. Li, *Complex symmetric weighted shift*, Trans. Amer. Math. Soc. **365**(2013), no.1, 511-530.

Generalized mean transform of CSOs

Indeed,

$$\text{if} \quad T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{on} \quad \mathbb{C}^4,$$

$$\text{then} \quad U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } |T| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, we obtain for $0 < t \leq 1$

$$\widetilde{T}(t) = |T|^t U |T|^{1-t} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2^{1-t} & 0 \\ 0 & 0 & 0 & 2^t \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Generalized mean transform of CSOs

Hence, by [LZ2], T is a complex symmetric operator and by [LZ2] and [GW, Lemma 1], $\widetilde{T}(t)$ is a complex symmetric operator if and only if $t = \frac{1}{2}$.
On the other hand, the generalized mean transform is

$$\widehat{T}(t) = \frac{1}{2}[\widetilde{T}(t) + \widetilde{T}(1-t)] = (2^{t-1} + 2^{-t}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, (t \neq 0).$$

Thus $\widehat{T}(t) = 0 \oplus (2^{t-1} + 2^{-t}) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is complex symmetric from [LZ2, Proposition 3.2] and [GW, Lemma 1].

Generalized mean transform of CSOs

Theorem 2.1.11

Let $T \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator. Then the generalized mean transform of T is complex symmetric for $t \in (0, \frac{1}{2}]$.

Generalized mean transform of CSOs

Lemma 2.1.12

Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{L}(\mathcal{H})$. Suppose that T is complex symmetric with a conjugation C. Then $T = U|T| = CJ|T| = C\tilde{J}|T|$, where $\tilde{J}$ is a conjugation that extends J and we have

1. $T^* = U^*|T^*| = JC|T^*| = \tilde{J}C|T^*|$.
2. $C\widetilde{T^*}(t)C = \widetilde{T}(t)$ for all $t \in (0, 1)$.

Theorem 2.1.13

Let $T \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator. Then

1. $\widetilde{T^*}(t)$ and $(\widetilde{T}(1-t))^*$ are unitarily equivalent for all $t \in (0, 1)$.
2. $\widehat{T^*}(t)$ and $(\widehat{T}(t))^*$ are unitarily equivalent for every $t \in (0, \frac{1}{2}]$.

Generalized mean transform of CSOs

Lemma 2.1.14

(Proposition 3.2 in [LZ2]) Let $\{e_j\}_{j=1}{}^n$ be an orthonormal basis of $\mathbb{C}^n$. If $T = \sum_{j=1}^{n-1} \lambda_j e_j \otimes e_{j+1}$ and $\lambda_j \neq 0$ for all j. Then T is complex symmetric if and only if $|\lambda_j| = |\lambda_{n-j}|$ for all $1 \leq j \leq n-1$.

Proposition 2.1.15

Suppose that $T = \sum_{j=1}^{n-1} \lambda_j e_j \otimes e_{j+1}$ where $n \geq 3$ and λ_j are complex numbers such that $\lambda_1 \neq 0$ and $|\lambda_j| + |\lambda_{j+1}| \neq 0$ for all j. Then $\widehat{T}$ is complex symmetric if and only if

$$|\lambda_1| = |\lambda_{n-2}| + |\lambda_{n-1}| \text{ and } |\lambda_j| + |\lambda_{j+1}| = |\lambda_{n-j-1}| + |\lambda_{n-j-2}|$$

for all $1 \leq j \leq n-2$.

Generalized mean transform of CSOs

Example 2.1.16

If $T = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^4$, then $\widehat{T} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is complex symmetric by Lemma 2.1.14 and Proposition 2.1.15. But, T is not complex symmetric by Lemma 2.1.14.

Remark

1) In general, the mean transform $\widehat{T}$ of a complex symmetric operator T may not be complex symmetric ([Theorem 5.2(2) in [Ben]).

Ben C. Benhida, *Mind Duggal transform*, http://arxiv.org/abs/1804.00877, Filomat, to appear.

Generalized mean transform of CSOs

Remark

2) In general, let n be an odd number. If the mean transform $\widehat{T}$ of T is complex symmetric, then

$$\begin{cases} |\lambda_1| = |\lambda_{n-2}| + |\lambda_{n-1}| \\ \qquad\qquad\qquad\qquad \vdots \\ |\lambda_{\frac{n-1}{2}-1}| + |\lambda_{\frac{n-1}{2}}| = |\lambda_{\frac{n-1}{2}}| + |\lambda_{\frac{n-1}{2}+1}| \\ \qquad\qquad\qquad\qquad \vdots \\ |\lambda_1| + |\lambda_2| = |\lambda_{n-3}| + |\lambda_{n-2}|. \end{cases}$$

This implies that $|\lambda_{\frac{n-1}{2}-1}| = |\lambda_{\frac{n-1}{2}+1}|$ and so on. From this, we deduce $|\lambda_{n-1}| = 0$.

Generalized mean transform of skew CSOs

- Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *skew complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $CTC = -T^*$.
- A map K on $\mathcal{H}$ is called an *anti-conjugation* if K is conjugate-linear, $K^2 = -I$, and $\langle Kx, Ky \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$.
- For a subspace $\mathcal{M}$ of $\mathcal{H}$, a conjugate-linear map K on $\mathcal{H}$ is called a *partial anti-conjugation* supported on $\mathcal{M}$ if $\ker(K) = \mathcal{M}$ reduces K and $K|_{\mathcal{M}^\perp}$ is an anti-conjugation.

Generalized mean transform of skew CSOs

Theorem 2.1.17

Let $T = U|T|$ be the polar decomposition $T \in \mathcal{L}(\mathcal{H})$. Suppose that T is skew complex symmetric with a conjugation C. If dim ker(T) is even or ∞, then there exists an anti-conjugation K such that $T = U|T| = CK|T|$ and we have

1. $K|T|^t = |T|^t K$ for all $0 \leq t \leq 1$.
2. $K\widetilde{T}(t)K = \left(\widetilde{T}(1-t)\right)^*$ for all $0 < t < 1$.
3. $T^* = U^*|T^*| = -KC|T^*|$.
4. $C\widetilde{T^*}(t)C = -\widetilde{T}(t)$ for all $0 < t < 1$.
5. $\widetilde{T^*}(t)$ and $(\widetilde{T}(1-t))^*$ are unitarily equivalent for all $t \in (0,1)$.
6. $\widehat{T^*}(t)$ and $(\widehat{T}(t))^*$ are unitarily equivalent for every $t \in (0, \frac{1}{2}]$.

Generalized mean transform of skew CSOs

Corollary 2.1.18

With the same hypothesis as above, we have

1. $K\widetilde{T}K = \left(\widetilde{T}\right)^*$.
2. $\widetilde{T^*}$ and $(\widetilde{T})^*$ are unitarily equivalent.

Theorem 2.1.19

Let $T \in \mathcal{L}(\mathcal{H})$ be a skew complex symmetric operator, i.e., $T = -CT^*C$ for a conjugation C. If dim ker(T) is even or ∞, there exists an anti-conjugation K such that

$$K\widehat{T}(t)K = \left(\widehat{T}(t)\right)^* \quad \text{for} \quad t \in (0, \frac{1}{2}].$$

m-Complex symmetric operators

2.2. m-complex symmetric operators

CKL M. Chō, E. Ko and J. Lee, *On m-complex symmetric operators*, Mediterranean Journal of Mathematics, 13(4)(2016), 2025-2038.

CKL2 M. Chō, E. Ko and J. Lee, *On m-complex symmetric operators II*, 13(5)(2016), 3255-3264.

BCKL C. Benhida, M. Cho,, E. Ko and J. E. Lee, *On symmetric and skew-symmetric operators*, Filomat, 32:1(2018), 293-303.

m-Complex symmetric operators

Motivation

In 1970, J. W. Helton initiated the study of operators $T \in \mathcal{L}(\mathcal{H})$ which satisfy an identity of the form

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} T^{m-j} = 0. \tag{11}$$

He J. W. Helton, *Operators with a representation as multiplication by x on a Sobolev space*, Colloquia Math. Soc. Janos Bolyai **5**, Hilbert Space Operators, Tihany, Hungary (1970), 279-287.

m-Complex symmetric operators

- A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$.
- Note that $(CTC)^k = CT^k C$ and $(CTC)^* = CT^* C$ for every positive integer k, and $\|C\| = 1$.

Definition 2.2.1

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-complex symmetric operator* if there exists some conjugation C such that

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} C T^{m-j} C = 0$$

for some positive integer m. In this case, we say that T is m-complex symmetric with conjugation C.

m-Complex symmetric operators

- Set $\Delta_m(T) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j} T^{*j}CT^{m-j}C$.
- T is an m-complex symmetric operator with conjugation C if and only if $\Delta_m(T) = 0$.
- Note that

$$T^*\Delta_m(T) - \Delta_m(T)(CTC) = \Delta_{m+1}(T). \qquad (12)$$

- If T is m-complex symmetric with conjugation C, then T is n-complex symmetric with conjugation C for all $n \geq m$.
- A 1-complex symmetric operator is complex symmetric.

Examples of m-CSOs

- Normal operators, Algebraic operator of order 2, Aluthge transform of CSO, Truncated Toeplitz operator, Finite Toeplitz matrices, and Hankel matrices.
- (S. R. Garcia and M. Putinar, 2006);
  If T is nilpotent of order 2, then T is complex symmetric.

Examples of m-CSOs

Example 2.2.2

Let $T \in \mathcal{L}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. If T is nilpotent of order $k > 2$ and $T^* \neq CTC$, then T is a $(2k-1)$-complex symmetric operator with conjugation C. Indeed, since T is nilpotent of order k, it gives that $CT^jC = T^{*j} = 0$ for all $j \geq k$. Then since $\max\{j, 2k-1-j\} \geq k$ for any j $(j = 0, 1, 2, ..., 2k-1)$, we get

$$\sum_{j=0}^{2k-1} (-1)^{2k-1-j} \binom{2k-1}{j} T^{*j} C T^{2k-1-j} C = 0.$$

Hence T is a $(2k-1)$-complex symmetric operator with conjugation C.

m-Complex symmetric operators

Example 2.2.3

Let C be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$ on $\mathbb{C}^3$.
If $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$, then $T^3 = 0$ and T is a not complex symmetric operator by [GP]. Hence T is a 5-complex symmetric operator with conjugation C. However, since $T^3 = 0$, we have

$$\sum_{j=0}^{4} (-1)^{4-j} \left(\begin{array}{c} 4 \\ j \end{array} \right) T^{*j} C T^{4-j} C = 6 T^{*2} C T^2 C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 24 \end{pmatrix} \neq 0.$$

So it is not a 4-complex symmetric operator.

m-Complex symmetric operators

Example 2.2.4

Let C be a conjugation on $\mathcal{H}$ and a self-adjoint operator $R \in \mathcal{L}(\mathcal{H})$ be complex symmetric with C, i.e., $R = CRC$. If $RQ = QR$ and $Q^k = 0$ for some $k > 2$ with $Q^* \neq CQC$, then an operator $T = R + Q$ is $(2k-1)$-complex symmetric with conjugation C.

Some spectrums

- The spectrum of $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

- The point spectrum of $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : ker(T - \lambda) \neq (0)\}.$$

- The approximate point spectrum of $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$\sigma_a(T) := \{\lambda \in \mathbb{C} : \ \exists \ \{x_n\} \in \mathcal{H} \ s.t. \ \|x_n\| = 1 \ \forall n \text{ and}$$

$$\lim_{n\to\infty} \|(T - \lambda)x_n\| = 0\}.$$

m-Complex symmetric operators

- Two vectors x and y are C-orthogonal if $\langle Cx, y\rangle = 0$.

Theorem 2.2.5

Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C.
(i) If λ is an eigenvalue of T, then $\overline{\lambda}$ is an eigenvalue of T^*.
(ii) Eigenvectors of T corresponding to distinct eigenvalues are C-orthogonal.
(iii) If $\lambda \in \sigma_{ap}(T)$, then $\overline{\lambda} \in \sigma_{ap}(T^*)$.
(iv) Let $\lambda \neq \mu$. If $\{x_n\}$, $\{y_n\}$ are sequences of unit vectors such that $\lim_{n\to\infty}(T-\lambda)x_n = 0$ and $\lim_{n\to\infty}(T-\mu)y_n = 0$, then $\lim_{n\to\infty}\langle Cx_n, y_n\rangle = 0$.

m-Complex symmetric operators

- $T \in \mathcal{L}(\mathcal{H})$ is said to be *isoloid* if for any $\lambda \in \operatorname{iso}\sigma(T)$, $\lambda \in \mathbb{C}$ is an eigenvalue of T, where isoΔ denotes the set of all isolated points of Δ.

Corollary 2.2.6

Let $T \in \mathcal{L}(\mathcal{H})$ be m-complex symmetric with conjugation C. If T is isoloid, then T^* is also isoloid.

Proof.

Assume that T is isoloid. If $\overline{\lambda} \in \text{iso}\,\sigma(T^*) = \text{iso}\,\sigma(T)^*$, then $\lambda \in \text{iso}\,\sigma(T)$ and hence $\lambda \in \sigma_p(T)$. By Theorem 2.2.5, $\overline{\lambda} \in \sigma_p(T^*)$. So, T^* is also isoloid. □

m-Complex symmetric operators

Theorem 2.2.7

If $\{T_k\}$ is a sequence of m-complex symmetric operators with conjugation C such that $\lim_{k\to\infty}\|T_k - T\| = 0$, then T is also m-complex symmetric with conjugation C.

Proposition 2.2.8

Let $T \in \mathcal{L}(\mathcal{H})$ be invertible and let C be a conjugation on $\mathcal{H}$.
(i) If $T^{*j}CT^{m-j}C = CT^{m-j}CT^{*j}$ for $j = 0, 1, \cdots, m$, then T is m-complex symmetric with conjugation C if and only if $CT^{*-1}C$ is m-complex symmetric with conjugation C.
(ii) T is m-complex symmetric with conjugation C if and only if T^{-1} is m-complex symmetric with conjugation C.

m-Complex symmetric operators

Theorem 2.2.9

If $T \in \mathcal{L}(\mathcal{H})$ is an m-complex symmetric operator with conjugation C, then T^n is also m-complex symmetric with conjugation C for some $n \in \mathbb{N}$.

Corollary 2.2.10

Let $T \in \mathcal{L}(\mathcal{H})$ be m-complex symmetric with conjugation C. If $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$, then $\lim_{n\to\infty} \|T^{*mn} Cx\|^{\frac{1}{n}} = 0$.

SVEP

Single-valued extension property

- We say that an operator T has the *single-valued extension property at λ* (abbreviated SVEP at λ) if for every open set U containing λ the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U.
- T has SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Property (β)

Property (β) [1959, E. Bishop]

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G.

Decomposable

Decomposable [1963, C. Foias]

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{H} = \mathcal{M} + \mathcal{N}$, $\sigma(T|_{\mathcal{M}}) \subset \overline{U}$, and $\sigma(T|_{\mathcal{N}}) \subset \overline{V}$.

Decomposable $\Rightarrow$ Property (β) $\Rightarrow$ SVEP.

ISP, Invariant subspace problem

VON NEUMANN (1932)

Does every bounded linear operator T on a separable Hilbert space $\mathcal{H}$ over complex $\mathbb{C}$ have a non-trivial invariant subspace?

- $\mathcal{M}$ is *nontrivial* if it is different from (0) and $\mathcal{H}$.
- A closed subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* for T if $T\mathcal{M} \subset \mathcal{M}$.
- $\mathcal{M}$ is *hyperinvariant* for T if it is invariant for every operator in $\{T\}' = \{S \in \mathcal{L}(\mathcal{H}) : TS = ST\}$ of T.

m-Complex symmetric operators

Theorem 2.2.11

((2011, JMAA) Jung, Ko, Lee, and Lee) Let $T \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator with conjugation C. Then T has property (β) if and only if T is decomposable.

Theorem 2.2.12

Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C. Then T^* has the property (β) if and only if T is decomposable.

m-Complex symmetric operators

Corollary 2.2.13

Let $T \in \mathcal{L}(\mathcal{H})$ be m-complex symmetric operators.
(i) If T^* is hyponormal, i.e. $TT^* \geq T^*T$, then T is decomposable.
(ii) If T^* has the property (β) and $\sigma(T)$ has nonempty interior, then T has a nontrivial invariant subspace.

m-Complex symmetric operators

Proof.
(i) Since T^* is hyponormal, then it has the property (β). So, the proof follows from Theorem 2.2.12.
(ii) If T^* has the property (β), then T is decomposable from Theorem 2.2.12. So, in this case, T has the property (β) by [LN]. Since $\sigma(T)$ has nonempty interior, we get this result from Theorem 2.1 in [Es]. □

Es J. Eschmeier, *Invariant subspaces for operators with Bishop's property (β) and thick spectrum*, J. Funct. Anal. **94**(1990), 196-222.

m-Complex symmetric operators

- $\rho_T(x) = \{\lambda_0 \in \mathbb{C} : \exists$ an $\mathcal{H}$-valued analytic function f defined in a neighborhood of λ_0 s.t. $(\lambda I - T)f(\lambda) \equiv x\}$: the *local resolvent set of* x.
- $\sigma_T(x) = \mathbb{C} \backslash \rho_T(x)$: the *local spectrum of* T *at* x.
- $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ where $F \subset \mathbb{C}$: the *local spectral subspace of* T.

Theorem 2.2.14

Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C. If T^* has the single-valued extension property, then T has the single-valued extension property. Moreover, in this case, $\sigma_{T^*}(x) \subset \sigma_T(Cx)^*$ for all $x \in \mathcal{H}$.

m-Complex symmetric operators

Corollary 2.2.15

Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C. If T^* has the single-valued extension property, then

$$CH_T(F) \subset H_{T^*}(F^*)$$

where $F^* := \{\bar{z} : z \in F\}$ for any set F in $\mathbb{C}$.

Proof.

If $x \in CH_T(F)$, then $Cx \in H_T(F)$ and so $\sigma_T(Cx) \subset F$. Thus $\sigma_T(Cx)^* \subset F^*$. Since $\sigma_{T^*}(x) \subset \sigma_T(Cx)^*$ by Theorem 2.2.14, it ensures that $\sigma_{T^*}(x) \subset F^*$ and so $x \in H_{T^*}(F^*)$. Hence $CH_T(F) \subset H_{T^*}(F^*)$. □

m-Complex symmetric operators

- Assume that T has the single-valued extension property. If there exists a constant k such that for every $x, y \in \mathcal{H}$ with $\sigma_T(x) \cap \sigma_T(y) = \emptyset$ we have

$$\| x \| \leq k \| x + y \|$$

where k is independent of x and y, we say that an operator T satisfies *Dunford's boundedness condition* (B).

Corollary 2.2.16

Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C. If T^* has the single-valued extension property and the Dunford's boundedness condition (B), then T also has the Dunford's boundedness condition (B).

m-Complex symmetric operators

Proof.

By Theorem 2.2.14, we know that T has the single-valued extension property. Assume that x and y are any vectors in $\mathcal{H}$ such that $\sigma_T(x) \cap \sigma_T(y) = \emptyset$. Since $C\mathcal{H} = \mathcal{H}$, there exist $x_1, y_1 \in \mathcal{H}$ such that $x = Cx_1$ and $y = Cy_1$. Hence $\sigma_T(Cx_1) \cap \sigma_T(Cy_1) = \emptyset$, i.e., $\sigma_T(Cx_1)^* \cap \sigma_T(Cy_1)^* = \emptyset$. By Theorem 2.2.14, we have

$$\sigma_{T^*}(x_1) \cap \sigma_{T^*}(y_1) = \emptyset.$$

m-Complex symmetric operators

Proof.

Since T^* has the Dunford's boundedness condition (B), there exists a constant k such that

$$\| x_1 \| \leq k \| x_1 + y_1 \|$$

where k is independent of x_1 and y_1. Moreover, since $x_1 = Cx$ and $y_1 = Cy$, there is a constant k such that

$$\| Cx \| \leq k \| Cx + Cy \|, \text{ i.e., } \| x \| \leq k \| x + y \| .$$

Hence T also has the Dunford's boundedness condition (B).

m-Complex symmetric operators

- Set $\Delta_m(T) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j} T^{*j}CT^{m-j}C$.
- For $0 < p \leq 1$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$.

Theorem 2.2.17

Let T be an operator on $\mathcal{H}$ and C be a conjugation on $\mathcal{H}$.
(i) If m is even, then $\Delta_m(T)$ is complex symmetric with the conjugation C. In this case, if $\Delta_m(T)$ is p-hyponormal, then it is normal.
(ii) If m is odd, then $\Delta_m(T)$ is skew complex symmetric with the conjugation C. In this case, if $\Delta_m(T) = 0$ and $\Delta_{m-1}(T)$ is p-hyponormal, then $T^*\Delta_{m-1}(T) = \Delta_{m-1}(T)CTC$ and $\Delta_{m-1}(T)$ is normal.

m-Complex symmetric operators

Theorem

(iii) Let

$$K_m(T) := \bigcap_{n \geq m} ker(\Delta_n(T)).$$

If $K_1(T) \neq \{0\}$ and $K_m(T) \neq \mathcal{H}$, then the subspace $C(K_m(T))$ is a nontrivial invariant subspace for T.

m-Complex symmetric operators

Corollary 2.2.18

Let T be an operator on $\mathcal{H}$ and C be a conjugation on $\mathcal{H}$.
(i) If m is even, then $\sigma(\Delta_m(T)) = \sigma_{ap}(\Delta_m(T))$.
(ii) If m is odd, then $\sigma(\Delta_m(T)) = \sigma_{ap}(\Delta_m(T)) \cup [-\sigma_{ap}(\Delta_m(T))]$.
(iii) If m is odd and $\Delta_m(T)$ has finite rank k, then the rank of $\Delta_m(T)$ is even.
(iv) If $K_1(T) \neq \{0\}$ and $1 \notin \sigma_p(CTC)$, then $C(K_1(T))$ has at least two distinct elements of $\mathcal{H}$.
(v) Put $F_n(T) := \bigcap_{n \leq j \leq m-1} ker(\Delta_j(T))$ for $n = 1, 2, \cdots, m-1$. If T is a strict m-complex symmetric operator and $F_1(T) \neq \{0\}$, then $CF_n(T)$ is a nontrivial invariant subspace for T where $n = 1, 2, \cdots, m-1$.

m-Complex symmetric operators

- $T \in \mathcal{L}(\mathcal{H})$ is said to be a *normaloid* operator if $\|T\| = r(T)$ where $r(T)$ is the spectral radius of T.
- A vector $x \in \mathcal{H}$ is said to be *isotropic* if $\langle x, Cx \rangle = 0$ ([GPP]).

Theorem 2.2.19

Let $T \in \mathcal{L}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Suppose $\Delta_{m+1}(T) = 0$, $\Delta_m(T)$ is normaloid, and an eigenvector corresponding to every eigenvalue in $\sigma_p(\Delta_m(T))$ is not isotropic. Assume that one of the following statements holds;
(i) When m is even, for every $\mu \in \sigma_{ap}(\Delta_m(T))$ there exist $\lambda \in \sigma(\Delta_1(T))$ and a sequence $\{x_n\}$ of unit vectors such that $|\lambda|^m = |\mu|$ and

$$\lim_{n\to\infty} \|(\Delta_m(T) - \mu)x_n\| = \lim_{n\to\infty} \|(\Delta_1(T) - \lambda)x_n\| = 0.$$

m-Complex symmetric operators

Theorem

(ii) When m is odd, for every $\mu \in \sigma_{ap}(\Delta_m(T))$ there exist $\lambda \in \sigma(T^* + CTC)$ and a sequence $\{x_n\}$ of unit vectors such that $|\lambda|^m = |\mu|$ and

$$\lim_{n\to\infty} \|(\Delta_m(T) - \mu)x_n\| = \lim_{n\to\infty} \|((T^* + CTC) - \lambda)x_n\| = 0.$$

Then $\Delta_m(T) = 0$.

m-Complex symmetric operators

Corollary 2.2.20

Let C be a conjugation on $\mathcal{H}$ and let $T \in \mathcal{L}(\mathcal{H})$ be a strict $(m+1)$-complex symmetric operator, and an eigenvector corresponding to every eigenvalue in $\sigma_p(\Delta_m(T))$ be not isotropic. If one of the following statements holds;

(i) When m is even, for every $\mu \in \sigma_{ap}(\Delta_m(T))$, there exist $\lambda \in \sigma(\Delta_1(T))$ and a sequence $\{x_n\}$ of unit vectors such that $|\lambda|^m = |\mu|$ and

$$\lim_{n\to\infty} \|(\Delta_m(T) - \mu)x_n\| = \lim_{n\to\infty} \|(\Delta_1(T) - \lambda)x_n\| = 0,$$

m-Complex symmetric operators

Corollary

(ii) *When m is odd, for every $\mu \in \sigma_{ap}(\Delta_m(T))$, there exist $\lambda \in \sigma(T^* + CTC)$ and a sequence $\{x_n\}$ of unit vectors such that $|\lambda|^m = |\mu|$ and*

$$\lim_{n\to\infty} \|(\Delta_m(T) - \mu)x_n\| = \lim_{n\to\infty} \|((T^* + CTC) - \lambda)x_n\| = 0,$$

then $\Delta_m(T)$ is not normaloid.

m-Complex symmetric operators

Using (12), we know that if $\Delta_m(T) = 0$, then $\Delta_{m+1}(T) = 0$. Unlike Theorem 2.2.19, we need a simple condition in the following theorem.

Theorem 2.2.21

For an operator $T \in \mathcal{L}(\mathcal{H})$, let $\Delta_2(T) = 0$. If T is Hermitian or $\Delta_1(T)$ is p-hyponormal, then $\Delta_1(T) = 0$.

Corollary 2.2.22

Let C be a conjugation operator on $\mathcal{H}$, H and K be Hermitian operators. Suppose that $T = H + iK \in \mathcal{L}(\mathcal{H})$ satisfies $HCK = KCH$ and $CRC \geq R$, where $R = i(HK - KH)$. If $\Delta_2(T) = 0$, then $\Delta_1(T) = 0$.

Some useful spectral properties

An operator T on $\mathcal{H}$ is *antilinear* if for all $x, y \in \mathcal{H}$

$$T(\alpha x + \beta y) = \overline{\alpha} Tx + \overline{\beta} Ty$$

holds for all $\alpha, \beta \in \mathbb{C}$.

Lemma 2.2.23

Let B and C be antilinear operators on $\mathcal{H}$. Then the following properties hold;

- BC and CB are linear operators.
- $\gamma B + \delta C$ is an antilinear operator for any $\gamma, \delta \in \mathbb{C}$.
- If D is a linear operator, then BD, DB, CD, and DC are antilinear operators.
- If B^{-1} exists, then B^{-1} is an antilinear operator.

Some useful spectral properties

For an antilinear operator T, a *Hermitian adjoint* operator of T on $\mathcal{H}$ is an antilinear operator $T^\dagger : \mathcal{H} \to \mathcal{H}$ with the property;

$$\langle Tx, y\rangle = \overline{\langle x, T^\dagger y\rangle} \tag{13}$$

for all $x, y \in \mathcal{H}$. If an antilinear operator T is bounded, then, by the Riesz representation theorem, the Hermitian adjoint of T exists and is unique ([CVLL, Page 90]). For antilinear operators T and R, we get immediately from (13) that $(T^\dagger)^\dagger = T$, $(T+R)^\dagger = T^\dagger + R^\dagger$ and $(TR)^\dagger = R^\dagger T^\dagger$.

CVLL G. Cassinelli, E. Vito, A. Levrero, P. J. Lahti, *The Theory of Symmetry Actions in Quantum Mechanics*, Springer.

Some useful spectral properties

Let's start by the following result which is a slight variation of Jacobson's lemma.

Proposition 2.2.24

Let B and C be two antilinear bounded operators on $\mathcal{H}$. Then BC and CB are in $\mathcal{L}(\mathcal{H})$ and

$$I - CB \quad \text{is invertible} \iff I - BC \quad \text{is invertible.} \tag{14}$$

Global spectral properties

For $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$, $\sigma_r(T)$, and $\sigma_c(T)$ for the spectrum, the point spectrum, the approximate point spectrum, the surjective spectrum, the residual spectrum, and continuous spectrum of T, respectively.

Proposition 2.2.25

Let B and C be two antilinear bounded operators on $\mathcal{H}$. Then the following statements hold;

- $\sigma(BC) \setminus \{0\} = \sigma(CB)^* \setminus \{0\}$
- $\sigma_p(BC) \setminus \{0\} = \sigma_p(CB)^* \setminus \{0\}$
- $\sigma_{ap}(BC) \setminus \{0\} = \sigma_{ap}(CB)^* \setminus \{0\}$
- $\sigma_r(BC) \setminus \{0\} = \sigma_r(CB)^* \setminus \{0\}$
- $\sigma_c(BC) \setminus \{0\} = \sigma_c(CB)^* \setminus \{0\}$

where $E^* := \{\bar{\lambda} : \lambda \in E\}$ for $E \subset \mathbb{C}$.

Global spectral properties

We define Weyl spectrum, $\sigma_w(T)$ and Browder spectrum, $\sigma_b(T)$, by

$$\begin{aligned} \sigma_w(T) &= \bigcap_{\{K \text{ is compact}\}} \sigma(T+K) \\ &= \{\lambda \in \mathbb{C} \ : \ T-\lambda \text{ is not Fredholm with index zero}\}, \end{aligned}$$

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Fredholm of finite ascent and descent}\}$$

An operator T in $\mathcal{L}(\mathcal{H})$ is said to satisfy *Weyl's theorem* if

$$\sigma_w(T) = \sigma(T) \setminus \pi_{00}(T)$$

where

$$\pi_{00}(T) = \{\lambda \in iso(\sigma(T)) : 0 < \dim(\ker(T-\lambda)) < \infty\}$$

and $iso(E)$ is the set of all isolated points of E. We say that *Browder's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma_b(T) = \sigma_w(T)$.

Global spectral properties

Notice also that if B and C are antilinear, then C is naturally a mapping of various objects related to BC into those related to CB. For example,

$$C : \ker(BC - \lambda)^p \longrightarrow \ker(CB - \bar{\lambda})^p.$$

Proposition 2.2.26

Let B and C be two antilinear bounded operators on $\mathcal{H}$.
If $0 \in \pi_{00}(BC) \cap \pi_{00}(CB)$ or $0 \notin \pi_{00}(CB) \cup \pi_{00}(BC)$ then;
(i) BC satisfies Weyl's theorem if and only if CB satisfies Weyl's theorem.
(ii) BC satisfies Browder's theorem if and only if CB satisfies Browder's theorem.

Local spectral properties

- $\sigma_{svep}(T)$ denote the set of all points where T fails to have the SVEP.
- For an open set U in $\mathbb{C}$, we denote by $\mathcal{O}(U, \mathcal{H})$ and $\mathcal{E}(U, \mathcal{H})$ the Fréchet space of all $\mathcal{H}$-valued analytic functions on U and the Fréchet space of all $\mathcal{H}$-valued C^∞-functions on U, respectively.
- An operator $T \in \mathcal{L}(\mathcal{H})$ is said to satisfy *Bishop's property (β) at $\lambda \in \mathbb{C}$ (resp. $(\beta)_\epsilon$)* if there exists $r > 0$ such that for every open subset $U \subset D(\lambda, r)$ and for any sequence (f_n) in $\mathcal{O}(U, \mathcal{H})$ (resp. in $\mathcal{E}(U, \mathcal{H})$) such that whenever, $(T - z)f_n(z) \longrightarrow 0$ in $\mathcal{O}(U, \mathcal{H})$ (resp. in $\mathcal{E}(U, \mathcal{H})$), then $f_n \longrightarrow 0$ in $\mathcal{O}(U, \mathcal{H})$ (resp. in $\mathcal{E}(U, \mathcal{H})$).
- $\sigma_\beta(T)$ (resp. $\sigma_{\beta_\epsilon}(T)$) is the set of all points where T does not have property (β) (resp. $(\beta)_\epsilon$).

Local spectral properties

- The operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *spectral decomposition property* (δ) at λ if there exists an open neighborhood U of λ such that for every finite open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$, with $\sigma(T) \setminus U \subseteq U_1$, we have

$$\mathcal{X}_T(\bar{U}_1) + \cdots + \mathcal{X}_T(\bar{U}_n) = \mathcal{H}, \tag{15}$$

  where $\mathcal{X}_T(F)$ is the set of elements $x \in \mathcal{H}$ such that the equation $(T - \lambda)f(\lambda) = x$ has a global analytic solution on $\mathbb{C} \setminus F$.
- The δ-spectrum $\sigma_\delta(T)$ and the decomposability spectrum $\sigma_{dec}(T)$ are defined in a similar way.

Local spectral properties

Proposition 2.2.27

Let B and C be antilinear bounded operators on $\mathcal{H}$. Then BC and CB are in $\mathcal{L}(\mathcal{H})$ and

- $\sigma_{svep}(BC) = \sigma_{svep}(CB)^*$
- $\sigma_{\beta}(BC) = \sigma_{\beta}(CB)^*$
- $\sigma_{\beta_\epsilon}(BC) = \sigma_{\beta_\epsilon}(CB)^*$
- $\sigma_{\delta}(BC) = \sigma_{\delta}(CB)^*$
- $\sigma_{dec}(BC) = \sigma_{dec}(CB)^*$.

Local spectral properties

- An antilinear bounded operator A on $\mathcal{H}$ is called *normal* if A and $A^\dagger$ commute where $A^\dagger$ satisfies (13) (see [Uh, Section 4.1, Page 27]).

Proposition 2.2.28

Let B and C be antilinear bounded operators on $\mathcal{H}$. Then $\sigma(BC) = \sigma(CB)^*$ in the following cases;

1. C and B are injective.
2. C and $C^\dagger$ are injective.
3. C or B is injective with dense range.
4. C and B are not injective.
5. C and $C^\dagger$ are not injective.
6. C or B is normal.

Applications

- Recall that C is a conjugation on $\mathcal{H}$ if $C : \mathcal{H} \longrightarrow \mathcal{H}$ is an antilinear operator that satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$.

Theorem 2.2.29

Let C be a conjugation on $\mathcal{H}$. Then the Hermitian adjoint of C is the conjugation C, i.e., $C^\dagger = C$. Conversely, assume that C is antilinear with $C^2 = I$. If $C^\dagger = C$, then C is a conjugation on $\mathcal{H}$.

Corollary 2.2.30

([GP2]) Let B and C be conjugations on $\mathcal{H}$. Then BC and CB are unitary.

Applications

Theorem 2.2.31

Let T be in $\mathcal{L}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then

- $\sigma_{svep}(CTC) = \sigma_{svep}(T)^*$
- $\sigma_{\beta}(CTC) = \sigma_{\beta}(T)^*$
- $\sigma_{\beta_\epsilon}(CTC) = \sigma_{\beta_\epsilon}(T)^*$
- $\sigma_{\delta}(CTC) = \sigma_{\delta}(T)^*$
- $\sigma_{dec}(CTC) = \sigma_{dec}(T)^*$.

Applications

Theorem 2.2.32

Let T be in $\mathcal{L}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then

$$\sigma_\bullet(CTC) = \sigma_\bullet(T)^*$$

when $\sigma_\bullet \in \{\sigma, \sigma_p, \sigma_{ap}, \sigma_c, \sigma_r, \sigma_{su}, \sigma_e, \sigma_w, \ldots\}$.

Theorem 2.2.33

Let T be in $\mathcal{L}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then T satisfies Weyl's (or Browder's) theorem if and only if CTC satisfies Weyl's (or Browder's) theorem.

Helton classes

Let A and B be two given operators in $\mathcal{L}(\mathcal{H})$. Recall the definition of the usual derivation operator $\delta_{A,B}(X)$ given by

$$\delta_{A,B}(X) = AX - XB \quad \text{for} \quad X \in \mathcal{L}(\mathcal{H}).$$

For every positive integer k, we have

$$\delta_{A,B}^{k}(X) = \delta_{A,B}(\delta_{A,B}^{k-1}(X)) \quad \text{for} \quad X \in \mathcal{L}(\mathcal{H}).$$

Definition 2.2.34

Let A and B be in $\mathcal{L}(\mathcal{H})$. An operator B is said to be in H$elton_k(A)$ if $\delta_{A,B}^{k}(I) = 0$.

Helton classes

Theorem 2.2.35

[2008, Lee] Let A and B be in $\mathcal{L}(\mathcal{H})$. If B is in $Helton_k(A)$ then $\sigma_p(B) \subset \sigma_p(A)$, $\sigma_{ap}(B) \subset \sigma_{ap}(A)$, and $\sigma_{su}(A) \subset \sigma_{su}(B)$. In particular, $\sigma(A) \subset \sigma(B)$ when A has the SVEP. Moreover, if A and B^* have the SVEP, then $\sigma(A) = \sigma(B)$.

Theorem 2.2.36

[2008, Lee] Let A and B be in $\mathcal{L}(\mathcal{H})$. If B is in $Helton_k(A)$, then

- A has the SVEP at $\lambda \Longrightarrow B$ has the SVEP at λ.
- A has (β) at $\lambda \Longrightarrow B$ has (β) at λ.
- A has $(\beta)_\epsilon$ at $\lambda \Longrightarrow B$ has $(\beta)_\epsilon$ at λ.

m-Complex and m-skew complex symmetric operators

Let m be a positive integer. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-complex symmetric operator* if there exists some conjugation C such that $\Delta_m(T) = 0$ where

$$\Delta_m(T) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} C T^{m-j} C. \tag{16}$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-skew complex symmetric operator* if there exists some conjugation C such that $\Gamma_m(T) = 0$ where

$$\Gamma_m(T) := \sum_{j=0}^{m} \binom{m}{j} T^{*j} C T^{m-j} C. \tag{17}$$

m-Complex and m-skew complex symmetric operators

Remark

It is easy to see that

- $T \in \mathcal{L}(\mathcal{H})$ is an *m-complex symmetric operator* which means that $CTC \in \text{Helton}_m(T^*)$.
- $T \in \mathcal{L}(\mathcal{H})$ is an *m-skew complex symmetric operator* which means that $-CTC \in \text{Helton}_m(T^*)$.

Theorem 2.2.37

Let T be in $\mathcal{L}(\mathcal{H})$. If T is an m-complex symmetric operator, then

- T^* has the SVEP at $\lambda \Longrightarrow T$ has the SVEP at $\bar{\lambda}$.
- T^* has (β) at $\lambda \Longrightarrow T$ has (β) at $\bar{\lambda}$.
- T^* has $(\beta)_\epsilon$ at $\lambda \Longrightarrow T$ has $(\beta)_\epsilon$ at $\bar{\lambda}$.

m-Complex and m-skew complex symmetric operators

Corollary 2.3.38 [2016, CKL, Theorem 4.7]

Let T be in $\mathcal{L}(\mathcal{H})$. If T is an m-complex or m-skew complex symmetric operator, then

$$T^* \text{ has } (\beta) \iff T \textit{ is decomposable}.$$

For example, if T is a nilpotent operator of order $k > 2$, then T^* is nilpotent of order $k > 2$ and so T^* is $(2k-1)$-complex symmetric from Example 3.1 in [CKL]. Moreover, in this case, T^* has the property (β). Hence T is decomposable from Corollary 2.3.38.

Complex symmetric and skew-complex symmetric operators

One could wonder why we are considering this special case separately. There are at least two reasons.
-The first one is:
For an arbitrary conjugation C and an operator T on $\mathcal{H}$, one can write T as a sum of a complex symmetric operator and a skew-complex symmetric operator. Namely, $T = A + B$ where $A = -\frac{1}{2}\Gamma_1(T^*)$ and $B = -\frac{1}{2}\Delta_1(T^*)$ where $A = CA^*C$, $B = -CB^*C$.
-The second one is:
$\text{Helton}_1(A) = \{A\}$. Thus we have the coincidence of many spectra (instead of the inclusion).

Complex symmetric and skew-complex symmetric operators

Corollary 2.2.39

Let T be in $\mathcal{L}(\mathcal{H})$.

1. If T is a complex symmetric operator, then
   - T^* has the SVEP at $\lambda \Longleftrightarrow T$ has the SVEP at $\bar{\lambda}$.
   - T^* has (β) at $\lambda \Longleftrightarrow T$ has (β) at $\bar{\lambda}$.
   - T^* has $(\beta)_\epsilon$ at $\lambda \Longleftrightarrow T$ has $(\beta)_\epsilon$ at $\bar{\lambda}$.
2. If T is a skew complex symmetric operator, then
   - T^* has the SVEP at $\lambda \Longleftrightarrow T$ has the SVEP at $-\bar{\lambda}$.
   - T^* has (β) at $\lambda \Longleftrightarrow T$ has (β) at $-\bar{\lambda}$.
   - T^* has $(\beta)_\epsilon$ at $\lambda \Longleftrightarrow T$ has $(\beta)_\epsilon$ at $-\bar{\lambda}$.

Complex symmetric and skew-complex symmetric operators

Corollary 2.2.40

Let T be in $\mathcal{L}(\mathcal{H})$.

1. If T is a complex symmetric operator, then
$$\sigma_\bullet(T^*) = \sigma_\bullet(T)^*.$$
2. If T is a skew complex symmetric operator, then
$$\sigma_\bullet(T^*) = -\sigma_\bullet(T)^*$$

when $\sigma_\bullet \in \{\sigma, \sigma_p, \sigma_{ap}, \sigma_c, \sigma_r, \sigma_{su}, \sigma_e, \sigma_w, \ldots\}$.

Complex symmetric and skew-complex symmetric operators

Corollary 2.2.41 [2011, JKLL, Theorem 4.4]

Let T be in $\mathcal{L}(\mathcal{H})$. If T is a complex symmetric or a skew complex symmetric operator, then T satisfies Weyl's (or Browder's) theorem if and only if T^* satisfies Weyl's (or Browder's) theorem.

∞-Complex symmetric operators

2.3. ∞-complex symmetric operators

CKL3 M. Chō, E. Ko and J. Lee, *On ∞-complex symmetric operators*, Glasgow Mathematical Journal, 60(1)(2018), 35-50.

∞-Complex symmetric operators

Definition 2.3.1

An operator $T \in \mathcal{L}(\mathcal{H})$ is called an *∞-complex symmetric operator* with conjugation C if $\limsup_{m\to\infty} \|\Delta_m(T)\|^{\frac{1}{m}} = 0$.

$$\begin{aligned}\{1-CSO\} \subset \{2-CSO\} &\subset \{3-CSO\} \subset \cdots \\ &\subset \{m-CSO\} \subset \cdots \subset \{\infty-CSO\}.\end{aligned}$$

Examples

Example 2.3.2

Let C be the canonical conjugation on $\mathcal{H}$ given by

$$C(\sum_{n=0}^{\infty} x_n e_n) = \sum_{n=0}^{\infty} \overline{x_n} e_n$$

where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$. Given any $\epsilon > 0$, choose a positive integer N such that $\frac{1}{N} < \epsilon$. Fix any $m > N$. If W is the weighted shift on $\mathcal{H}$ defined by $We_n = \frac{1}{2^{m+n}} e_{n+1}$ $(n = 0, 1, 2, ...)$ for such m, then $T = I + W$ is an ∞-complex symmetric operator.

Examples

Indeed, since W is a quasinilpotent operator, $\sigma(W) = \{0\}$, and $\Delta_m(T) = \Delta_m(W)$, it follows that

$$\begin{aligned} \|\Delta_m(T)\|^{\frac{1}{m}} &= \|\Delta_m(W)\|^{\frac{1}{m}} \\ &\leq \Big(\sum_{j=0}^{m} \binom{m}{j} \|W^{*j}\| \|CW^{m-j}C\|\Big)^{\frac{1}{m}} \\ &\leq \Big(\sum_{j=0}^{m} \binom{m}{j} \|W^{*}\|^{j} \|W\|^{m-j}\Big)^{\frac{1}{m}} \leq \frac{1}{2^{m-1}} < \frac{1}{N} < \epsilon. \end{aligned}$$

By taking limsup as $m \to \infty$ in the above inequality, we get that

$$\limsup_{m\to\infty} \|\Delta_m(T)\|^{\frac{1}{m}} \leq \epsilon.$$

Since ϵ is arbitrary, it follows that T is an ∞-complex symmetric operator (cf. Theorem 2.3.4).

∞-complex symmetric operators

- Two vectors x and y are C-orthogonal if $\langle Cx, y\rangle = 0$.

Theorem 2.3.3

Let $T \in \mathcal{L}(\mathcal{H})$ be an ∞-complex symmetric operator with conjugation C and let λ and μ be any distinct eigenvalues of T.
(i) Eigenvectors of T corresponding to λ and μ are C-orthogonal.
(ii) If $\{x_n\}$ and $\{y_n\}$ are sequences of unit vectors such that $\lim_{n\to\infty}(T-\lambda)x_n = 0$ and $\lim_{n\to\infty}(T-\mu)y_n = 0$, then $\lim_{k\to\infty}\langle Cx_{n_k}, y_{n_k}\rangle = 0$ where $\langle Cx_{n_k}, y_{n_k}\rangle$ is any convergent subsequence of $\langle Cx_n, y_n\rangle$.

Theorem 2.3.4

Let Q be a quasinilpotent operator. Then $T = aI + Q$ is an ∞-complex symmetric operator for all $a \in \mathbb{C}$.

∞-complex symmetric operators

Theorem 2.3.5 [CKL II, 2016]

Let T be an m-complex symmetric operator with a conjugation C. If λ is an eigenvalue of T, then $\overline{\lambda}$ is an eigenvalue of T^*.

- However, if T is an ∞-complex symmetric operator, this does not hold.

∞-complex symmetric operators

Example 2.3.6

Let C be the conjugation on $\mathcal{H}$ given by

$$C(\sum_{n=0}^{\infty} x_n e_n) = \sum_{n=0}^{\infty} (-1)^{n+1} \overline{x_n} e_n$$

where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$ and let W be the weighted shift on $\mathcal{H}$ defined by $We_n = \frac{1}{n+1} e_{n+1}$ $(n = 0, 1, 2, \ldots)$. If $T = \lambda I + W^*$, then T is an ∞-complex symmetric operator. Moreover, $(T - \lambda I)e_0 = W^* e_0 = 0$, but

$$(T^* - \overline{\lambda} I)Ce_0 = WCe_0 = We_0 = e_1 \neq 0.$$

∞-complex symmetric operators

Theorem 2.3.7

If $\{T_n\}$ is a sequence of commuting ∞-complex symmetric operators with conjugation C such that $\lim_{n\to\infty}\|T_n - T\| = 0$, then T is also ∞-complex symmetric with conjugation C.

∞-complex symmetric operators

Proposition 2.3.8

Let C be a conjugation on $\mathcal{H}$. Assume that $T \in \mathcal{L}(\mathcal{H})$ is a complex symmetric operator with conjugation C and $R \in \mathcal{L}(\mathcal{H})$ commutes with T.
(i) RT is an m-complex symmetric operator with conjugation C if and only if R is an m-complex symmetric operator on $\overline{ran(T^m)}$.
(ii) If R is an ∞-complex symmetric operator with conjugation C, then RT is an ∞-complex symmetric operator with conjugation C.

Corollary 2.3.9

If T is normal or algebraic operator of order 2 and $R = I + Q$ where Q is quasinilpotent with $QT = TQ$, then $QT + T$ is an ∞-complex symmetric operator.

∞-complex symmetric operators

Theorem 2.3.10

Let S and T be in $\mathcal{L}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. Suppose that $TS = ST$ and $S^*(CTC) = (CTC)S^*$ for a conjugation C.

(i) If T and S are m-complex symmetric and n-complex symmetric, respectively, then $T + S$ is $(m + n - 1)$-complex symmetric.

(ii) If T is complex symmetric and S is an ∞-complex symmetric operator, then $T + S$ is ∞-complex symmetric operator.

∞-complex symmetric operators

- $\mathcal{X}_T(F)$ is the set of elements $x \in \mathcal{H}$ such that the equation $(T-\lambda)f(\lambda) = x$ has a global analytic solution on $\mathbb{C} \setminus F$: the *glocal spectral subspace* of T.

Theorem 2.3.11 [CKL]

Let T be an m-complex symmetric operator with a conjugation C. Then T^* has the property (β) if and only if T is decomposable.

Theorem 2.3.12

Let $T \in \mathcal{L}(\mathcal{H})$ be an ∞-complex symmetric operator with conjugation C. Then the following statements hold:
(i) $\mathcal{X}_{CTC}(F) \subset \mathcal{X}_{T^*}(F)$ for every closed set F in $\mathbb{C}$.
(ii) T has the decomposition property (δ) if and only if T is decomposable.

∞-complex symmetric operators

- $X \in \mathcal{L}(\mathcal{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range.
- $S \in \mathcal{L}(\mathcal{H})$ is said to be a *quasiaffine transform* of an operator $T \in \mathcal{L}(\mathcal{H})$ if there is a quasiaffinity $X \in \mathcal{L}(\mathcal{H})$ such that $XS = TX$.
- Two operators S and T are *quasisimilar* if there are quasiaffinities X and Y such that $XS = TX$ and $SY = YT$.

Corollary 2.3.13

Let $T \in \mathcal{L}(\mathcal{H})$ be an ∞-complex symmetric operator and T have the decomposition property (δ).
(i) If T has real spectrum on $\mathcal{H}$, then $exp(iT)$ is decomposable.
(ii) If $\sigma(T)$ is not singleton and $S \in \mathcal{L}(\mathcal{H})$ is quasisimilar to T, then S has a nontrivial hyperinvariant subspace.

∞-complex symmetric operators

Corollary 2.3.14

(iii) If $F \subset \mathbb{C}$ is closed, then the operator $S =: T/_{H_T(F)}$, induced by T, on the quotient space $\mathcal{H}/H_T(F)$ satisfies $\sigma(S) \subset \overline{\sigma(T) \setminus F}$.
(iv) If $\mathcal{M}$ is a spectral maximal space of T, then $\mathcal{M} = H_T(\sigma(T|_{\mathcal{M}}))$.
(v) $f(T)$ is decomposable where f is any analytic function on some open neighborhood of $\sigma(T)$.
(vi) $\sigma(T) = \sigma_{ap}(T) = \sigma_{su}(T) = \cup\{\sigma_T(x) : x \in \mathcal{H}\}$.

Tensor products of ∞-complex symmetric operators

- Let $\mathcal{H}_1 \otimes \mathcal{H}_2$ denote the completion (endowed with a sensible uniform cross-norm) of the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable complex Hilbert spaces.
- For operators $T \in \mathcal{L}(\mathcal{H}_1)$ and $S \in \mathcal{L}(\mathcal{H}_2)$, we define the *tensor product* operator $T \otimes S$ on $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by

$$(T \otimes S)(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j) = \sum_{j=1}^{n} \alpha_j Tx_j \otimes Sy_j.$$

- Then it is well known that $T \otimes S \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

Tensor products of ∞-complex symmetric operators

- Since $T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$ and $T \otimes I = \oplus_n^\infty T$, it is clear that an operator T is an m-complex symmetric operator with conjugation C if and only if $T \otimes I$ and $I \otimes T$ are m-complex symmetric operators with conjugation C.
- We replace the notation $\Delta_m(T; C)$ with $\Delta_m(T)$ as follows if necessary;

$$\Delta_m(T; C) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} C T^{m-j} C.$$

Tensor products of ∞-complex symmetric operators

- Similarly, for conjugations C and D on $\mathcal{H}$, we define $C \otimes D$ on $\mathcal{H} \otimes \mathcal{H}$ by

$$(C \otimes D)(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j) = \sum_{j=1}^{n} \overline{\alpha_j} Cx_j \otimes Dy_j.$$

- Then $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$ (see Lemma 3.2.15 or [GP, Lemma 6]).

Tensor products of ∞-complex symmetric operators

Lemma 2.3.15 [2015, Chō, Lee, and Motoyoshi]

If C and D are conjugations on $\mathcal{H}$, then $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$.

Lemma 2.3.16

Let T and S be m-complex symmetric and n-complex symmetric with conjugation C, respectively. If T commutes with S and $S^*(CTC) = (CTC)S^*$, then TS is $(m+n-1)$-complex symmetric with conjugation C.

Theorem 2.3.17

Let T and S be an m-complex symmetric operator and n-complex symmetric operator with conjugations C and D, respectively. Then $T \otimes S$ is an $(m+n-1)$-complex symmetric operator with conjugation $C \otimes D$.

Tensor products of ∞-complex symmetric operators

- $T \in \mathcal{L}(\mathcal{H})$ is called a *2-normal* operator if T is unitarily equivalent to an operator matrix of the form $\begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ where N_i are mutually commuting normal operators for $i = 1, 2, 3, 4$.

Corollary 2.3.18

If T is an m-complex symmetric operator with a conjugation C and S is a 2-normal operator with $TS = ST$, then $T \otimes U^*NU$ is an m-complex symmetric operator where $S = U^*NU$ with $N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}$ and a unitary operator U.

Tensor products of ∞-complex symmetric operators

Example 2.3.19

Let C be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$ on $\mathbb{C}^3$.

If N is normal and $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$ with $TN = NT$, then T is a 5-complex symmetric operator with conjugation C from [CKL]. Hence $T \otimes N = \begin{pmatrix} 0 & N & 0 \\ 0 & 0 & 2N \\ 0 & 0 & 0 \end{pmatrix}$ is 5-complex symmetric from Theorem 2.3.17.

Tensor products of ∞-complex symmetric operators

Lemma 2.3.20

Let T and S be ∞-complex symmetric operators with conjugation C. Assume that $TS = ST$ and $S^*(CTC) = (CTC)S^*$. Then TS is an ∞-complex symmetric operator with conjugation C.

Theorem 2.3.21

Let T and S be ∞-complex symmetric operators with conjugations C and D, respectively. Then $T \otimes S$ is an ∞-complex symmetric operator with conjugation $C \otimes D$.

Corollary 2.3.22

Let T and S be ∞-complex symmetric operators with conjugations C and D, respectively. Then $(T \otimes S)^*$ has the property (β) if and only if $T \otimes S$ is decomposable.

CSOs and their applications

3. Complex symmetric operators and their applications

Complex symmetric Toeplitz operators on the Hardy space

3.1. Complex symmetric Toeplitz operators on the Hardy space

KL Eungil Ko and Ji Eun Lee, *On complex symmetric Toeplitz operators*, J. Math. Anal. Appl. **434**(2016), 20-34.

Hardy space

- $L^2 := L^2(\partial\mathbb{D})$ is the usual Lebesque space on the unit circle $\partial\mathbb{D}$.
- L^∞ is the Banach space consisting of all essentially bounded functions on $\partial\mathbb{D}$.
- $\{z^n : n = 0, \pm 1, \pm 2, \pm 3, \cdots\}$ is an orthonormal basis for L^2.
- The *Hilbert Hardy space*, denoted by H^2, consists of all analytic functions f on $\mathbb{D}$ with power series representation $f(z) = \sum_{n=0}^\infty a_n z^n$ and $\sum_{n=0}^\infty |a_n|^2 < \infty$, or equivalently, with

$$\sup_{0<r<1} \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\Big) < \infty.$$

- $H^2 = \overline{\text{span}\{z^n : n = 0, 1, 2, 3, \cdots\}}$.
- H^∞ is the space of bounded analytic functions on $\mathbb{D}$.

Toeplitz operators

Definition 3.1.1

For any $\varphi \in L^\infty$, the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by the formula

$$T_\varphi f = P(\varphi f), f \in H^2$$

where P denotes the orthogonal projection of L^2 onto H^2.

- Note that T_φ is bounded if and only if $\varphi \in L^\infty$ and $\|T_\varphi\| = \|\varphi\|_\infty$.
- T_φ is a Toeplitz operator if and only if $S^* T_\varphi S = T_\varphi$ where S is the unilateral shift on H^2, i.e., $Sf(z) = zf(z)$ for $f \in H^2$.
- T_φ is called *analytic* if $\varphi \in H^\infty$, i.e., φ is a bounded analytic function on the unit disc $\mathbb{D}$
- T_φ is called *coanalytic* if $\overline{\varphi} \in H^\infty$ where $\overline{\varphi}$ denotes the complex conjugate of φ.

Complex symmetric Toeplitz operators

(1963-64), A. Brown and P. Halmos

T_φ is normal if and only if $\varphi = \alpha + \beta\rho$ where ρ is a real valued function in L^∞ and $\alpha, \beta \in \mathbb{C}$.

(2006), S. R. Garcia and M. Putinar

Truncated Toeplitz operators (i.e., $A^u_\varphi f = P_u(\varphi f), P_u : H^2 \to \mathcal{K}_u := H^2 \ominus uH^2$) are CSOs.

- K. Guo and S. Zhu ([GZ]) have raised the following question.
- **Question** Characterize a complex symmetric Toeplitz operator on the Hardy space H^2 of the unit disk.

GZ K. Guo and S. Zhu, *A canonical decomposition of complex symmetric operators*, J. Oper. Theory, **72**(2014), 529-547.

Complex symmetric Toeplitz operators

- For $u \in H^2$ with $u(z) = \sum_{n=0}^{\infty} a_n z^n$, we define the function $\widetilde{u}$ on the boundary of $\mathbb{D}$ by $\widetilde{u}(e^{i\theta}) := \sum_{n=0}^{\infty} a_n e^{in\theta}$.
- A function $u \in H^2$ is called *inner* if $|\widetilde{u}(e^{i\theta})| = 1$ for almost all θ.

Theorem 3.1.2

For $\varphi \in L^\infty$, let T_φ be a complex symmetric operator on H^2. If T_φ is analytic or coanalytic, then φ is either identically zero on $\mathbb{D}$ or a nonzero constant function on $\mathbb{D}$.

Corollary 3.1.3

If φ is a nonconstant inner function on $\mathbb{D}$, then T_φ is not a complex symmetric operator with conjugation C.

Complex symmetric Toeplitz operators

Lemma 3.1.4

Let $C_{\xi,\theta} : H^2 \to H^2$ be defined by $C_{\xi,\theta} f(z) = e^{i\xi}\overline{f(e^{i\theta}\overline{z})}$ for all ξ and θ. Then $C_{\xi,\theta}$ is a conjugation on H^2. Moreover, $C_{\xi,\theta}$ and $C_{\widetilde{\xi},\widetilde{\theta}}$ are unitarily equivalent where $(\widetilde{\xi},\widetilde{\theta})$ satisfies the equation $\widetilde{\xi} - k\widetilde{\theta} = -\xi + k\theta - 2n\pi$ for every $k \in \mathbb{N}$ and $n \in \mathbb{Z}$.

Complex symmetric Toeplitz operators

- Put $\varphi_+(z) = \sum_{n=1}^{\infty} \hat{\varphi}(n) z^n, \varphi_-(z) = \sum_{n=1}^{\infty} \overline{\hat{\varphi}(-n)} z^n$, and $\varphi_0(z) = \hat{\varphi}(0) e_0$. Hence $\varphi = \varphi_+ + \varphi_0 + \overline{\varphi_-}$.

Theorem 3.1.5

For any $\varphi \in L^\infty$, let T_φ be a Toeplitz operator on H^2 and let $\hat{\varphi}(n)$ be the nth Fourier coefficient of φ. Then the following assertions are equivalent:
(i) T_φ is complex symmetric with the conjugation $C_{\xi,\theta}$.
(ii) $\hat{\varphi}(-n) = \hat{\varphi}(n)\lambda^n$ for all $n \in \mathbb{Z}$ with $|\lambda| = 1$.
(iii) $\varphi(z) = \varphi_0 + \sum_{n=1}^{\infty} \hat{\varphi}(n)(z^n + \lambda^n \overline{z}^n)$ with $|\lambda| = 1$.
(iv) $\varphi(z) = \varphi_+(z) + \varphi_0 + \varphi_+(e^{i\theta}\overline{z})$ for $\varphi_+ \in zH^2$ and some θ.

Complex symmetric Toeplitz operators

Put $C_{00}f(z) = \overline{f(\overline{z})}$ and $C_{0,\pi}f(z) = \overline{f(-\overline{z})}$ for $f \in H^2$.

Corollary 3.1.6

For $\varphi(z) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n) z^n \in L^\infty$, let T_φ be a Toeplitz operator on H^2. Then the following statements hold:

(i) T_φ is complex symmetric with the conjugation $C_{0,0}$

$$\Leftrightarrow \varphi(z) = \varphi_0 + 2\sum_{n=1}^{\infty} \hat{\varphi}(n) Re\{z^n\}.$$

(ii) T_φ is complex symmetric with the conjugation $C_{0,\pi}$

$$\Leftrightarrow \varphi(z) = \varphi_0 + 2\sum_{k=1}^{\infty} \hat{\varphi}(2k) Re\{z^{2k}\} + 2i\sum_{k=1}^{\infty} \hat{\varphi}(2k-1) Im\{z^{2k-1}\}.$$

Complex symmetric Toeplitz operators

Corollary 3.1.7

Under the same hypotheses as in Theorem 3.1.5, the following assertions hold.
(i) If T_φ is a complex symmetric operator with both conjugations $C_{0,0}$ and $C_{0,\pi}$, then $\hat{\varphi}(2k-1) = 0$ for all positive integer k.
(ii) If $\varphi(z) = \phi(z) + \alpha + \phi(\overline{z})$ and $\psi(z) = \phi(z) + \beta + \phi(-\overline{z})$ for $\phi \in zH^2$ and $\alpha, \beta \in \mathbb{C}$, then T_φ and T_ψ are complex symmetric operators on H^2.

Complex symmetric Toeplitz matrices

We know from Corollary 3.1.6 that the matrices (a_{ij}) and $(\widetilde{a_{ij}})$ for complex symmetric operators T_φ with conjugation $C_{0,0}$ and $C_{0,\pi}$ with respect to the basis $\{z^n : n = 0, 1, 2, \cdots\}$ are given by

$$(a_{ij}) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & \cdots \\ a_1 & a_0 & a_1 & a_2 & \ddots & \cdots \\ a_2 & a_1 & a_0 & a_1 & \ddots & \ddots \\ a_3 & a_2 & a_1 & a_0 & \ddots & \ddots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

Complex symmetric Toeplitz matrices

$$(\widetilde{a_{ij}}) = \begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 & \cdots & \cdots \\ a_1 & a_0 & -a_1 & a_2 & \ddots & \cdots \\ a_2 & a_1 & a_0 & -a_1 & \ddots & \ddots \\ a_3 & a_2 & a_1 & a_0 & \ddots & \ddots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $a_k = \hat{\varphi}(k)$ for positive integer k. In this case, $(a_{ij})^t = (a_{ij})$ and $(\widetilde{a_{ij}})^t \neq (\widetilde{a_{ij}})$ where t denotes the transpose.

Complex symmetric Toeplitz operators

Corollary 3.1.8

If (a_{ij}) and $(\widetilde{a_{ij}})$ are matrices in the previous notes, then the following properties hold.
(i) (a_{ij}) is self-adjoint if and only if $a_k = \overline{a_k}$ for all $k = 0, 1, 2, \cdots$.
(ii) $(\widetilde{a_{ij}})$ is self-adjoint if and only if $a_{2k} = \overline{a_{2k}}$ and $a_{2k+1} = -\overline{a_{2k+1}}$ for all $k = 0, 1, 2, \cdots$.

Complex symmetric normal Toeplitz operators

- T_φ is normal if and only if $\varphi = \alpha + \beta\rho$ where ρ is a real valued function in L^∞ and $\alpha, \beta \in \mathbb{C}$.
- Put $\varphi_+(z) = \sum_{n=1}^{\infty} \hat{\varphi}(n)e_n$, $\varphi_-(z) = \sum_{n=1}^{\infty} \overline{\hat{\varphi}(-n)}e_n$, and $\varphi_0(z) = \hat{\varphi}(0)e_0$.

Theorem 3.1.9

Let φ be in L^∞ such that $\varphi = \varphi_+ + \varphi_0 + \overline{\varphi_-}$ where φ_+ and φ_- are in zH^2. If T_φ is complex symmetric with the conjugation $C_{\xi,\theta}$, then T_φ is normal if and only if $\overline{\hat{\varphi}(n)} = e^{i(\zeta+n\theta)}\hat{\varphi}(n)$ for all positive integer n and for some ζ, θ.

Complex symmetric normal Toeplitz operators

Corollary 3.1.10

Let φ be in L^∞ such that $\varphi = \varphi_+ + \varphi_0 + \overline{\varphi_-}$. Then the following statements hold:
(i) If T_φ is complex symmetric with the conjugation $C_{0,0}$, then T_φ is normal if and only if
$\varphi(z) = \varphi_0 + 2e^{-i\frac{\theta}{2}} Re\{\sum_{n=1}^{\infty} e^{i\frac{\theta}{2}} \hat{\varphi}(n) Re(z^n)\}$ for some θ.
(ii) If T_φ is complex symmetric with the conjugation $C_{0,\pi}$, then T_φ is normal if and only if a symbol function φ has the form;

$$\begin{aligned}\varphi(z) \;&=\; \varphi_0 + 2e^{-i\frac{\theta}{2}} Re\{\sum_{k=1}^{\infty} e^{i\frac{\theta}{2}} \hat{\varphi}(2k) Re(z^{2k})\} \\ &\quad + 2ie^{-i\frac{\theta}{2}} Im\{\sum_{k=1}^{\infty} e^{i\frac{\theta}{2}} \hat{\varphi}(2k-1) Im(z^{2k-1})\}.\end{aligned}$$

Complex symmetric normal Toeplitz operators

Corollary 3.1.11

For $\varphi(z) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n) z^n \in L^\infty$, let T_φ be a Toeplitz operator on H^2. If T_φ is complex symmetric with the conjugation $C_{\xi,\theta}$, then T_φ is unitary if and only if for some ζ, $\overline{\hat{\varphi}(n)} = e^{i(\zeta+n\theta)}\hat{\varphi}(n)$ for all positive integers n and

$$\sum_{n=-k}^{\infty} \hat{\varphi}(n)\overline{\hat{\varphi}(n+k-l)} = \sum_{n=-k}^{\infty} \hat{\varphi}(-n)\overline{\hat{\varphi}(-(n+k-l))} = \begin{cases} 1 \text{ if } k = l \\ 0 \text{ if } k \neq l \end{cases}$$

for all positive integers l, k.

CSO and nonnormal Toeplitz operators

Corollary 3.1.12

For any $\varphi \in L^\infty$, let T_φ be a Toeplitz operator on H^2 and let $\hat{\varphi}(n)$ be the nth Fourier coefficient of φ. Then the following statements hold.

(i) If $\hat{\varphi}(-n) = \hat{\varphi}(n)$ for all $n \geq 1$ and $\overline{\hat{\varphi}(k)} \neq e^{i\theta}\hat{\varphi}(k)$ for some positive integer k and for all θ, then T_φ is nonnormal and complex symmetric.

(ii) If $\hat{\varphi}(-n) = \hat{\varphi}(n)(-1)^n$ for all $n \geq 1$ and $(-1)^n\overline{\hat{\varphi}(k)} \neq e^{i\theta}\hat{\varphi}(k)$ for some positive integer k and for all θ, then T_φ is nonnormal and complex symmetric.

A general conjugation

Theorem 3.1.13

Let φ be in L^∞ such that $\varphi = \varphi_+ + \varphi_0 + \overline{\varphi_-}$ where φ_+ and φ_- are in zH^2. If C is a conjugation on H^2 and T_φ is a complex symmetric operator with conjugation C, then

$$\sum_{i=0}^{k-1}[\hat{\varphi}(k-i)a_i-\hat{\varphi}(-(k-i))\overline{\widetilde{a_k}}\gamma_k]=\sum_{n=1}^{\infty}[\hat{\varphi}(n)\overline{\widetilde{a_{n+k}}}\gamma_k-a_{n+k}\hat{\varphi}(-n)] \tag{18}$$

for all k where $a_k = \langle f, z^k\rangle$, $\widetilde{a_k} = \langle Cf, z^k\rangle$, and $\gamma_k = \langle Cz^j, z^k\rangle$ for all $f \in H^2$.

A general conjugation

Corollary 3.1.14

Under the same hypotheses as in Theorem 3.1.13, if T_φ is a complex symmetric operator with the conjugation $C_{\xi,\theta}$, then

$$\sum_{i=0}^{k-1}[\lambda^{k-i}\hat{\varphi}(-(k-i))-\hat{\varphi}(k-i)]a_i=\sum_{n=1}^{\infty}[\hat{\varphi}(-n)-\overline{\lambda}^n\hat{\varphi}(n)]a_{n+k} \tag{19}$$

where $|\lambda|=1$.

A general conjugation

Remark that if $C_{\xi,\theta} = C_{0,0}$ in Corollary 3.1.14, then $\widetilde{a_j} = \overline{a_i}$, $\widetilde{\alpha_k} = \sum_{n=1}^{\infty} a_{n+k}\hat{\varphi}(n)$, and $\widetilde{\beta_k} = \sum_{i=0}^{k-1} \hat{\varphi}(-(k-i))a_i$. An equation (18) implies that

$$\sum_{i=0}^{k-1}[\hat{\varphi}(-(k-i)) - \hat{\varphi}(k-i)]a_i = \sum_{n=1}^{\infty}[\hat{\varphi}(-n) - \hat{\varphi}(n)]a_{n+k}. \quad (20)$$

Since T_{φ} is complex symmetric with the conjugation $C_{0,0}$, by Corollary 3.1.14, we know that the equation (20) always holds.

A general conjugation

Example 3.1.15

If T_z is a unilateral shift on H^2, then T_z is not a complex symmetric Toeplitz operator with conjugation $C_{0,0}$. Indeed, since $\varphi(z) = z$, it follows that $\hat{\varphi}(1) = 1$ and $\hat{\varphi}(n) = 0$ for all $n \neq 1$. Then we obtain from (20) that

$$\begin{aligned} & \sum_{i=0}^{k-1}[\hat{\varphi}(-(k-i)) - \hat{\varphi}(k-i)]a_i - \sum_{n=1}^{\infty}[\hat{\varphi}(-n) - \hat{\varphi}(n)]a_{n+k} \\ = \; & [\hat{\varphi}(-(k)) - \hat{\varphi}(k)]a_0 + \cdots + [\hat{\varphi}(-1) - \hat{\varphi}(1)]a_{k-1} \\ - \; & [\hat{\varphi}(-1) - \hat{\varphi}(1)]a_{1+k} - [\hat{\varphi}(-2) - \hat{\varphi}(2)]a_{2+k} - \cdots \\ = \; & \hat{\varphi}(1)(a_{k+1} - a_{k-1}) \neq 0 \end{aligned}$$

for some k. Thus (20) does not hold.

CSO Toeplitz operators with a finite symbol

Theorem 3.1.16

Let $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ where $N \geq m > 0$ and $a_n \in \mathbb{C}$ with nonzero a_{-m}, a_N. Then T_φ is complex symmetric with the conjugation $C_{\xi,\theta}$ if and only if $m = N$ and $a_{-n} = a_n e^{in\theta}$ for all $n = 1, 2, \cdots, N$ and some θ. In particular, in this case, T_φ is normal if and only if $a_{-m} = a_m e^{im\theta}$ and $\overline{a_m} a_k = e^{i(m-k)\theta} a_m \overline{a_k}$ for all $k = 1, 2, \cdots, m-1$.

CSO Toeplitz operators with a finite symbol

Corollary 3.1.17

Let $\varphi(z) = \sum_{n=-m}^{m} a_n z^n$ for $a_n \in \mathbb{C}$ with nonzero a_{-m}, a_m. If T_φ is complex symmetric with the conjugations $C_{0,0}$ and $C_{0,\pi}$ and it is normal, then $\overline{a_m} a_k = a_m \overline{a_k}$ when $k+m$ is even, $a_k = 0$ for $k = 2, 4, 6, \cdots, m-1$ when m is odd, or $a_k = 0$ for $k = 1, 3, 5, \cdots, m-1$ when m is even.

Example 3.1.18

Let $\varphi(z) = e^{i\theta}(3z^{n+1} + z^n + \overline{z}^n + 3\overline{z}^{n+1})$ for some θ. Then T_φ is complex symmetric and normal from Theorem 3.1.16.

Example 3.1.19

Suppose that

$$\varphi(z) = iz^3 + z + \overline{z} + i\overline{z}^3 \text{ and } \psi(z) = 2z^2 + z + i - \overline{z} + 2\overline{z}^2.$$

Then the matrices for Toeplitz operators T_φ and T_ψ with respect to the basis $\mathcal{B} = \{z^n : n = 0, 1, 2, \cdots\}$ are given by

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & i & 0 & \ddots \\ 1 & 0 & 1 & 0 & i & \ddots \\ 0 & 1 & 0 & 1 & 0 & \ddots \\ i & 0 & 1 & 0 & 1 & \ddots \\ 0 & i & 0 & 1 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

Example

$$[T_\psi]_{\mathcal{B}} = \begin{pmatrix} i & -1 & 2 & 0 & 0 & \ddots \\ 1 & i & -1 & 2 & 0 & \ddots \\ 2 & 1 & i & -1 & 2 & \ddots \\ 0 & 2 & 1 & i & -1 & \ddots \\ 0 & 0 & 2 & 1 & i & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Hence, by Theorem 3.1.16, T_φ and T_ψ are complex symmetric with respect to the conjugations $C_{0,0}$ and $C_{0,\pi}$, respectively. But, since $\overline{a_3}a_{-1} - a_{-3}\overline{a_1}$ and $\overline{a_2}a_{-1} - a_{-2}\overline{a_1}$ are nonzero, both T_φ and T_ψ are not normal from Theorem 3.1.16 or [FL].

Complex symmetric Toeplitz operators on the weighted Bergman space

3.2. Complex symmetric Toeplitz operators on the weighted Bergman space

KLL Eungil Ko, Ji Eun Lee, and Jongrak Lee, Complex symmetric Toeplitz operators on the weighted Bergmann space, preprint.

Weighted Bergmann space

- For $-1 < \alpha < \infty$, the *weighted Bergman space* $A^2_\alpha(\mathbb{D})$ is the space of analytic functions in $L^2(\mathbb{D}, dA_\alpha)$ where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$.
- Here, $L^2(\mathbb{D}, dA_\alpha)$ is a Hilbert space with the following inner product

$$\langle f,\ g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z)$$

where $f,\ g \in L^2(\mathbb{D}, dA_\alpha)$.
- For any nonnegative integer n, let

$$e_n(z) = \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}}\ z^n \quad (z \in \mathbb{D}),$$

where $\Gamma(s)$ is the usual Gamma function, i.e., $\Gamma(s) = (s-1)!$. Then $\{e_n\}$ is an orthonormal basis for $A^2_\alpha(\mathbb{D})$ ([HKZ]).

Toeplitz operators on $A^2_\alpha(\mathbb{D})$

- For $\varphi \in L^\infty(\mathbb{D})$, the *Toeplitz operator* T_φ on $A^2_\alpha(\mathbb{D})$ is defined by

$$T_\varphi f := P(\varphi \cdot f).$$

where P is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha(\mathbb{D})$.

- The *reproducing kernel* in $A^2_\alpha(\mathbb{D})$ is given by

$$K_z(\omega) = \frac{1}{(1 - z\overline{\omega})^{\alpha+2}},$$

for $z, \omega \in \mathbb{D}$. Thus we have

$$(T_\varphi f)(z) = \int_{\mathbb{D}} \frac{\varphi(\omega) f(\omega)}{(1 - z\overline{\omega})^{\alpha+2}} dA_\alpha(\omega),$$

for $f \in A^2_\alpha(\mathbb{D})$ and $\omega \in \mathbb{D}$.

Toeplitz operators on $A^2_\alpha(\mathbb{D})$

Basic properties of Toeplitz operators

Let f and g be bounded functions and $\alpha, \beta \in \mathbb{C}$. Then from the definition of Toeplitz operator, we can easily check that
(i) $T_{\alpha f+\beta g} = \alpha T_f + \beta T_g$.
(ii) $T_f^* = T_{\overline{f}}$.
(iii) If $T_f = 0$, then $f = 0$.
(iv) If $f \in H^\infty(\mathbb{D})$, then $T_g T_f = T_{gf}$ and $T_{\overline{f}} T_g = T_{\overline{f}g}$.

Question

Characterize a complex symmetric Toeplitz operator on the weighted Bergman space $A^2_\alpha(\mathbb{D})$.

Complex symmetric Toeplitz operators

Proposition 3.2.1

Let C be a conjugation on $A^2_\alpha(\mathbb{D})$. Suppose that T_φ is complex symmetric with the conjugation C. Then the following assertions hold.

(i) If φ is not a constant function in $L^\infty(\mathbb{D})$, then

$$ker(T_\varphi - \lambda I) = ker(T_\varphi^* - \overline{\lambda} I) = \{0\}$$

for some $\lambda \in \mathbb{C}$.

(ii) If $\varphi \in H^\infty(\mathbb{D})$ is not identically zero on $\mathbb{D}$, then T_φ is a quasiaffinity, i.e., it has trivial kernel and dense range.

Complex symmetric Toeplitz operators

Lemma 3.2.2

([AC] and [You]) *Let φ be a bounded harmonic function on $\mathbb{D}$. Then the following statements are equivalent;*
(i) T_φ is normal.
(ii) There exists a nonzero pair $(a, b) \in \mathbb{C}^2$ such that $a\varphi + b\overline{\varphi}$ is a constant on $\mathbb{D}$.
(iii) The set $\varphi(\mathbb{D})$ lies on some line in $\mathbb{C}$.

AC S. Axler and Z. Cuckovic, *Commuting Toeplitz operators with harmonic symbols*, Int. Eq. Op. Th. **14** (1991), 1-12.

You A. Yousef, *Two problems in the theory of Toeplitz operators on the Bergman space*, (2009), Theses and Dissertations.

Complex symmetric Toeplitz operators

Theorem 3.2.3

Let T_φ be complex symmetric with a conjugation C on $A^2_\alpha(\mathbb{D})$. Then the following statements hold;
(i) Assume that ψ is a bounded analytic functions and φ is a bounded measurable function. If $[T_\psi, T_\varphi] = 0$ where $[R, S] = RS - SR$, then the set $\varphi(\mathbb{D})$ lies on some line in $\mathbb{C}$.
(ii) If $\varphi \in L^\infty$, then

$$P(\varphi\widetilde{K_\lambda}) = \varphi(\lambda)\widetilde{K_\lambda} \tag{21}$$

holds where $\widetilde{K_\lambda} := CK_\lambda$ and $K_\lambda = \frac{1}{1-\overline{\lambda}z}$. In particular, if $\varphi \in H^\infty$, then φ is a constant on $\mathbb{D}$.

Complex symmetric Toeplitz operators

Proposition 3.2.4

Let C be a conjugation on $A^2_\alpha(\mathbb{D})$. Then the following statements hold;
(i) The Parseval's identity holds, i.e., $\sum_{n=0}^{\infty} |\langle f, Ce_n\rangle|^2 = \|f\|_2^2$ for every $f \in A^2_\alpha(\mathbb{D})$.
(ii) Then the set of functions

$$\{Ce_n(z) := \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}}Cz^n\}$$

forms an orthonormal basis for $A^2_\alpha(\mathbb{D})$.

Complex symmetric Toeplitz operators

Lemma 3.2.5

([HLP]) *For nonnegative integers n, m,*

$$P(z^n\overline{z}^m) = \begin{cases} \frac{\Gamma(n+1)\Gamma(n-m+\alpha+2)}{\Gamma(n+\alpha+2)\Gamma(n-m+1)} z^{n-m} & \text{if } n \geq m; \\ 0 & \text{if } n < m. \end{cases}$$

HLP I. S. Hwang, J. Lee and S. W. Park, *Hyponormal Toeplitz operators with polynomial symbols on the weighted Bergman spaces*, J. Inequal. Appl. **2014**, (2014) 8 pp.

Complex symmetric Toeplitz operators

Theorem 3.2.6

Let φ be in $L^\infty(\mathbb{D})$ such that

$$\varphi(z) = \sum_{n=1}^{\infty} \overline{\hat{\varphi}(-n)}\bar{z}^n + \sum_{n=0}^{\infty} \hat{\varphi}(n) z^n.$$

If C is a conjugation on $A^2_\alpha(\mathbb{D})$, then T_φ is a complex symmetric operator with conjugation C if and only if $\hat{\varphi}(-k) = C\hat{\varphi}(k)$ for all $k \in \mathbb{N} \cup \{0\}$.

Complex symmetric Toeplitz operators

Example 3.2.7

Let C be a conjugation on $A^2_\alpha(\mathbb{D})$. If $\varphi(z) = \overline{a}C\overline{z} + az$ for some $a \in \mathbb{C}$, then T_φ is a complex symmetric operator on $A^2_\alpha(\mathbb{D})$ from Theorem 3.2.6.

Remark

We observe from Theorem 3.2.6 and [KL, Theorem 2.14] that the necessary and sufficient conditions for the complex symmetric Toeplitz operator T_φ on $A^2_\alpha(\mathbb{D})$ and on $H^2(\mathbb{T})$ are the same.

Complex symmetric of Toeplitz operators

- It is known from [CC] that if $\varphi(z) = \overline{z}^2 + 2z$, then T_φ is not hyponormal on the Hardy space $H^2(\mathbb{T})$, but T_φ is hyponormal on the Bergman space $A^2_\alpha(\mathbb{D})$.
- Let $\varphi(z) = \overline{z} + z^2$. Then T_φ is hyponormal on $H^2(\mathbb{T})$ but is not hyponormal on $A^2_\alpha(\mathbb{D})$.
- Hence, there is no relation between the hyponormality of T_φ on $H^2(\mathbb{T})$ and on $A^2_\alpha(\mathbb{D})$.

CC Z. Cuckovi and R. E. Curto, *A New Necessary Condition for the Hyponormality of Toeplitz Operators on the Bergman Space*, preprint.

Complex symmetric Toeplitz operators

Lemma 3.2.8

([CC], Theorem 4.2]) *Let* $\varphi = \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q$ *with* $n < m, p < q, n - m = q - p$ *and for nonzero* $\alpha, \beta, \gamma, \delta$ *and* $\varphi + \overline{\varphi}$ *is a constant. Then* T_φ *is normal if and only if* φ *is one of exactly three types;*

(i) $\varphi = \alpha z^n - \lambda \overline{\alpha} \overline{z}^n$

(ii) $\varphi = \alpha z^n + \beta z^m - \lambda(\overline{\alpha} \overline{z}^n + \overline{\beta} \overline{z}^m)$

(iii) $\varphi = \beta z^m - \lambda \overline{\beta} \overline{z}^m$.

Complex symmetric Toeplitz operators

Lemma 3.2.9

Let $\varphi(z) = \overline{g(z)} + f(z)$, *where*

$$f(z) = a_m z^m + a_N z^N \quad \text{and} \quad g(z) = a_{-m} z^m + a_{-N} z^N \ (m < N).$$

Then T_φ *is normal on* $A^2_\alpha(\mathbb{D})$ *if and only if* $|a_N| = |a_{-N}|$ *and* $\overline{a_m} a_N = \overline{a_{-m}} a_{-N}$.

Complex symmetric Toeplitz operators

Lemma 3.2.10

[**Cowen's Theorem [Cow]**] For $\varphi \in L^\infty(\mathbb{T})$, write

$$\mathcal{E}(\varphi) := \{k \in H^\infty(\mathbb{T}) : \|k\|_\infty \leq 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

Cow C. C. Cowen, *Hyponormal and subnormal Toeplitz operators*, Proc. Amer. Math. Soc. **103**(1988), 809-812.

Complex symmetric Toeplitz operators

Lemma 3.2.11

([KL-J]) *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where*

$$f(z) = a_m z^m + a_N z^N \quad \text{and} \quad g(z) = a_{-m} z^m + a_{-N} z^N.$$

If $a_m \overline{a_N} = a_{-m} \overline{a_{-N}}$ *and* $|a_{-N}| \le |a_N|$, *then* T_φ *on* $A^2_\alpha(\mathbb{D})$ *is hyponormal if and only if*

$$\frac{|a_N|^2 - |a_{-N}|^2}{(m+\alpha+1)(m+\alpha+2)\cdots(N+\alpha+1)} \ge \frac{|a_{-m}|^2 - |a_m|^2}{(m+1)(m+2)\cdots N}. \tag{22}$$

Complex symmetric Toeplitz operators

Theorem 3.2.12

Let $\varphi(z) = \overline{g(z)} + f(z)$, where

$$f(z) = a_m z^m + a_N z^N \quad \text{and} \quad g(z) = a_{-m} z^m + a_{-N} z^N \ (m < N).$$

If T_φ is complex symmetric on $A^2_\alpha(\mathbb{D})$, then the following statements hold.

(i) T_φ is hyponormal on $A^2_\alpha(\mathbb{D})$.
(ii) T_φ is hyponormal on $H^2(\mathbb{T})$.
(iii) T_φ is normal on $A^2_\alpha(\mathbb{D})$.
(iv) T_φ is normal on $H^2(\mathbb{T})$.

Complex symmetric Toeplitz operators

Corollary 3.2.13

Let $\varphi(z) = \overline{g(z)} + f(z)$, where

$$f(z) = a_m z^m + a_N z^N \quad \text{and} \quad g(z) = a_{-m} z^m + a_{-N} z^N.$$

If T_φ is complex symmetric on $A^2_\alpha(\mathbb{D})$, then T_φ is hyponormal on $A^2_\alpha(\mathbb{D})$ if and only if $\mathcal{E}(\varphi)$ is nonempty where $\mathcal{E}(\varphi)$ is defined in Lemma 3.2.10.

Complex symmetric Toeplitz operators

Example 3.2.14

Let $\varphi(z) = \overline{z}^2 + 2z$. Then from [CC], T_φ is not hyponormal on $H^2(\mathbb{T})$, but is hyponormal in $A^2_\alpha(\mathbb{D})$. Hence T_φ is not complex symmetric from Theorem 3.2.12. Similarly, if $\psi(z) = \overline{z} + z^2$, then T_ψ is not complex symmetric from Theorem 3.2.12.

Example 3.2.15

Let $\varphi(z) = \overline{z}^2 + \overline{z} + \frac{1}{2}z + 2z^2$. Then by Lemma 3.2.10, $k(z) = \frac{1}{2} + \frac{3}{8}z \in \mathcal{E}(\varphi)$ and so T_φ is hyponormal on $H^2(\mathbb{T})$, but by Lemma 3.2.11, T_φ is hyponormal in $A^2_\alpha(\mathbb{D})$ if and only if $\alpha \leq 5$. Therefore, for $\alpha > 5$, T_φ is not hyponormal and hence T_φ is not complex symmetric from Theorem 3.2.12.

Complex symmetric Toeplitz operators with the special conjugation

Lemma 3.2.16

For every μ and λ with $|\mu| = |\lambda| = 1$, let $C_{\mu,\lambda} : A^2_\alpha(\mathbb{D}) \to A^2_\alpha(\mathbb{D})$ be given by

$$C_{\mu,\lambda} f(z) = \mu \overline{f(\lambda \overline{z})}.$$

Then $C_{\mu,\lambda}$ is a conjugation on $A^2_\alpha(\mathbb{D})$.

Lemma 3.2.17

Let φ be in $L^\infty(\mathbb{D})$ such that $\varphi(z) = \sum_{n=1}^\infty \overline{\hat{\varphi}(-n)} \overline{z}^n + \sum_{n=0}^\infty \hat{\varphi}(n) z^n$. Then the following statements are equivalent:

(i) T_φ on $A^2_\alpha(\mathbb{D})$ is complex symmetric with the conjugation $C_{\mu,\lambda}$.
(ii) $\hat{\varphi}(-n) = \lambda^n \hat{\varphi}(n)$ for all $n \in \mathbb{N} \cup \{0\}$ and $|\lambda| = 1$.
(iii) T_φ on H^2 is complex symmetric with the conjugation $C_{\mu,\lambda}$.

Complex symmetric Toeplitz operators with the special conjugation

We denote by φ_+ and φ_- as the positive and negative parts of φ, respectively:

$$\varphi_+(z) = \sum_{n=1}^{\infty} \hat{\varphi}(n) z^n, \ \varphi_-(z) = \sum_{n=1}^{\infty} \overline{\hat{\varphi}(-n)} \overline{z}^n, \ \textit{and} \ \ \varphi_0(z) = \hat{\varphi}(0).$$

Hence $\varphi = \varphi_+ + \varphi_0 + \varphi_-$.

Theorem 3.2.18

Let $\varphi \in L^\infty(\mathbb{D})$. If T_φ is a Toeplitz operator on $A^2_\alpha(\mathbb{D})$, then the following statements are equivalent.
(i) *T_φ is complex symmetric with the conjugation $C_{\mu,\lambda}$.*
(ii) *$\varphi(z) = \varphi_0 + \sum_{n=1}^{\infty} \hat{\varphi}(n)(z^n + \lambda^n \overline{z}^n)$ with $|\lambda| = 1$.*
(iii) *$\varphi(z) = \varphi_+(z) + \varphi_0 + \varphi_+(\lambda \overline{z})$.*

Complex symmetric Toeplitz operators with the special conjugation

Corollary 3.2.19

Let $\varphi(z) = \overline{g(z)} + f(z)$, *where*

$$f(z) = a_m z^m + a_N z^N \quad \textit{and} \quad g(z) = a_{-m} z^m + a_{-N} z^N.$$

If T_φ *on* $A^2_\alpha(\mathbb{D})$ *is complex symmetric with the conjugation* $C_{\mu,\lambda}$, *then* T_φ *is normal.*

Complex symmetric Toeplitz operators with the special conjugation

Put $C_1 f(z) = \overline{f(\overline{z})}$ and $C_2 f(z) = \overline{f(-\overline{z})}$ for all $f \in A^2_\alpha(\mathbb{D})$. Then C_1 and C_2 are clearly conjugations on $A^2_\alpha(\mathbb{D})$.

Corollary 3.2.20

For any $\varphi \in L^\infty$, let T_φ be a Toeplitz operator on $A^2_\alpha(\mathbb{D})$. Then the following statements hold.
(i) T_φ is complex symmetric with the conjugation C_1 if and only if $\overline{\hat{\varphi}(-n)} = \hat{\varphi}(n)$ for all $n \in \mathbb{Z}$.
(ii) T_φ is complex symmetric with the conjugation C_2 if and only if $\overline{\hat{\varphi}(-n)} = \hat{\varphi}(n)(-1)^n$ for all $n \in \mathbb{N} \cup \{0\}$.

Complex symmetry Toeplitz operators with the special conjugation

Proposition 3.2.21

If T_φ on $A^2_\alpha(\mathbb{D})$ is complex symmetric with the conjugation $C_{\mu,\lambda}$, then T_φ is normal if and only if $\gamma\lambda^n\hat{\varphi}(n) = \overline{\hat{\varphi}(n)} = \lambda^n\hat{\varphi}(-n)$ for all $n \in \mathbb{N} \cup \{0\}$ with $|\lambda| = |\gamma| = 1$.

Complex symmetry Toeplitz operators with the special conjugation

Remark

- The authors in [KL] gave the necessary and sufficient condition for complex symmetric Toeplitz operators with conjugation $C_{\mu,\lambda}$ in the Hardy space $H^2(\mathbb{T})$ as follows :
- For $\varphi(z) \in L^\infty$, $\varphi(z) = \varphi_+(z) + \varphi_0 + \varphi_+(e^{i\theta}\overline{z})$ for $\varphi_+ \in zH^2$ for some θ.
- By Theorem 3.2.18, T_φ is complex symmetric Toeplitz operators with conjugation $C_{\mu,\lambda}$ on the Hardy space $H^2(\mathbb{T})$ if and only if it is complex symmetric Toeplitz operators with conjugation $C_{\mu,\lambda}$ on the weighted Bergman space $A^2_\alpha(\mathbb{D})$.

Complex symmetry Toeplitz operators with the special conjugation

- In Hardy space $H^2(\mathbb{T})$, $\overline{z}^n z^m$ is equal to z^{m-n}
- In the weighted Bergman space $A^2_\alpha(\mathbb{D})$, $\overline{z}^n z^m \neq z^{m-n}$ since $z \in \mathbb{D}$.

Theorem 3.2.22

Let $\varphi(z) = a\overline{z}^n z^m + b\overline{z}^s z^t$ where $a, b \in \mathbb{C}$ and $n - m = t - s$. Then T_φ on $A^2_\alpha(\mathbb{D})$ is complex symmetric with the conjugation $C_{\mu,\lambda}$ if and only if $s = m$, $t = n$, and $a = b\lambda^{n-m}$.

Complex symmetry Toeplitz operators with the special conjugation

Corollary 3.2.23

Let $\varphi(z) = a\overline{z}^n z^m + bz^\ell$ where $n > m$ and $a, b \in \mathbb{C}$ with $|a| \neq |b|$. Then T_φ on $A^2_\alpha(\mathbb{D})$ is never complex symmetric with the conjugation $C_{\mu,\lambda}$.

Example 3.2.24

Let $\varphi(z) = \overline{z}^2 z + az$ for $a \in \mathbb{C}$ with $a \neq 1$. By Corollary 3.2.23, T_φ is never complex symmetric with the conjugation $C_{\mu,\lambda}$ in the weighted Bergman space $A^2_\alpha(\mathbb{D})$ or $H^2(\mathbb{T})$.

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Thank you for your attention !

Happy new year 2020!!!