Notes on quantized semisimple Lie groups and quantum flag manifolds

Bob Yuncken
Université Clermont Auvergne

December 9, 2019

Contents

Introduction

The goal of these lectures is to introduce the basic ideas of quantum groups and their representation theory. Our motivation will come from the representation theory of Lie groups, particularly semisimple Lie groups, although we won't have time to delve very far into that enormous subject.

The lectures with be organized around the simple example of the group $\mathrm{SU}(2)$ and its quantum analogue $\mathrm{SU}_q(2)$, in which we can already see much of the power and depth of unitary representation theory.

Chapter 1

Compact quantum groups

1.1 Example: The discrete Fourier transform

Let's start with some classical harmonic analysis. Consider the circle group:

$$\mathbb{T} = S^1 \cong \mathbb{R}/2\pi\mathbb{Z}.$$

We'll identify $\mathbb{T}$ with the set of complex numbers of modulus 1, and we'll write $\mathbf{z} : \mathbb{T} \to \mathbb{C}$ for the inclusion:

$$\mathbf{z}(e^{i\theta}) = e^{i\theta}.$$

Definition 1.1.1. A *trigonometric polynomial* is a function of the form $a = \sum_{k\in\mathbb{Z}} a_k \mathbf{z}^k$ with $a_k \in \mathbb{C}$ and $a_k = 0$ for all but finitely many k.

The set of trigonometric polynomials is a countable dimensional vector space and also an algebra over $\mathbb{C}$. In fact it is an algebra in two ways. Firstly, we can equip it with the structure of *pointwise multiplication*,

$$a.b(e^{i\theta}) := a(e^{i\theta})b(e^{i\theta}).$$

Secondly, we could equip it with the structure of *convolution*,

$$a * b(e^{i\theta}) := \int_0^{2\pi} a(e^{i(\theta-\phi)})b(e^{i\phi})\,\frac{d\phi}{2\pi}.$$

To distinguish the two structures we'll use separate notation for the two.

Definition 1.1.2. We write

- $\mathcal{A}(\mathbb{T})$ for the algebra of trigonometric polynomials with pointwise multiplication,

- $\mathcal{D}(\mathbb{T})$ for the algebra of trigonometric polynomials with the convolution product.

Remark 1.1.3. In general in these notes, we will use $\mathcal{A}$ to denote algebras with pointwise product and $\mathcal{D}$ for algebras with convolution product. The letter $\mathcal{D}$ is chosen to suggest smooth 'distributions' or 'densities' (see below).

Basic Fourier theory says that both these algebras can be understood in terms of the sequence of Fourier coefficients $\hat{a} = (\hat{a}_k)_{k\in\mathbb{Z}}$ of a trigonometric polynomial a. Let us write

$$\mathbb{C}[\mathbb{Z}] = \{(c_k)_{k\in\mathbb{Z}} \mid c_k \in \mathbb{C} \text{ and } c_k = 0 \text{ for all but finitely many } k \in \mathbb{Z}\}.$$

Again, this is a countable dimensional vector space which can be equipped with two different algebra structures, namely pointwise multiplication or convolution. Again, we distinguish these with separate notation:

- $\mathcal{A}(\mathbb{Z})$ denotes the space $\mathbb{C}[\mathbb{Z}]$ equipped with pointwise multiplication, so that
$$(c.d)_k = c_k.d_k$$
- $\mathcal{D}(\mathbb{Z})$ denotes the space $\mathbb{C}[\mathbb{Z}]$ equipped with convolution product,
$$(c * d)_k = \sum_{j\in\mathbb{Z}} c_{k-j}d_j.$$

Theorem 1.1.4 (Fourier transform for trigonometric polynomials)**.** *The* Fourier transform
$$\mathcal{F} : \sum_{k\in\mathbb{Z}} a_k\mathbf{z}^k \mapsto (a_k)_{k\in\mathbb{Z}}$$
is an isomorphism of algebras
$$\mathcal{F} : \mathcal{A}(\mathbb{T}) \xrightarrow{\cong} \mathcal{D}(\mathbb{Z}),$$
and also of algebras
$$\mathcal{F} : \mathcal{D}(\mathbb{T}) \xrightarrow{\cong} \mathcal{A}(\mathbb{Z}).$$

Theorem 1.1.4 is purely algebraic. It can be extended to a result in functional analysis in many different ways. For instance, the Fourier transform $\mathcal{F} : \mathcal{A}(\mathbb{T}) \to \mathcal{D}(\mathbb{Z})$ extends to the following isomorphisms of topological vector spaces and topological algebras:

$L^2(\mathbb{T}) \cong \ell^2(\mathbb{Z})$ (These are not algebras)

$C^\infty(\mathbb{T}) \cong \mathcal{S}(\mathbb{Z}) := \{(a_k)_{k\in\mathbb{Z}} \mid \text{for every } n \in \mathbb{N}, \text{ the sequence } (k^n a_k)_{k\in\mathbb{Z}} \text{ is bounded}\}$
(Elements of $\mathcal{S}(\mathbb{Z})$ are sequences of *rapid decay*)

$A(\mathbb{T}) \cong \ell^1(\mathbb{Z})$ ($A(T)$ is the Fourier algebra of $\mathbb{T}$)

$C(\mathbb{T}) \cong C^*(\mathbb{Z})$ ($C^*(\mathbb{Z})$ is the C^*-algebra of $\mathbb{Z}$)

$L^\infty(\mathbb{T}) \cong \mathrm{vN}(\mathbb{Z})$ ($\mathrm{vN}(\mathbb{Z})$ is the von Neumann algebra of $\mathbb{Z}$)

and many others. All of the topological algebras on the left are the completions of the dense subalgebra $\mathcal{A}(\mathbb{T})$ with respect to some locally convex topology. In every case, the proof of the isomorphism starts with Theorem 1.1.4.

The groups $\mathbb{T}$ and $\mathbb{Z}$ are called *Pontryagin duals* of one another. One of the goals of these lecture notes will be to generalize Pontryagin duality as far as we can.

Consider, for instance, the *special unitary group*

$$\mathrm{SU}(N) = \{g \in M_n(\mathbb{C}) \mid g^* g = I \text{ and } \mathrm{Det}(g) = 1\}$$

It is a compact Lie group. We denote its Haar measure by μ.

Let (e_i) be the standard orthonormal basis for $\mathbb{C}^N$. Consider the matrix coefficient functions

$$u_{ij} = \langle e_i | \cdot | e_j \rangle : \mathrm{SU}(N) \to \mathbb{C}; \qquad g \mapsto g_{ij} := \langle e_i, g e_j \rangle.$$

These are continuous functions on $\mathrm{SU}(N)$. Inside the space of continuous functions $C(\mathrm{SU}(N))$, we have the subspace of polynomial functions in the matrix coefficients u_{ij}. This subspace is closed under both pointwise multiplication,

$$a.b(g) := a(g)b(g)$$

and convolution,

$$a * b(g) := \int_G a(gh^{-1}) b(h)\, d\mu(h).$$

Thus, we again have two possible algebra structures on this dense subspace, which we denote by $\mathcal{A}(\mathrm{SU}(N))$ and $\mathcal{D}(\mathrm{SU}(N))$, respectively.

Note though, that since $\mathrm{SU}(N)$ is not abelian, the algebra $\mathcal{D}(\mathrm{SU}(N))$ is not commutative. Therefore, there is no hope of finding an algebra isomorphism $\mathcal{D}(\mathrm{SU}(N)) \cong \mathcal{A}(\hat{G})$ for some algebra of functions on a Pontryagin dual group $\hat{G}$. Instead, the Pontryagin dual $\hat{G}$ will be, by definition, a quantum group.

1.2 Distributions

To put the above example into a general context, it is better to see the algebras $\mathcal{D}(\mathbb{T})$ and $\mathcal{D}(\mathbb{Z})$ as algebras of distributions. We begin by recalling some definitions.

Let M be a smooth manifold. We write $C^\infty(M)$ for the space of smooth $\mathbb{C}$-valued functions on M, and $C_c^\infty(M)$ for the subspace of compactly supported smooth functions. These have natural locally convex vector space topologies; see for instance [Fri98] or [Trè67].

A *compactly supported distribution* on M is a continuous linear functional

$$u : C^\infty(M) \to \mathbb{C}.$$

The space of compactly supported distributions on M is denoted $\mathcal{E}'(M)$. We thus get a bilinear pairing:

$$(\ ,\ ) : \mathcal{E}'(M) \times C^\infty(M) \to \mathbb{C}, \qquad (u, f) = u(f).$$

Example 1.2.1. Let $M = G$ be a Lie group, and let μ be the left-invariant Haar measure on G. Associated to any compactly supported smooth function $f \in C_c^\infty(G)$ there is a compactly supported distribution $f\mu \in \mathcal{E}'(G)$ defined by

$$(f\mu, a) := \int_M f(x)a(x)\, d\mu(x).$$

In this way, we get a continuous embedding of $C_c^\infty(G)$ as a dense subspace of $\mathcal{E}'(G)$. Distributions of the form $f\mu$ are called *compactly supported smooth densities*.

We will denote the space of compactly supported smooth densities on G by $C_c^\infty(G; |\Omega^1|)$. It is isomorphic to $C_c^\infty(G)$ as a space, although the topology of $C_c^\infty(G)$ does not agree with the topology of $C_c^\infty(G; |\Omega^1|)$ as a subspace of $\mathcal{E}'(G)$. The notation $C_c^\infty(G; |\Omega^1|)$ comes from the fact that the smooth densities on G can be seen as the space of sections of a certain canonical line bundle $|\Omega^1|$ over G called the *bundle of densities*, or 1-densities more explicitly. We won't need this description.

It's common in the literature to identify the spaces $C_c^\infty(G)$ and $C_c^\infty(G; |\Omega^1|)$, but it can be extremely helpful to maintain a distinction. For instance, they have opposite functoriality (see below).

Example 1.2.2. We don't exclude the possibility that M is a 0-dimensional manifold, *i.e.*, a discrete space. In this case, every function is smooth, so that

$$C^\infty(M) = \{a : M \to \mathbb{C}\},$$
$$C_c^\infty(M) = \{a : M \to \mathbb{C} \text{ with finite support}\}.$$

If μ denotes counting measure on a discrete group G, then the analogue of the construction from Example 1.2.1 gives the pairing

$$(f\mu, a) = \sum_{x \in M} f(x)a(x),$$

for $f \in C_c^\infty(G)$ and $a \in C^\infty(G)$.

Smooth functions are contravariant. Specifically, if $\varphi : M \to N$ is a smooth map then we can pull back smooth functions via φ,

$$\varphi^* : C^\infty(N) \to C^\infty(M); \qquad \varphi^* f = f \circ \varphi.$$

Compactly supported distributions are covariant, and the push-forward map $\varphi_* : \mathcal{E}'(M) \to \mathcal{E}'(N)$ is defined by duality:

$$(\varphi_* u, a) := (u, \varphi^* a),$$

for $u \in \mathcal{E}'(M),\ a \in C^\infty(N)$.

We finish this section with a quick reminder about tensor products of function spaces. This part will be light on details.

If M is a finite set, and $\mathcal{A}(M)$ denotes the set of all complex valued functions on M, then there is a canonical isomorphism

$$\mathcal{A}(M) \otimes \mathcal{A}(M) \cong \mathcal{A}(M \times M),$$

such that $(f \otimes g)(x,y) = f(x)g(y)$. If M is a manifold then this formula gives an embedding of $C_c^\infty(M) \otimes C_c^\infty(M)$ into $C_c^\infty(M)$ which is dense but not surjective. To get an isomorphism, we need to use a completed tensor product. The usual choice is the projective tensor product, see [Trè67], which we will denote by $\bar{\otimes}$. Similar statements are possible for the spaces of distributions and smooth densities, with appropriate completed tensor products:

$$\begin{aligned} C_c^\infty(M) \bar{\otimes} C_c^\infty(M) &\cong C_c^\infty(M \times M), \\ C_c^\infty(M; |\Omega^1|) \bar{\otimes} C_c^\infty(M; |\Omega^1|) &\cong C_c^\infty(M \times M; |\Omega^1|), \\ \mathcal{E}'(M) \bar{\otimes} \mathcal{E}'(M) &\cong \mathcal{E}'(M \times M). \end{aligned}$$

In fact, we won't make much use of these topological tensor products, since very shortly we'll restrict our attention to certain countable dimensional dense subalgebras, for which the algebraic tensor product will suffice for our needs.

1.3 Algebras associated to Lie groups

Let G be a Lie group. Let us write, temporarily,

$$\begin{aligned} \mathcal{A}^\infty(G) &:= C_c^\infty(G), \\ \mathcal{D}^\infty(G) &:= C_c^\infty(G; |\Omega^1|). \end{aligned}$$

We'll change this notation shortly. Again, we recall that these are isomorphic as vector spaces, but equipped with different algebra structures. If we equip both with the standard locally convex topology on $C_c^\infty(G)$, then they are topological algebras.

Let us examine the algebra structures from a functorial point of view. The pointwise product on $\mathcal{A}^\infty(G)$ is given by the pull-back by the diagonal embedding

$$\mathrm{Diag} : G \to G \times G; \qquad \mathrm{Diag}(x) = (x, x).$$

That is, for all $a, b \in \mathcal{A}^\infty(G)$,

$$a.b = \mathrm{Diag}^*(a \otimes b).$$

On the other hand, the convolution product on $\mathcal{D}^\infty(G)$ is the push-forward by the group law

$$\mathrm{Mult} : G \times G \to G; \qquad \mathrm{Mult}(x, y) = xy.$$

That is, for all $u, v \in C_c^\infty(G; |\omega^1|)$,

$$u * v = \mathrm{Mult}_*(u \otimes v).$$

Exercise 1.3.1. Check these formulas.

Both the pointwise multiplication and the convolution are associative products. For the convolution on $\mathcal{D}^\infty(G)$, this follows immediately from the associativity of the group law:

$$\mathrm{Mult} \circ (\mathrm{Mult} \times \mathrm{id}) = \mathrm{Mult} \circ (\mathrm{id} \times \mathrm{Mult}) : G \times G \times G \to G.$$

For the pointwise multiplication on $\mathcal{A}^\infty(G)$, this follows from the *coassociativity* of the diagonal embedding:

$$(\mathrm{Diag} \times \mathrm{id}) \circ \mathrm{Diag} = (\mathrm{id} \times \mathrm{Diag}) \circ \mathrm{Diag} : G \to G \times G \times G.$$

This symmetry between the maps $\mathrm{Mult} : G \times G \to G$ and $\mathrm{Diag} : G \to G \times G$ is striking, and this is the fundamental observation underlying quantum groups. The notions of associativity and coassociativity can both be expressed by a commuting diagram of maps, and the difference between the two is just the direction of the arrows. This will be true of many dual notions to come: algebras and coalgebras, units and counits, etc.

We should observe, though, that there is one important way in which this symmetry fails for the maps Diag and Mult on a classical group. Let us write $\mathfrak{S}$ for the flip map

$$\mathfrak{S} : G \times G \to G \times G; \qquad (x, y) \mapsto (y, x).$$

Then the diagonal embedding is always *cocommutative*, in the sense that

$$\mathfrak{S} \circ \mathrm{Diag} = \mathrm{Diag}.$$

But the multiplication map is not always *commutative*. Indeed, we have

$$\mathrm{Mult} \circ \mathfrak{S} = \mathrm{Mult},$$

only if the group G is abelian. The theory of quantum groups allows us to rectify this asymmetry.

1.4 Hopf algebras

Given that the algebra maps on $\mathcal{A}^\infty(G)$ and $\mathcal{D}^\infty(G)$ are defined by Diag^* and Mult_*, respectively, it becomes clear that these spaces also both admit a second, less well-known, algebraic structure, called a *coproduct*. Morally, we would like to say that we have coproducts

$$\begin{aligned}\Delta = \mathrm{Mult}^* :&\mathcal{A}^\infty(G) \to \mathcal{A}^\infty(G)\bar{\otimes}\mathcal{A}^\infty(G),\\ \hat{\Delta} = \mathrm{Diag}_* :&\mathcal{D}^\infty(G) \to \mathcal{D}^\infty(G)\bar{\otimes}\mathcal{D}^\infty(G).\end{aligned}$$

For a general Lie group this is not quite right. For instance on a non-compact group G, the pull back $\mathrm{Mult}^* a$ of a compactly supported function $a \in C_c^\infty(M)$ is not compactly supported, so that Mult^* maps $\mathcal{A}^\infty(G)$ into $C^\infty(G \times G)$, not into $\mathcal{A}^\infty(G)\bar{\otimes}\mathcal{A}^\infty(G) = C_c^\infty(G \times G)$. Likewise, for any Lie group of dimension greater than 0, the push-forward of a compactly supported smooth density by the diagonal embedding is not smooth, so that Diag_* maps $\mathcal{D}^\infty(G)$ to $\mathcal{E}'(G \times G)$.

We will avoid this technical issue almost entirely, by restricting our attention to finite and compact quantum groups, where the problem won't arise. But, for the sake of completeness, let us remark that there are analogues of the definitions to follow which can take account of this problem. Specifically, we can observe that $C^\infty(G \times G)$ and $\mathcal{E}'(G \times G)$ are multiplier algebras of $C_c^\infty(G \times G)$ and $C^\infty(G \times G; |\Omega^1|)$, respectively. Taking this into account leads to the notion of a *bornological quantum group* [Voi08], generalizing the notion of *algebraic quantum groups* of Van Daele [VD98]. Another approach, is to use C^* and von Neumann algebras, giving the notion of *locally compact quantum group* of Kustermans and Vaes [KV00].

To return to the purely algebraic situation, the following definition will underpin the entire theory of quantum groups.

Definition 1.4.1. A Hopf algebra (over $\mathbb{C}$) is a complex vector space $\mathcal{A}$ equipped with linear maps as follows:

- **Product** $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, written as usual as $a \otimes b \mapsto ab$.
- **Unit** $\eta : \mathbb{C} \to \mathcal{A}$, determined equivalently by a unit element $1 := \eta(1)$.
- **Coproduct** $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, denoted sometimes by the Sweedler notation $a \mapsto \sum a_{(1)} \otimes a_{(2)}$, often with the summation suppressed.
- **Counit** $\epsilon : \mathcal{A} \to \mathbb{C}$.
- **Antipode (or coinverse)** $S : \mathcal{A} \to \mathcal{A}$.

These must satisfy the following axioms:

- $(\mathcal{A}, m, 1)$ **is a unital algebra.**

- $(\mathcal{A}, \Delta, \epsilon)$ **is a counital coalgebra.** This means Δ is coassociative,

$$(\Delta \otimes \mathrm{id})\Delta = (\Delta \otimes \mathrm{id})\Delta,$$

and the counit satisfies

$$(\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id})\Delta,$$

under the natural identification $\mathcal{A} \otimes \mathbb{C} = \mathcal{A} = \mathbb{C} \otimes \mathcal{A}$.

- $(\mathcal{A}, m, 1, \Delta, \epsilon)$ **is a bialgebra.** This means the coalgebra maps are unital algebra homomorphisms,

$$\begin{aligned}\Delta(ab) &= a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)},\\ \epsilon(ab) &= \epsilon(a)\epsilon(b),\\ \Delta(1) &= 1 \otimes 1.\end{aligned}$$

Equivalently, the algebra maps are counital coalgebra homomorphisms.

- **The antipode condition.**

$$m \circ (S \otimes \mathrm{id}) \circ \Delta(a) = \epsilon(a)1 = m \circ (\mathrm{id} \otimes S) \circ \Delta(a),$$

for all $a \in \mathcal{A}$.

It follows automatically that S is an algebra antihomomorphism and a coalgebra antihomomorphism:

$$\begin{aligned}S(ab) &= S(b)S(a),\\ (S \times S)(\Delta(a)) &= \Delta^{\mathrm{cop}}(S(a)),\end{aligned}$$

where $\Delta^{\mathrm{cop}} = \mathfrak{S} \circ \Delta$.

We note that the axioms of a counital coalgebra are obtained from those of a unital algebra by drawing the commuting diagrams representing the algebra axioms in terms of the maps m and η, and reversing all the arrows.

To illustrate this definition, consider the two Hopf algebras naturally associated to a finite group.

Example 1.4.2. Let G be a finite group and let $\mathcal{A}(G) = C(G)$ be the space of all complex valued functions on G. Then $\mathcal{A}(G)$ is a Hopf algebra with:

$$\begin{aligned}ab(g) &= a(g)b(g), & 1(g) &= 1,\\ \Delta a(g,h) &= a(gh), & \epsilon(a) &= a(e),\\ (Sa)(g) &= a(g^{-1}),\end{aligned}$$

for $a, b \in \mathcal{A}(G)$, $g, h \in G$.

Example 1.4.3. Still with G a finite group, let $\mathcal{D}(G) = \mathbb{C}[G]$ be the vector space of with basis $(\delta_g)_{g\in G}$. It is a Hopf algebra with operations

$$\delta_g * \delta_h = \delta_{gh}, \qquad \hat{1} = \delta_e,$$
$$\hat{\Delta}(\delta_g) = \delta_g \otimes \delta_g, \qquad \hat{\epsilon}(\delta_g) = 1,$$
$$S(\delta_g) = \delta_{g^{-1}}.$$

Of course, there is an identification of $\mathbb{C}[G]$ with $C(G)$ by sending a function $u : G \to \mathbb{C}$ to the element $u = \sum_{g\in G} u(g)\delta_g$. In this picture, the above operations become,

$$u * v(g) = \sum_{h\in G} u(gh^{-1})v(h), \qquad \hat{1}(g) = \begin{cases} 1, & \text{if } g = e, \\ 0, & \text{else,} \end{cases},$$
$$\hat{\Delta}u(g,h) = \begin{cases} u(g), & \text{if } g = h, \\ 0, & \text{else,} \end{cases} \qquad \hat{\epsilon}(a) = \sum_{g\in G} u(g),$$
$$(\hat{S}u)(g) = u(g^{-1}),$$

for $u, v \in \mathcal{A}(G)$, $g, h \in G$.

Exercise 1.4.4. Consider the maps Mult : $G \times G \to G$, Diag : $G \to G \times G$, e : $\{e\} \to G$ the inclusion of the trivial subgroup, $\pi : G \to \{e\}$ the quotient to the trivial subgroup, and $\iota : G \to G$ the group inverse. Show that the Hopf algebra operations on $\mathcal{A}(G)$ are given by

$$m = \text{Diag}^*, \quad \eta = \pi^*, \quad \Delta = \text{Mult}^*, \quad \epsilon = \text{e}^*, \quad S = \iota^*,$$

and those on $\mathcal{D}(G)$ are

$$\hat{m} = \text{Mult}_*, \quad \hat{\eta} = \text{e}_*, \quad \hat{\Delta} = \text{Diag}_*, \quad \hat{\epsilon} = \pi_*, \quad \hat{S} = \iota_*.$$

Deduce the veracity of the Hopf algebra axioms for $\mathcal{A}(G)$ and $\mathcal{D}(G)$.

1.5 $*$-structures and Haar integrals

Recall that the Diagonal embedding Diag : $G \to G \times G$ is always a cocommutative map for a classical finite group, so that the product on $\mathcal{A}(G)$ is always abelian. Following Connes' philosophy, this suggests that a non-abelian Hopf algebra might provide a reasonable notion of 'group' in noncommutative geometry. However, noncommutative geometry really relies on $*$-algebras, so we need to add a little more structure.

Definition 1.5.1. A $*$-Hopf algebra is a Hopf algebra $\mathcal{A}$ equipped with an antilinear involution $a \mapsto a^*$ which is an algebra anti-automorphism and coalgebra automorphism. That is, for all $a, b \in \mathcal{A}$,

$$(ab)^* = b^*a^*$$

and

$$\Delta(a^*) = \Delta(a)^* = \sum a^*_{(1)} \otimes a^*_{(2)}.$$

Example 1.5.2. Let G be a compact group. The Hopf algebra $\mathcal{A}(G)$ of Example 1.4.2 is a $*$-Hopf algebra if equipped with the involution of pointwise conjugation,

$$a^*(g) = \overline{a(g)}, \qquad (g \in G).$$

Example 1.5.3. The convolution algebra $\mathcal{D}(G)$ of Example 1.4.3 is a $*$-Hopf algebra with involution determined by

$$\delta_g^* = \delta_{g^{-1}},$$

for all $g \in G$.

Proposition 1.5.4. *If $\mathcal{A}$ is a Hopf $*$-algebra then*

$$1^* = 1, \qquad \epsilon(a^*) = \overline{\epsilon(a)}, \qquad S(S(a^*)^*) = a,$$

for all $a \in \mathcal{A}$. In particular, the antipode S is invertible with $S^{-1} = * \circ S \circ *$.

As a final piece of structure, we need an analogue of the Haar measure.

Definition 1.5.5. A $*$-Hopf algebra $\mathcal{A}$ is called a *compact quantum group algebra* if it is equipped with a linear functional ϕ, called the *Haar state*, satisfying

Positivity: $\phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$,

Faithfulness: $\phi(a^*a) = 0$ iff $a = 0$.

Left-invariance: $(\mathrm{id} \otimes \phi)\Delta(a) = \phi(a)1$ for all $a \in \mathcal{A}$.

When $\mathcal{A}$ is finite dimensional, we call $\mathcal{A}$ a *finite quantum group algebra*.

Example 1.5.6. Let G be a finite group. The $*$-Hopf algebra $\mathcal{A}(G)$ admits a Haar state defined by

$$\phi(a) := \sum_{g \in G} a(g).$$

Note that this is really the integral with respect to Haar measure,

$$\phi(a) = \int_G a(g)\, d\mu(g),$$

since here the Haar measure μ is counting measure. When checking the axioms, one sees that the left-invariance of ϕ, as in Definition 1.5.5 corresponds exactly to the left-invariance of the Haar measure on G. Essentially the same argument works for the Hopf $*$-algebra $\mathcal{A}(G)$ of matrix coefficients on any compact group G, see the next chapter.

Example 1.5.7. Let G be a finite group. The convolution algebra $\mathcal{D}(G)$ of Example 1.4.3 is a compact quantum group algebra with Haar state

$$\hat{\phi}(u) := u(e).$$

Exercise 1.5.8. Check the above examples.

We have now obtained a reasonable notion of the noncommutative algebra of functions on a finite, or compact, quantum group. The notion of the *quantum group* itself is purely formal. It has no concrete existence on its own outside of these algebras. Nevertheless, we will use the notation $\mathcal{A} = \mathcal{A}(G)$ whenever we wish to interpret a compact quantum group algebra $\mathcal{A}$ as formally representing an algebra of functions on some hypothetical quantum group G.

1.6 Duality

Let $\mathcal{A}$ and $\mathcal{D}$ be Hopf $*$-algebras. We denote the Hopf algebra operations on $\mathcal{A}$ without hats and on $\mathcal{D}$ with hats.

Definition 1.6.1. A bilinear map

$$(\, , \,) : \mathcal{D} \times \mathcal{A} \to \mathbb{C}$$

is called a *skew-pairing* if for all $a, b \in \mathcal{A}$, $u, v \in \mathcal{D}$,

$$\begin{aligned} (u, ab) &= (\hat{\Delta}(u), b \otimes a), & (uv, a) &= (u \otimes v, \Delta(a)), \\ (u, 1) &= \hat{\epsilon}(u), & (1, a) &= \epsilon(a), \\ (u, a^*) &= \overline{(\hat{S}^{-1}(u)^*, a)}, & (u^*, a) &= \overline{(u, S(a)^*)}. \end{aligned}$$

Lemma 1.6.2. *Either of the two conditions in the last line of Definition 1.6.1 follows from the other. Moreover, we also have*

$$(u, S(a)) = (\hat{S}^{-1}(u), a), \qquad (\hat{S}(u), a) = (u, S^{-1}(a)),$$

for all $a \in \mathcal{A}$, $u \in \mathcal{D}$.

Theorem 1.6.3. *Let $\mathcal{A}$ be a finite quantum group algebra and denote its dual space by $\mathcal{D} = \mathcal{A}^*$. Then $\mathcal{D}$ admits a unique structure of a Hopf $*$-algebra such that the canonical pairing*

$$\mathcal{D} \times \mathcal{A} \to \mathbb{C}$$

is a skew-pairing of $*$-Hopf algebras. Moreover, the Haar integral on $\mathcal{A}$ induces a linear isomorphism $\mathcal{F} : \mathcal{A} \to \mathcal{D}$ defined by

$$\mathcal{F}(a)(b) := \phi(ba),$$

and the linear form $\hat{\phi} : \mathcal{D} \to \mathbb{C}$ defined on $\mathcal{D}$ by

$$\hat{\phi}(\mathcal{F}(a)) = \epsilon(a)$$

is a Haar integral. Therefore $\mathcal{D}$ is a finite quantum group algebra.

If we are interpreting $\mathcal{A} = \mathcal{A}(G)$ as the algebra of functions on a hypothetical quantum group G, then we will write $\mathcal{D}(G)$ for the dual space of $\mathcal{A}(G)$ with the Hopf $*$-algebra structure of the above theorem.

Remark 1.6.4. Given the presentation in these notes, we can't quite extend Theorem 1.6.3 to general compact quantum groups. To understand the problem, consider the algebra of smooth compactly supported densities $\mathcal{D}^\infty(G)$ on a classical compact group. Under the coproduct

$$\Delta = \mathrm{Diag}_* : \mathcal{E}'(G) \to \mathcal{E}'(G \times G),$$

a smooth density is mapped to a distribution supported on the diagonal $\mathrm{Diag}(G) \subset G \times G$. This never a smooth density, unless it is zero. In particular, we don't have $\Delta : \mathcal{D}^\infty(G) \to \mathcal{D}^\infty(G) \otimes \mathcal{D}^\infty(G)$.

It is possible to extend the framework of compact quantum groups so that Δ maps $\mathcal{A}$ into the multiplier algebra of $\mathcal{A} \otimes \mathcal{A}$. With some additional axioms, this yields the notion of a *multiplier Hopf algebra* and then an *algebraic quantum group*, due to Van Daele [VD98] which is nicely adapted to duals of compact quantum groups. Unfortunately, we won't have time to study this.

In the context of finite quantum groups, and in particular Theorem 1.6.3, Pontryagin Duality is reduced to the level of a formal definition. Since $\mathcal{D}(G)$ is a finite quantum group algebra, we can formally define it to be the algebra of functions $\mathcal{A}(\hat{G})$ on some hypothetical finite quantum group $\hat{G}$. We can't do any better than this, because quantum groups only exist at the formal level of algebras anyway.

Classical Pontryagin Duality is recovered from this abstract nonsense only when we realize that when G is abelian, the commutative algebra $\mathcal{A}(\hat{G}) := \mathcal{D}(G)$ is isomorphic to the algebra of functions on some finite space $\hat{G}$. That is, we have replaced Pontryagin Duality by Gelfand Duality. Moreover, Gelfand duality says that the comultiplication $\hat{\Delta} : \mathcal{A}(\hat{G}) \to \mathcal{A}(\hat{G}) \otimes \mathcal{A}(\hat{G}) = \mathcal{A}(\hat{G} \times \hat{G})$ is given by the pull-back by a map $\hat{G} \times \hat{G} \to \hat{G}$, and the Hopf algebra axioms imply that this defines an associative group law on $\hat{G}$.

In the literature, the linear isomorphism $\mathcal{F} : \mathcal{A}(G) \to \mathcal{D}(G)$ is sometimes referred to as the "Fourier transform". This is a little bit misleading, since in the case of a classical Lie group, $\mathcal{F}$ is really just the map

$$\mathcal{F} : C_c^\infty(G) \to C_c^\infty(G; |\Omega^1|)$$

which sends a smooth function a to the smooth density $a\mu$. This map is fairly trivial. The Fourier transform is obtained from this only once we compose this map with the Gelfand transform.

Along the above lines, one can show that any commutative finite quantum group algebra is isomorphic to $\mathcal{A}(G)$ for some finite group G, and any cocommutative finite quantum group algebra is isomorphic to $\mathcal{D}(G)$ for some finite group G. So far, these are the only examples we have.

There are some simple constructions to obtain finite dimensional examples which are neither commutative nor cocommutative, such as the Drinfeld double. But our focus in these notes will be a more interesting family of quantum deformations of compact semisimple Lie groups, which we discuss in the coming chapters.

Chapter 2

Matrix coefficient algebras

2.1 Unitary representations and matrix coefficients

Let's return briefly to the discrete Fourier transform, Theorem 1.1.4. This is of course, really a result from unitary representation theory. Let's recall the definitions.

Definition 2.1.1.

- Let G be a locally compact topological group. A *unitary representation of G on a Hilbert space V* is a weakly continuous map

  $$\pi : G \to U(V)$$

  where $U(V)$ denotes the space of unitary operators on G.

- A *subrepresentation* of V is a subspace $W \leq V$ which is invariant under the representation:

  $$\pi(g)W \subseteq W, \qquad \text{for all } g \in G.$$

- A unitary representation is *irreducible* if it has no non-trivial proper subrepresentation $0 \subsetneqq W \subsetneqq V$.

Weakly continuous means that for any pair of vectors $\xi, \eta \in V$, the map

$$\langle \xi | \bullet | \eta \rangle : G \to \mathbb{C}; \qquad g \mapsto \langle \xi, \pi(g)\eta \rangle. \tag{2.1.1}$$

is continuous. Maps of this form are called *matrix coefficients* of the representation g, and they will play a major role here. We are using the physicists' bra-ket notation $\langle \xi | \bullet | \eta \rangle$ to denote these matrix coefficient functions.

If $V = V_1 \oplus V_2$ is a decomposition of V as a direct sum of two subrepresentations, then V_1 and V_2 are orthogonal. As a consequence, any matrix coefficient of V decomposes as a sum of matrix coefficients for V_1 and V_2.

For a compact group G, any unitary representation $\pi : G \to U(V)$ decomposes as a direct sum of irreducible representations, meaning

$$V = \bigoplus_i V_i$$

where V_i are mutually orthogonal subspaces, each invariant under π.

Moreover, if π is any continuous representation on a finite dimensional vector space V,

$$\pi : G \to \mathrm{Aut}(V)$$

then V admits an inner product which makes G unitary. Therefore, any finite dimensional representation decomposes as a direct sum of irreducibles. It follows that any matrix coefficient of a finite dimensional representation is a sum of matrix coefficients of irreducible representations.

It is often convenient to ignore the Hilbert space structure, and consider matrix coefficients associated to a vector $\xi \in V$ and a covector $\xi' \in V^*$:

$$\langle \xi' | \cdot | \xi \rangle : G \mapsto \langle \xi', \pi(g)\xi \rangle := \xi'(\pi(g)\xi).$$

Example 2.1.2. For every $n \in \mathbb{Z}$, the map

$$\pi_n : \mathbb{T} \to U(\mathbb{C}); \qquad \pi_n(e^{i\theta}) = e^{in\theta}$$

is a unitary representation of $\mathbb{T}$. These are all the irreducible representations of $\mathbb{T}$, up to isomorphism.

Therefore, the matrix coefficients of π_n are just the multiples of the functions $\mathbf{z}^n : e^{i\theta} \mapsto e^{in\theta}$. As a consequence, the set of matrix coefficients of all finite dimensional representations of $\mathbb{T}$ is precisely the space $\mathcal{A}(\mathbb{T})$ of trigonometric polynomials from Section 1.1.

Definition 2.1.3. Let G be a compact group. We write $\mathcal{A}(G)$ for the set of all matrix coefficients of finite dimensional representations of G.

Proposition 2.1.4. *The space $\mathcal{A}(G)$ is a compact quantum group algebra with Hopf algebra operations analogous to Example 1.4.2, namely*

$$\begin{aligned} (ab)(g) &= a(g)b(g), & 1(g) &= 1, \\ \Delta(a)(g,h) &= a(gh), & \epsilon(a) &= a(1), \end{aligned}$$
$$S(a)(g) = a(g^{-1}).$$

and Haar state

$$\phi(a) = \int_G a(g)\, dg.$$

One of the key points in this proposition is that all the maps are well-defined on $\mathcal{A}(G)$. Elements of the proof of this will be useful to us later. Before giving the proof, we need to recall the definition of a tensor product of representations.

Definition 2.1.5. Let $\pi : G \to \mathrm{Aut}(V)$ and $\tau : G \to \mathrm{Aut}(W)$ be two finite dimensional representations of G. The tensor product of π and τ is the representation defined by

$$\pi \otimes \tau : G \to \mathrm{Aut}(V \otimes W); \qquad (\pi \otimes \tau)(g)(v \otimes w) = \pi(g)v \otimes \pi(g)w.$$

Remark 2.1.6. If we want to be fussy, the notation $\pi \otimes \tau$ is really an abbreviation for the composition $(\pi \otimes \tau) \circ \mathrm{Diag} : g \mapsto \pi(g) \otimes \tau(g)$. The appearance of the diagonal embedding Diag here is of course relevant to the definition of the product in $\mathcal{A}(G)$.

Proof of Proposition 2.1.4. Let $a = \langle \xi' | \bullet | \xi \rangle$ be a matrix coefficient of $\pi : G \to \mathrm{End}(V)$ and $b = \langle \eta' | \bullet | \eta \rangle$ be a matrix coefficient of $\tau : G \to \mathrm{End}(W)$.

For the algebra structure, we note that

$$ab(g) = \langle \xi' | \pi(g) | \xi \rangle \langle \eta' | \tau(g) | \eta \rangle = \langle \xi' \otimes \eta' | (\pi \otimes \tau)(g) | \xi \otimes \eta \rangle,$$

where $\pi \otimes \sigma : G \to \mathrm{End}(V \otimes W)$ is the tensor product representation, so that ab is again in $\mathcal{A}(G)$. The constant function 1 is a matrix unit for the trivial representation of G on $\mathbb{C}$.

For the coalgebra structure, let (e_i) be a basis for V and (e^i) a dual basis for V^*. Then

$$\Delta(a)(g, h) = \langle \xi' | \pi(g)\pi(h) | \xi \rangle = \sum_i \langle \xi' | \pi(g) | e_i \rangle \langle e^i | \pi(h) | \xi \rangle.$$

In other words

$$\Delta(\langle \xi' | \bullet | \xi \rangle) = \sum_i \langle \xi' | \bullet | e_i \rangle \otimes \langle e^i | \bullet | \xi \rangle. \tag{2.1.2}$$

This shows that $\Delta : \mathcal{A}(G) \to \mathcal{A}(G) \otimes \mathcal{A}(G)$ is well-defined.

For the antipode, we recall the definition of the *contragredient representation*

$$\pi^{\mathrm{c}} : G \to \mathrm{End}(V^*); \pi^{\mathrm{c}}(g)\xi' = \xi' \circ \pi(g^{-1}).$$

With this, we have

$$S(a)(g) = a(g^{-1}) = \langle \xi' | \pi(g^{-1}) | \xi \rangle = \langle \xi | \pi^{\mathrm{c}}(g) | \xi' \rangle,$$

under the natural identification $\xi \in V \cong V^{**}$. Therefore $S(a)$ is a matrix coefficient of π^{c}.

To see that the involution applied to a matrix coefficient gives again a matrix coefficient, one needs to consider the *conjugate representation* of a representation $\pi : G \to \mathrm{End}(V)$. Recall that the conjugate vector space $\overline{V}$ is V with a modified $\mathbb{C}$-linear structure, given by

$$\begin{aligned} \overline{\xi} + \overline{\eta} &= \overline{\xi + \eta}, \\ \lambda \overline{\xi} &= \overline{\overline{\lambda} \xi}, \end{aligned}$$

for $\xi, \eta \in V$, $\lambda \in \mathbb{C}$. Then the representation π on V is also a representation on $\overline{V}$, which we denote by $\overline{\pi}$. One can then check that

$$\langle \xi' | \bullet | \xi \rangle^* = \langle \overline{\xi'} | \bullet | \overline{\xi} \rangle.$$

Once we know that these maps are well-defined, the axioms of a Hopf $*$-algebra axioms can be checked directly. The fact that ϕ is a Haar integral can be checked as in Exercise 1.5.6. □

The coproduct in equation (2.1.2) is called the *matrix coproduct*. It corresponds to the usual product law for matrix coefficients: $(AB)_{ik} = \sum_j A_{ij} B_{jk}$.

We can give a more algebraic description of the Haar state.

Lemma 2.1.7. *Let G be a compact group and $a = \langle \xi' | \bullet | \xi \rangle \in \mathcal{A}(G)$ a matrix coefficient of an irreducible finite dimensional representation $\pi : G \to \mathrm{Aut}(V)$. Show that the Haar state applied to a is*

$$\phi(a) = \begin{cases} \langle \xi', \xi \rangle, & \text{if } \pi \text{ is the trivial representation,} \\ 0, & \text{otherwise.} \end{cases}$$

2.2 Representations of SU(2)

Let's now consider the compact group $G = \mathrm{SU}(2)$. This is a 3-dimensional Lie group,

$$\mathrm{SU}(2) = \left\{ g = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \middle| \ \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}, \tag{2.2.1}$$

diffeomorphic to the unit sphere in $\mathbb{C}^2$.

The group SU(2) acts on $\mathbb{C}^2$ by its usual linear action. This defines a representation

$$\tau : \mathrm{SU}(2) \to \mathrm{Aut}(\mathbb{C}^2),$$

called the *fundamental representation*, which is unitary with respect to the standard inner product.

To get other representations, we can proceed as follows. Let V_n denote the set of homogeneous polynomials of degree n in two variables $z = (z_1, z_2)$. Then $\mathrm{SU}(2)$ acts by pull-back on V_n, giving us a representation

$$\pi_n : \mathrm{SU}(2) \to \mathrm{Aut}(V_n); \qquad (\pi_n(g)p)(z) = p(g^{-1}z).$$

This is a representation is of dimension $n+1$ since V_n is spanned by the polynomials $z_1^n, z_1^{n-1}z_2, \ldots, z_2^n$. It is unitary with respect to the inner product

$$\langle p, q \rangle = \int_{S^3} \overline{p(z)} q(z)\, dz$$

where S^3 is the unit sphere in $\mathbb{C}^2$ and dz is the unique rotation-invariant smooth measure.

Theorem 2.2.1. *The representations π_n above are all irreducible. Moreover, every irreducible unitary representation of* $\mathrm{SU}(2)$ *is isomorphic to one of these representations.*

For instance, the trivial representation of $\mathrm{SU}(2)$ is isomorphic to π_0 and the fundamental representation is isomorphic to π_1.

This theorem is of fundamental importance for the representation theory of semisimple Lie groups. To prove it, it's best to use the Lie algebra of $\mathrm{SU}(2)$, which we'll deal with in chapter **??**.

Therefore, the compact quantum group algebra $\mathcal{A}(\mathrm{SU}(2))$ is spanned by the matrix coefficients of the representations π_n. For instance, we have the matrix coefficients of the fundamental representation

$$\langle e_1 | \cdot | e_1 \rangle : \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mapsto \alpha,$$
$$\langle e_2 | \cdot | e_1 \rangle : \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mapsto \beta,$$

where α, β are the coefficients as in Equation 2.2.1. It follows that $\mathcal{A}(SU(2))$ contains all polynomials in α, β and their conjugates. In fact, $\mathcal{A}(\mathrm{SU}(2))$ is precisely the algebra of such polynomials, under pointwise multiplication.

2.3 The Lie algebra

As is often the case, it is much easier to work with linear spaces than smooth manifolds. This motivates us to pass to the Lie algebra $\mathfrak{su}(2)$. Let's recall the definitions, in the generality of $\mathrm{SU}(N)$. The reader familiar with Lie algebras can skip this section.

The Lie algebra $\mathfrak{g} = \mathfrak{su}(N)$ is the tangent space of $G = \mathrm{SU}(N)$ at the identity element I. This we can calculate. If $g : \mathbb{R} \to \mathrm{SU}(N)$ is a smooth curve with $g(0) = I$, then it has the form

$$g(t) = I + tX + o(t)$$

for some tangent vector $X = g'(0) \in \mathfrak{su}(N)$. Since $g(t)$ is unitary for all t we get

$$I = (I + tX + o(t))^*(I + tX + o(t)) = I + tX + tX^* + o(t)$$

and so

$$X + X^* = 0.$$

Similarly, using the fact that for any $X \in M_N(\mathbb{C})$ we have $\mathrm{Det}(I + tX + o(t^2)) = 1 + t\,\mathrm{Tr}(X) + o(t^2)$, we deduce that $\mathrm{Tr}(X) = 0$ for any $X \in \mathfrak{su}(N)$. As a consequence, we have

$$\mathfrak{su}(N) = \{X \in M_N(\mathbb{C}) \mid X = -X^* \text{ and } \mathrm{Tr}(X) = 0\}.$$

This linear space retains information about the group structure on $\mathrm{SU}(N)$ as follows. Let $g \in \mathrm{SU}(N)$. The conjugation map

$$\mathscr{A}d_g : G \to G; \quad \mathscr{A}d_g(h) = ghg^{-1}$$

is a smooth map which sends I to I, so it has a derivative on the tangent space at I, which we denote by

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}.$$

Again, we can calculate this by considering the action of $\mathscr{A}d_g$ on a smooth family of elements $h(t) = I + tY + o(t)$ as follows:

$$\mathscr{A}d_g(I + tY + o(t)) = I + tgYg^{-1} + o(t),$$

so

$$\mathrm{Ad}_g(Y) = gYg^{-1}.$$

Next we can differentiate this map $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ with respect to $g \in G$, to obtain a map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. Consider, as usual, a smooth curve $g(t) = I + tX + o(t) \in \mathrm{SU}(N)$. One can easily check that $g(t)^{-1} = I - tX + o(t)$, which is to say that the derivative of the inverse map $\iota : G \to G$ is

$$d\iota_I : X \to -X. \tag{2.3.1}$$

Therefore

$$\begin{aligned}\mathrm{Ad}_{g(t)}(Y) = g(t)Yg(t)^{-1} &= (I + tX + o(t))Y(I - tX + o(Y)) \\ &= Y - tXY - tYX + o(t),\end{aligned}$$

and hence the derivative of Ad is given by

$$\mathrm{ad}_X(Y) = XY - YX =: [X, Y].$$

This expression $[X, Y]$ is the *Lie bracket* of X and Y.

Definition 2.3.1. A real-linear subspace $\mathfrak{g} \subset M_k(\mathbb{R})$ is called a *Lie algebra* if it is closed under the Lie bracket, *i.e.*, for any $X, Y \in \mathfrak{g}$ we have $[X, Y] \in \mathfrak{g}$.

Remark 2.3.2. There is an abstract definition of Lie algebra as a vector space $\mathfrak{g}$ with a Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is bilinear, antisymmetric, and satisfies the Jacobi identity. Since we will only be concerned with matrix groups, Definition 2.3.1 will be enough for our needs.

Example 2.3.3. The Lie algebra $\mathfrak{su}(2)$ is

$$\left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \middle| \, a,b,c \in \mathbb{R} \right\}.$$

It is a real Lie algebra of dimension 3.

2.4 Representations of Lie algebras

Suppose $\pi : \mathrm{SU}(N) \to \mathrm{Aut}(V)$ is a smooth finite dimensional representation of the group $G = \mathrm{SU}(N)$. Once again, we can differentiate this at the identity to obtain a derived map $\pi : \mathfrak{g} \to \mathrm{End}(V)$. Explicitly, we have

$$\pi(I + tX + o(t)) = 1 + t.\pi(X) + o(t)$$

for any smooth curve $g(t) = I + tX + o(t)$ in $\mathrm{SU}(N)$. One can check that the derived map on $\mathfrak{g}$ is a Lie algebra representation in the following sense.

Definition 2.4.1. Let $\mathfrak{g}$ be a Lie algebra. A representation of $\mathfrak{g}$ on a vector space V is a linear map $\pi : \mathfrak{g} \to \mathrm{End}(V)$ which is a Lie algebra morphism, meaning

$$\pi([X,Y]) = [\pi(X), \pi(Y)]$$

for all $X, Y \in \mathfrak{g}$.

Theorem 2.4.2. *Let G be a connected and simply connected Lie group, for example* $\mathrm{SU}(N)$*. The above correspondence from smooth finite dimensional representations of G to Lie algebra representations of $\mathfrak{g}$ is bijective. That is, every representation of the Lie algebra can be 'exponentiated' to give a representation of the Lie group.*

Example 2.4.3. Let $G = \mathbb{T}$ be the circle group. We can see this as the group of complex numbers of modulus 1 in $M_1(\mathbb{C})$. The Lie algebra of $\mathbb{T}$ can be identified as $\mathfrak{t} \cong \mathbb{R}$, such that $x \in \mathbb{R}$ corresponds to the a curve of type

$$g(t) = e^{itx} = 1 + t.ix + o(t^2).$$

The differential of the group representation

$$\pi_n : \mathbb{T} \to \mathrm{Aut}(\mathbb{C}^1); \qquad \pi_n(z) \mapsto z^n$$

is then the Lie algebra representation

$$\pi_n : \mathbb{R} \to \mathrm{End}(\mathbb{C}^1); \qquad \pi_n(x) = inx.$$

As a consequence we have the following result.

Proposition 2.4.4. *For any finite dimensional representation of $\mathbb{T}$, the derived representation of $1 \in \mathfrak{t}$ is diagonal with all eigenvalues in $i\mathbb{Z}$.*

Remark 2.4.5. The integrality observed in Proposition 2.4.4 is of central importance in the theory of *weights* for representations of $\mathrm{SU}(N)$. But we won't have time to go into this.

Example 2.4.6. The *fundamental representation* of $\mathrm{SU}(N)$ is the representation on $\mathbb{C}^N$ defined by

$$\pi(g)\xi = g.\xi \qquad (g \in \mathrm{SU}(N),\ \xi \in \mathbb{C}^N).$$

The associated Lie algebra representation is given by

$$\pi(X)\xi = X.\xi \qquad (X \in \mathfrak{su}(N),\ \xi \in \mathbb{C}^N).$$

Example 2.4.7. The derivative of the trivial representation,

$$\hat{\epsilon} : G \to \mathrm{Aut}(\mathbb{C}); \qquad \hat{\epsilon}(g) = 1,$$

is the trivial Lie algebra representation,

$$\hat{\epsilon} : \mathfrak{g} \to \mathrm{End}(\mathbb{C}); \qquad \hat{\epsilon}(X) = 0. \tag{2.4.1}$$

It is important to note how tensor products of representations behave when converted into Lie algebra representations. Recall that if $\pi : G \to \mathrm{Aut}(V)$ and $\tau : G \to \mathrm{Aut}(W)$ are finite dimensional representations, then their tensor product is

$$\pi \otimes \tau : G \to \mathrm{Aut}(V \otimes W); \qquad (\pi \otimes \tau)(g) = \pi(g) \otimes \tau(g).$$

If $g(t) = I + tX + o(t)$ is a smooth curve in G, then

$$(\pi \otimes \tau)(g(t)) = (1 \otimes 1 + t\pi(X) \otimes 1 + 1 \otimes t\tau(X) + o(t))$$

so that the differential representation of $\mathfrak{g}$ is

$$(\pi \otimes \tau)(X) = \pi(X) \otimes 1 + 1 \otimes \tau(X). \tag{2.4.2}$$

This is the Leibniz rule for Lie algebra representations.

2.5 Complexification

The Lie algebra associated to a Lie group is a real vector space. It is usually easier to work with a complex vector space.

Definition 2.5.1. The complexification of a real Lie algebra $\mathfrak{g}$ is the space $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{g}$, viewed as $\mathfrak{g} + \mathbf{i}\mathfrak{g}$, equipped with the Lie bracket $[\cdot, \cdot]$ which extends the bracket of $\mathfrak{g}$ to a $\mathbb{C}$-bilinear map.

In the above definition, $\mathbf{i}$ is a formal notation, used to define an abstract $\mathbb{C}$-linear structure on $\mathfrak{g} \oplus \mathfrak{g}$. However, for the Lie algebra $\mathfrak{g} = \mathfrak{su}(N)$ we can be more explicit, since it is embedded as a real subspace of $M_N(\mathbb{C})$, namely the space of traceless skew-adjoint matrices. Multiplying this by i we get $i\mathfrak{su}(N)$ is the space of traceless self-adjoint matrices. Since these have intersection $\{0\}$, we get

$$\mathfrak{su}(N)_{\mathbb{C}} = \mathfrak{su}(N) + i\mathfrak{su}(N) = \mathfrak{sl}(N, \mathbb{C}) = \{X \in M_N(\mathbb{C}) \mid \mathrm{Tr}(X) = 0\}$$

with its usual Lie bracket, $[X, Y] = XY - YX$.

Any $\mathbb{R}$-linear representation of $\mathfrak{g}$ on a complex vector space extends canonically to a $\mathbb{C}$-linear representation of $\mathfrak{g}_{\mathbb{C}}$. This is important, since the eigenvalues of representations of elements $X \in \mathfrak{g}$ are purely imaginary, *cf.* Proposition 2.4.3.

Example 2.5.2. The complexification of $\mathfrak{su}(2)$ is $\mathfrak{sl}(2, \mathbb{C})$, which is a three-dimensional complex Lie algebra with basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Lie bracket is given in this basis by

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H. \tag{2.5.1}$$

These are the defining relations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, and they are fundamental importance in representation theory of semisimple Lie groups. Note, in particular, that the adjoint action of H,

$$\mathrm{ad}(H) = [H, \bullet] : \mathfrak{g} \to \mathfrak{g},$$

is diagonal with respect to the basis E, H, F with eigenvalues $2, 0, -2$, respectively.

2.6 The universal enveloping algebra

Theorem 2.6.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. There exists an associative algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ over $\mathbb{C}$, unique up to isomorphism, which contains $\mathfrak{g}_{\mathbb{C}}$ as an embedded Lie subalgebra and such that any Lie algebra morphism $\mathfrak{g}_{\mathbb{C}} \to \mathcal{A}$ of $\mathfrak{g}_{\mathbb{C}}$ into a complex associative algebra $\mathcal{A}$ factors as $\mathfrak{g}_{\mathbb{C}} \hookrightarrow \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{A}$ for some algebra morphism $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{A}$.*

The point of the enveloping algebra is that it allows us to define products of Lie algebra elements. For instance, elements of $\mathcal{U}(\mathfrak{su}(2)_{\mathbb{C}})$ are linear combinations of monomials in the generators E, H, F of Example 2.5.2. The product of two such monomials can be calculated by commuting the terms past one another using the relations (2.5.1). This is basically the contents of the Poincaré-Birkhoff-Witt Theorem.

We won't go into the details of the construction, although it isn't particularly difficult.

After all the above constructions, beginning with a smooth representation of G on a complex vector space V, we obtain a $\mathbb{C}$-linear algebra representation of $\mathcal{U}(\mathfrak{g})$ by the following steps:

$$\begin{aligned} & \pi : G \to \mathrm{Aut}(V) && \text{(Lie group representation)} \\ \xrightarrow{\text{differentiate}} \quad & \pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(V) && \text{($\mathbb{R}$-Lie alg. representation)} \\ \xrightarrow{\text{complexify}} \quad & \pi : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}_{\mathbb{R}}(V) && \text{($\mathbb{C}$-Lie alg. representation)} \\ \xrightarrow{\text{universality}} \quad & \pi : \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \to \mathrm{End}_{\mathbb{R}}(V) && \text{($\mathbb{C}$-alg. representation)} \end{aligned}$$

Algebra representations are much more manageable than group representations. But moreover, the enveloping algebra is not just an associative algebra, it is a Hopf $*$-algebra.

Theorem 2.6.2. *The universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ of a complexified Lie algebra admits a unique Hopf $*$-algebra such that the operations on elements $X \in \mathfrak{g}$ are given by*

coproduct: $\hat{\Delta}(X) = X \otimes 1 + 1 \otimes X$,

counit: $\hat{\epsilon}(X) = 0$,

antipode: $\hat{S}(X) = -X$,

involution: $X^* = -X$.

That is, these definitions can be extended $\mathbb{C}$-linearly (antilinearly in the case of $*$) and multiplicatively (anti-multiplicatively in the case of $\hat{S}$ and $*$) to all of $\mathcal{U}(\mathfrak{g})$. Moreover, the natural bilinear pairing

$$(\ ,\ ) : \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \times \mathcal{A}(G) \to \mathbb{C}; \qquad (X, \langle x'| \cdot |\xi\rangle) := (xi', \pi(X)\xi).$$

is a nondegenerate skew-pairing of Hopf $*$-algebras.

The proof follows from duality with the Hopf algebra operations on $\mathcal{A}(G)$. We won't give the details. But note that the definition of the coproduct is inspired by the Leibniz rule (2.4.2), the counit by the trivial representation (2.4.1), and the antipode by the derivative of the inverse (2.3.1).

Exercise 2.6.3. Let $a = \langle\xi'| \cdot |\xi\rangle \in \mathcal{A}(G)$ be a matrix coefficient of a smooth finite dimensional representation. Show that if $X \in \mathfrak{g}$ is tangent to a curve $\gamma(t) = I + tX + o(t)$ through the identity then the pairing is given by

$$(X, a) = \frac{d}{dt} a(\gamma(t))|_{t=0}.$$

Thus the pairing with X is equal to a derivative of the Dirac distribution at the identity.

This exercise shows that we have an embedding $\mathfrak{g} \hookrightarrow \mathcal{E}'(G)$ of the Lie algebra as distributions supported at the identity. Since convolution of distributions is given by Mult_*, the set of distributions supported at the identity is a subalgebra of $\mathcal{E}'(G)$. It follows from the skew-pairing that $\mathcal{U}(\mathfrak{g})$ embeds as the algebra distributions on G supported at the identity. This gives an alternative definition of the enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.

The enveloping algebra $\mathcal{U}(\mathfrak{g})$ is an alternative 'dual object' for the Hopf $*$-algebra $\mathcal{A}(G)$ of a compact Lie group, which is in many ways simpler to define than the algebra $\mathcal{D}(G)$, which we have still not actually defined in general. It has the small disadvantage that $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ is not a compact quantum group algebra, because it doesn't have a Haar integral, but this is a minor complaint.

Chapter 3

the quantum group $\mathrm{SU}_q(2)$

Finally, we can give the first example of a compact quantum group which is neither commutative nor cocommutative: the quantum group $\mathrm{SU}_q(2)$. The existence of this quantum group was first observed by mathematical physicists working on quantum scattering theory and the quantum Yang-Baxter equation. We'll skip all the historical context and move directly to their first mathematical observation—the existence of a quantum deformation of the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{R})$. This algebra is the first in a whole family of quantized enveloping algebras called *Drinfeld-Jimbo algebras* after their discoverers.

3.1 q-numbers

Throughout this section we fix a strictly positive real number $q \in \mathbb{R}_+^\times$ with $q \neq 1$. We will sometimes write $q = e^h$ with $h \in \mathbb{R} \setminus \{0\}$. This is the quantizing parameter. Taking the limit as $q \to 1$ gives the classical Hopf $*$-algebras $\mathcal{U}(\mathfrak{su}(2)_\mathbb{C})$ and $\mathcal{A}(\mathrm{SU}(2))$ from the previous chapter.

Definition 3.1.1. Let $z \in \mathbb{C}$. The q-number $[z]_q$ is defined as

$$[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}.$$

Exercise 3.1.2. Show that as $q \to 1$ we have $[z]_q \to z$.

We often suppress the index q from the notation.

These q-numbers are like non-linear versions of the classical numbers. For instance, we don't have $[a]_q + [b]_q = [a+b]_q$ in general, but we do have the identity $[a+b]_q[a-b]_q = [a]_q^2 - [b]_q^2$.

3.2 The quantized enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2)_\mathbb{C})$

Recall that the classical enveloping algebra $\mathcal{U}(\mathfrak{su}(2)_\mathbb{C})$ is generated by three elements E, H, F satisfying the relations

$$[H,E] = 2E, \qquad [H,F] = 2F, \qquad [E,F] = H.$$

The fundamental idea of the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2)_\mathbb{C})$ is to replace these relations by

$$[H,E] = 2E, \qquad [H,F] = 2F, \qquad [E,F] = [H]_q \tag{3.2.1}$$

where

$$[H]_q := \frac{q^H - q^{-H}}{q - q^{-1}}. \tag{3.2.2}$$

The inspiration for this comes from the theory of Poisson quantization. But it raises the technical problem of how to make sense of the expression (3.2.2), which is not an algebraic relation because q^H is a transcendental function of H.

There are many possible solutions. An operator algebraic approach is to define q^H by functional calculus, after representing E, F, H as unbounded operators on some Hilbert space. There is also an h-adic approach, by expanding $q^H = e^{hH}$ as a formal power series in the parameter h.

But the standard approach, which is entirely algebraic, is simply to replace the generator H by a new generator $K := q^H$. The side-effect of this is that we are forced also to replace the first two relations in (3.2.1). To motivate this, some functional calculus.

Exercise 3.2.1. Let H be a self-adjoint operator on a Hilbert space V. In this context, we can define the operator e^{tH} for all $t \in \mathbb{R}$. Show that if $X \in \mathcal{B}(V)$ and $\lambda \in \mathbb{R}$, we have $[H,X] = \lambda X$ if and only if $e^{tH} X e^{-tH} = e^{t\lambda} X$ for all $t \in \mathbb{R}$.

Putting $K = q^H = e^{hH}$ and using the above exercise, this suggests the following definition.

Definition 3.2.2. The quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2)_\mathbb{C})$ is the universal algebra generated by three elements E, K, F such that K is invertible, and subject to the relations

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F, \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{3.2.3}$$

We equip it with the involution defined by

$$E^* = KF, \qquad F^* = EK^{-1}, \qquad K^* = K.$$

Remark 3.2.3. In the above definition we have imposed that K is self-adjoint. As we will see later, this is not quite strong enough for our purposes. We are supposed to be thinking that $K = q^H$ with H self-adjoint, which means that K is not just self-adjoint but positive. But positivity is not a purely algebraic condition, so we will have to impose this later when we define the algebra $\mathcal{A}(\mathrm{SU}(2))$ of matrix coefficients.

As in the classical case, $\mathcal{U}_q(\mathfrak{su}(2)_\mathbb{C})$ is a Hopf $*$-algebra, compare Theorem 2.6.2.

Theorem 3.2.4. *The quantized enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2)_\mathbb{C})$ admits a unique Hopf $*$-algebra structure such that the operations on the generators are given by*

$$\begin{aligned}
&\hat{\Delta}(E) = E \otimes K + 1 \otimes E,\\
&\hat{\Delta}(F) = F \otimes 1 + K^{-1} \otimes F,\\
&\hat{\Delta}(K) = K \otimes K,\\
&\hat{\epsilon}(E) = \hat{\epsilon}(F) = 0, \qquad \hat{\epsilon}(K) = 1,\\
&\hat{S}(E) = -EK^{-1}, \qquad \hat{S}(F) = -KF, \qquad \hat{S}(K) = K^{-1}.
\end{aligned}$$

3.3 Finite dimensional representations of $\mathcal{U}_q(\mathfrak{su}(2)_\mathbb{C})$

Let π be a representation of $\mathcal{U}_q(\mathfrak{su}(2)_\mathbb{C})$ on a finite dimensional vector space V. Using the relations

$$KEK^{-1} = q^2 E \qquad \text{and} \qquad KFK^{-1} = q^{-2} F,$$

we can see that if $v \in V$ is an eigenvector for K with eigenvalue $\lambda = q^\alpha$, then $\pi(E)v$ and $\pi(F)v$ are eigenvectors for K with eigenvalues $q^{\alpha+2}$ and $q^{\alpha-2}$, respectively. It follows that if V is irreducible, then $\pi(K)$ acts diagonally on V with all eigenvalues in $q^{\alpha+2\mathbb{Z}}$.

The exponents $\alpha + 2\mathbb{Z}$ are called the *weights* of the representation, and the eigenvectors of K are *weight vectors*.

As mentioned in Remark 3.2.3, we are only really interested in representations in which $\pi(K)$ is a positive operator. Such representations are called *integrable* or *type 1*. With a little elementary linear algebra, one can obtain the following classification of the irreducible integrable representations.

Theorem 3.3.1 (Classification of)**.** *Let $n \in \mathbb{N}$. Let V_n be an $(n+1)$-dimensional vector space with basis $v_n, v_{n-2}, v_{n-4}, \ldots, v_{-n}$.*

There is an irreducible representation π_n of $\mathcal{U}_q(\mathfrak{su}(2)_\mathbb{C})$ on $\mathbb{C}^{n+1}$ defined by

$$\begin{aligned}
&\pi_n(K) : v_j \mapsto q^j v_j\\
&\pi_n(F) : v_j \mapsto v_{j-2}\\
&\pi_n(E) : v_j \mapsto [\tfrac{1}{2}(n-j)]_q [\tfrac{1}{2}(n+j+2)]_q v_{j+2},
\end{aligned}$$

and there is an inner product on V_n such that π_n is a $*$-representation. With respect to this inner product, (v_j) is an orthogonal but not orthonormal basis.

Moreover, every irreducible integrable representation of $\mathcal{U}_q(\mathfrak{su}(2)_{\mathbb{C}})$ is isomorphic to one of these representations π_n.

The representations π_n are the quantum analogues of the irreducible $\mathfrak{su}(2)$-representations π_n from Section 2.2. The trivial representation is π_0. It is equivalently given by the counit of $\mathcal{U}_q(\mathfrak{su}(2)_{\mathbb{C}})$,

$$\pi_0 = \hat{\epsilon} : \mathcal{U}_q(\mathfrak{su}(2)_{\mathbb{C}}) \to \mathrm{End}(\mathbb{C}).$$

Thanks to the existence of a coproduct $\hat{\Delta}$ on $\mathcal{U}_q(\mathfrak{su}(2)_{\mathbb{C}})$, we can also define the tensor product of two finite dimensional representations $\pi : \mathcal{U}_q(\mathfrak{su}(2)_{\mathbb{C}}) \to \mathrm{End}(V)$ and $\tau \to \mathrm{End}(W)$,

$$(\pi \otimes \tau)(X) = \pi(X_{(1)}) \otimes \tau(X_{(2)}).$$

This is the analogue of the Leibniz rule (2.4.2). In this way, the finite dimensional integrable representations of $\mathcal{U}_q(\mathfrak{su}(2)_{\mathbb{C}})$ form a monoidal category, which we denote by $\mathrm{Rep}(\mathrm{SU}_q(2))$.

Remark 3.3.2. Unlike the classical enveloping algebra $\mathcal{U}(\mathfrak{su}(2))$, the coproduct $\hat{\Delta}$ on $\mathcal{U}_q(\mathfrak{su}(2)_{\mathbb{C}})$ is not cocommutative. Therefore, the flip map $\mathfrak{S} : V \otimes W \to W \otimes V$ does not intertwine the representations $\pi \otimes \tau$ and $\tau \otimes \pi$ in general. Nevertheless, one can show that there is a natural family of intertwiners $V \otimes W \to W \otimes V$, making $\mathrm{Rep}(\mathrm{SU}_q(2))$ into a braided monoidal category. This braiding structure was important for the original applications to the quantum Yang-Baxter equation.

3.4 The compact quantum group $\mathrm{SU}_q(2)$

Definition 3.4.1. We define $\mathcal{A}(\mathrm{SU}_q(2))$ to be the subspace of the dual space of $\mathcal{U}_q(\mathfrak{su}(2)_{\mathbb{C}})$ consisting of matrix coefficients of finite dimensional integrable representations,

$$\mathcal{A}(\mathrm{SU}_q(2)) = \{\langle \xi' | \bullet | \xi \rangle \in (\mathcal{U}_q(\mathfrak{su}(2)_{\mathbb{C}}))^* \mid \xi \in V,\ \xi' \in V^* \text{ for some fin. dim. integrable rep. } V\}.$$

Theorem 3.4.2. *The space $\mathcal{A}(\mathrm{SU}_q(2))$ admits the structure of a compact quantum group algebra such that the canonical pairing*

$$(\ ,\ ) : \mathcal{U}_q(\mathfrak{su}(2)_{\mathbb{C}}) \times \mathcal{A}(\mathrm{SU}_q(2)) \to \mathbb{C}; \qquad (X, \langle \xi' | \bullet | \xi \rangle) = \xi'(\pi(X)\xi)$$

is a skew-pairing of $*$-Hopf algebras.

Proof. Since we know that $\mathcal{U}_q(\mathfrak{su}(2)_\mathbb{C})$ is a $*$-Hopf algebra, the proof is essentially the same as that for the algebra $\mathcal{A}(G)$ of matrix coefficients for a classical group in Proposition 2.1.4. We define the Hopf $*$-algebra operations on $\mathcal{A}(\mathrm{SU}_q(2))$ by skew-duality with the operations on $\mathcal{U}_q(\mathfrak{su}(2)_\mathbb{C})$. To check these are well-defined it suffices to compute how they act on matrix coefficients:

- $\langle \xi' | \bullet | \xi \rangle \langle \eta' | \bullet | \eta \rangle = \langle \eta' \otimes \xi' | \bullet | \eta \otimes \xi \rangle$,
- $1 = \langle 1 | \bullet | 1 \rangle$, as a matrix coefficient for the trivial rep. $\hat{\epsilon}$,
- $\Delta \langle \xi' | \bullet | \xi \rangle = \sum_j \langle \xi' | \bullet | e_j \rangle \otimes \langle e^j | \bullet | \xi \rangle$, where (e_j) and (e^j) are dual bases for V and V^*,
- $\epsilon(\langle \xi' | \bullet | \xi \rangle) = (\xi', \xi)$,
- $S(\langle \xi' | \bullet | \xi \rangle) = \langle \xi | \bullet | \xi' \rangle$, as a matrix coefficient for the precontragredient representation ${}^{\mathrm{c}}\pi(X) = \pi(\hat{S}^{-1}(X))^{\mathrm{t}}$,
- $\langle \xi' | \bullet | \xi \rangle^* = \langle \overline{\xi'} | \bullet | \overline{\xi} \rangle$, as a matrix coefficient for the conjugate representation $\pi \circ \hat{S} \circ *$ on $\overline{V}$.

The Haar state is given on matrix coefficients for an irreducible representation π by

$$\phi(\langle \xi' | \bullet | \xi \rangle) = \begin{cases} (\xi', \xi), & \text{if } \pi \text{ is the trivial rep.} \\ 0, & \text{else.} \end{cases}$$

□

Remark 3.4.3. In the definition of the product, the tensor factors are flipped on the right-hand side in order to have a *skew*-pairing of Hopf $*$-algebras. This wasn't necessary in the classical case, because the product is commutative.

3.5 Quantum groups and noncommutative geometry

Theorem 3.4.2 provides us with a whole family of compact quantum groups $\mathrm{SU}_q(2)$ with $0 < q < \infty$. We declare the algebra $\mathcal{A}(\mathrm{SU}_1(2))$ to be the classical algebra of matrix coefficients for $\mathrm{SU}(2)$.

Note that the algebras $\mathcal{A}(G_q)$ are all canonically isomorphic as linear spaces. This is because for different values of q, the irreducible representations π_n ($n \in \mathbb{N}$) are all defined on the same vector space $V_n = \mathbb{C}^{n+1}$. Therefore, the space of matrix coefficients is

$$\mathcal{A}(\mathrm{SU}_q(2)) = \bigoplus_{n \in \mathbb{N}} V_n^* \otimes V_n = \bigoplus_{n \in \mathbb{N}} \mathrm{End}(V_n)^*$$

for every $q \in (0, \infty)$. Further, they are all isomorphic as coalgebras, since the coproduct in each case is given by

$$\hat{\Delta} : \langle \xi' | \bullet | \xi \rangle \mapsto \sum_i \langle \xi' | \bullet | e_i \rangle \otimes \langle e^i | \bullet | \xi \rangle.$$

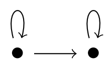
In fact, the only thing that varies between them is the algebra structure. Since the coproduct is supposed to represent the group law, $\hat{\Delta} = \mathrm{Mult}^*$, and the product is supposed to represent the topology, via Gelfand's Theorem, this is to say that the quantum groups $\mathrm{SU}_q(2)$ are all identical *as groups*, but we have changed the topology. That topology, for $q \neq 1$, is noncommutative, in the sense of Connes.

To look further into this noncommutative topology, we can complete the $*$-algebras $\mathcal{A}(\mathrm{SU}_q(2))$ to C^*-algebras. Specifically, using the Haar state ϕ on $\mathcal{A}(G_q)$, we can define a faithful inner product on $\mathcal{A}(\mathrm{SU}_q(2))$ by

$$\langle a, b \rangle = \phi(a^* b).$$

The Hilbert space completion is denoted $L^2(\mathrm{SU}_q(2))$, and the GNS construction yields a representation of $\mathcal{A}(\mathrm{SU}_q(2))$ on $L^2(\mathrm{SU}_q(2))$ by left multiplication. The norm-closure of $\mathcal{A}(\mathrm{SU}_q(2))$ in $\mathcal{B}(L^2(\mathrm{SU}_q(2)))$ is denoted $C(\mathrm{SU}_q(2))$ and called the *algebra of continuous functions* on the compact quantum group $\mathrm{SU}_q(2)$. Note that the same construction for the classical group $\mathrm{SU}(2)G$ yields the algebra $C(\mathrm{SU}(2))$ by the Stone-Weierstrass Theorem.

It turns out that these C^*-algebras are all isomorphic. In fact, they are all isomorphic to a particularly simple C^*-algebra: the graph C^*-algebra associated to the graph



This was observed by Woronowicz [Wor87], with generalizations by Hong-Szymanski [HS02]. On the other hand, the dense subalgebras $\mathcal{A}(\mathrm{SU}_q(2))$ are *not* isomorphic, which suggests that the quantum groups $\mathrm{SU}_q(2)$ are the same as noncommutative topological spaces, but not as noncommutative smooth manifolds.

Similar quantum group deformations are possible for any simply connected compact semisimple Lie group G. Drinfeld and Jimbo defined a quantization $\mathcal{U}_q^{\mathbb{R}}(\mathfrak{g})$ of the enveloping algebra, and the algebra of matrix coefficients of finite dimensional integrable representations is again a compact quantum group algebra $\mathcal{A}(G_q)$ for all $q \in (0, \infty)$, $q \neq 1$.

These quantum groups and their homogeneous spaces are natural candidates for noncommutative smooth manifolds, in the sense of Connes. But they don't fulfil Connes' original axioms. Chakraborty-Pal [CP03] managed to define nice spectral triples on $\mathrm{SU}_q(2)$, but the problem of incorporating the higher rank examples into Connes' framework has turned out to be surprisingly difficult.

Bibliography

[CP03] Partha Sarathi Chakraborty and Arupkumar Pal. Equivariant spectral triples on the quantum SU(2) group. *K-Theory*, 28(2):107–126, 2003.

[Fri98] F. G. Friedlander. *Introduction to the theory of distributions*. Cambridge University Press, Cambridge, second edition, 1998. With additional material by M. Joshi.

[HS02] Jeong Hee Hong and Wojciech Szymański. Quantum spheres and projective spaces as graph algebras. *Comm. Math. Phys.*, 232(1):157–188, 2002.

[KV00] Johan Kustermans and Stefaan Vaes. Locally compact quantum groups. *Ann. Sci. École Norm. Sup. (4)*, 33(6):837–934, 2000.

[Trè67] François Trèves. *Topological vector spaces, distributions and kernels*. Academic Press, New York-London, 1967.

[VD98] A. Van Daele. An algebraic framework for group duality. *Adv. Math.*, 140(2):323–366, 1998.

[Voi08] Christian Voigt. Bornological quantum groups. *Pacific J. Math.*, 235(1):93–135, 2008.

[Wor87] S. L. Woronowicz. Twisted SU(2) group. An example of a noncommutative differential calculus. *Publ. Res. Inst. Math. Sci.*, 23(1):117–181, 1987.